ON THE THEORY OF UNBIASED TESTS OF SIMPLE STATISTICAL HYPOTHESES SPECIFYING THE VALUES OF TWO OR MORE PARAMETERS[1]

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1. Summary. Unbiased critical regions of type D for testing simple hypotheses specifying the values of several parameters are defined and their properties studied. These regions constitute a natural generalization of the Neyman-Pearson regions of type A for testing simple hypotheses specifying the value of one parameter. A theorem is obtained which plays the role of the Neyman-Pearson fundamental lemma in the type A case. Illustrative examples of type D regions are given.

2. Introduction. The parameter space Ω will, in this paper, be a subset of a k-dimensional Euclidean space ($k \geq 1$), and $\theta = (\theta_1, \cdots, \theta_k)$ will denote a point in Ω. A simple statistical hypothesis is one which specifies the values of all unknown parameters. When we refer to a statistical test we mean a Borel measurable set in an n-dimensional sample space such that if the sample point falls in this critical region we reject the null hypothesis. In this paper the term *region* will always mean a Borel measurable set. The probability of rejecting a true hypothesis when using a given test is called the size of this test. A test is unbiased if the power function of the test has a relative minimum for the value $\theta = \theta^0$, where θ^0 is the value of θ specified by the hypothesis to be tested.

A locally best unbiased region for testing a simple hypothesis specifying the value of one parameter is called *type A* by Neyman and Pearson [1]. It is obtained by maximizing the curvature of the power curve at the point $\theta = \theta^0$ specified by the hypothesis, subject to the conditions of given size and unbiasedness. Geometrically speaking, the power curve of a region of type A is above the power curves of all other unbiased regions of the same size in an infinitesimal neighborhood of θ^0. For the purpose of generalization to the k-parameter case it is useful to note that if we consider the power curve of the type A region and the power curves of any other unbiased regions of the same size, then the length of a horizontal chord at a fixed infinitesimal distance above the minimum point is a minimum when compared with the length of this chord on the power curves of the other unbiased regions of the same size. We note that the definition of type A regions does not use any information about the relative importance of errors of type II. (An error of type II is made when we accept a false hypothesis.)

Type A regions remain invariant under transformations of the parameter which are locally one-to-one and twice differentiable. Regions of type A can

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be proved to exist under quite weak assumptions on the joint density function of the sample.

When we proceed to consider simple hypotheses specifying the values of two or more parameters, we are immediately faced with a more complicated situation. (For the sake of simplicity in statement, we confine ourselves in this introductory section to the two-parameter theory; the extension of our discussion to three or more parameters is direct.) In the two-parameter case the geometrical picture of the power function is a surface, and if we require of a locally best unbiased region that its power surface have maximum curvature along every cross-section at the point $(\theta_1, \theta_2) = (\theta_1^0, \theta_2^0)$ specified by the hypothesis, subject to the conditions of size and unbiasedness, then it develops that this requirement cannot be met even in the simplest cases; for if we maximize the curvature of the power surface along one of its cross-sections, we find that in general this causes the curvature to diminish along other cross-sections and so we cannot maximize the curvature along all cross-sections at once.

To handle the two-parameter theory, Neyman and Pearson [2] considered *type C* regions. They require of a critical region not only that it be of given size and unbiased but also that it have constant power in an infinitesimal neighborhood of (θ_1^0, θ_2^0) along a *given* family of concentric ellipses with the same shape and orientation; the type C region is then defined as the one among this class of regions which gives best local power. When the *given* family of ellipses consists of circles, the region of type C is called *regular*; otherwise it is called *nonregular*. One can choose the family of ellipses if and essentially only if one knows the relative importance of errors of type II in an infinitesimal neighborhood of (θ_1^0, θ_2^0). In the absence of such information one cannot proceed to find a region of type C. Regions of type C retain their property of unbiasedness under transformations of the parameter space which are locally one-to-one and twice differentiable, but in general *regular* unbiased critical regions of type C become *nonregular* under such transformations. Hence if one is inclined in the absence of advance information about errors of type II to favor the regular unbiased region of type C as a region fulfilling "good" intuitive requirements, then the objection can be raised that these regular regions of type C are not invariant under transformations of the parameter space.

There is an approach to the problem of finding a "good" critical region which overcomes the objections raised to the type C theory; i.e., it will provide us with a criterion for choosing a critical region without using any advance knowledge as to the relative importance of errors of type II, and this type of critical region will be invariant under transformations of the parameters. This type of critical region, which will be a natural generalization of the type A region of the one-parameter theory, will maximize the Gaussian curvature of the power surface at $(\theta_1, \theta_2) = (\theta_1^0, \theta_2^0)$, subject to the conditions of size and unbiasedness. In the next two sections we shall develop this theory for simple hypotheses specifying the values of two parameters, and then in Section 5 we shall extend it to the case of simple hypotheses specifying the values of three or more parameters.

3. Definition of unbiased critical regions of type D in the two-parameter case. We introduce for the power function of a region w the symbol

$$\beta(\theta_1, \theta_2 \mid w) = Pr(E \varepsilon w \mid \theta_1, \theta_2), \tag{3.1}$$

where $E = (x_1, \cdots, x_n)$ is the sample point in an n-dimensional sample space. Here the joint probability distribution of the sample depends on the parameter $\theta = (\theta_1, \theta_2)$, and we are testing the hypothesis $(\theta_1, \theta_2) = (\theta_1^0, \theta_2^0)$. We make a translation of the parameter space to bring the point (θ_1^0, θ_2^0) to the origin, so that we may consider the test of the hypothesis $(\theta_1, \theta_2) = (0, 0)$. The size of the critical region is then

$$\beta(0, 0 \mid w) = Pr(E \varepsilon w \mid 0, 0). \tag{3.2}$$

We also introduce the following notation:

$$\beta_i(w) = \left.\frac{\partial \beta(\theta_1, \theta_2 \mid w)}{\partial \theta_i}\right|_{(\theta_1, \theta_2) = (0,0)}, \qquad i = 1, 2, \tag{3.3}$$

$$\beta_{ij}(w) = \left.\frac{\partial^2 \beta(\theta_1, \theta_2 \mid w)}{\partial \theta_i \, \partial \theta_j}\right|_{(\theta_1, \theta_2) = (0,0)}, \qquad i, j = 1, 2. \tag{3.4}$$

We assume these derivatives exist. We shall write β_i and β_{ij} for $\beta_i(w)$ and $\beta_{ij}(w)$, respectively, whenever our doing so will cause no ambiguity. We note that the derivatives are taken at $(\theta_1, \theta_2) = (0, 0)$, though this fact does not show up in our notation.

In books on differential geometry, such as Eisenhart [3], it is shown that if we consider a surface in three-dimensional Euclidean space and a point (x_0, y_0) at which the second partial derivatives of the function $z = f(x, y)$ which describes the surface exist and are continuous, then the so-called Gaussian or total curvature K of the surface $z = f(x, y)$ at the point (x_0, y_0) is given by:

$$K = \frac{\left.\dfrac{\partial^2 z}{\partial x^2}\right|_{(x_0, y_0)} \left.\dfrac{\partial^2 z}{\partial y^2}\right|_{(x_0, y_0)} - \left[\left.\dfrac{\partial^2 z}{\partial x \partial y}\right|_{(x_0, y_0)}\right]^2}{\left(1 + \left[\left.\dfrac{\partial z}{\partial x}\right|_{(x_0, y_0)}\right]^2 + \left[\left.\dfrac{\partial z}{\partial y}\right|_{(x_0, y_0)}\right]^2\right)^2}. \tag{3.5}$$

The Gaussian curvature is invariant under translation and rotation of the coordinate axes. Applying (3.5) to the power surface $\beta = \beta(\theta_1, \theta_2 \mid w)$ at the point $(\theta_1, \theta_2) = (0, 0)$, imposing the condition of unbiasedness on w, and noting that necessary conditions for an unbiased region are that $\beta_1(w) = \beta_2(w) = 0$, we have

$$K = \frac{\beta_{11}(w)\beta_{22}(w) - \beta_{12}^2(w)}{(1 + 0 + 0)^2} = \begin{vmatrix} \beta_{11}(w) & \beta_{12}(w) \\ \beta_{12}(w) & \beta_{22}(w) \end{vmatrix} = \det B_w, \tag{3.6}$$

where

$$B_w = \begin{pmatrix} \beta_{11}(w) & \beta_{12}(w) \\ \beta_{12}(w) & \beta_{22}(w) \end{pmatrix}.$$

As a natural generalization of the type A region of the one-parameter theory, we now propose as a critical region for testing $(\theta_1, \theta_2) = (0, 0)$ that critical region which maximizes the Gaussian curvature of the power surface at (0, 0), subject to the conditions of size and unbiasedness. This leads us to the following

DEFINITION. *A region w_0 is said to be an unbiased critical region of type D for testing H_0 if:*

I. $\beta(0, 0 \mid w_0) = \alpha$;

II. $\beta_i(w_0) = 0, \qquad i = 1, 2$;

III. B_{w_0} *is positive definite*;

IV. *det* $B_{w_0} \geq$ *det* B_w *for any other region w satisfying* I–III.

Condition I specifies the size of the test. Conditions II and III insure the existence of a relative minimum at $(\theta_1, \theta_2) = (0, 0)$ and so imply the condition of unbiasedness. Condition IV specifies that the region of type D has maximal Gaussian curvature among all unbiased regions of the prescribed size.

Let us consider the geometrical interpretation of a region of type D. In the one-parameter theory we noted that the type A region minimizes the length of a certain infinitesimal chord on the power curve. We shall now see that the type D region minimizes the area of a certain infinitesimal ellipse, subject to the conditions of size and unbiasedness. Consider a Taylor expansion of the power function in an infinitesimal neighborhood of $(\theta_1, \theta_2) = (0, 0)$. We have, neglecting infinitesimals of the third and higher orders,

$$\begin{aligned} \beta(\theta_1, \theta_2 \mid w) &= \beta(0, 0 \mid w) + \theta_1\beta_1 + \theta_2\beta_2 + \tfrac{1}{2}(\theta_1^2\beta_{11} + 2\theta_1\theta_2\beta_{12} + \theta_2^2\beta_{22}) \\ &= \alpha + \tfrac{1}{2}(\theta_1^2\beta_{11} + 2\theta_1\theta_2\beta_{12} + \theta_2^2\beta_{22}). \end{aligned} \tag{3.7}$$

Consider the ellipse $\theta_1^2\beta_{11} + 2\theta_1\theta_2\beta_{12} + \theta_2^2\beta_{22} = \delta$, where δ is a positive constant; this ellipse is a horizontal cross-section of the power surface at an infinitesimal distance above the minimum point $(\theta_1, \theta_2) = (0, 0)$. It is well known that the area of this ellipse is given by

$$\frac{\pi\delta}{\sqrt{\begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{vmatrix}}} = \frac{\pi\delta}{\sqrt{\det B}}. \tag{3.8}$$

We have just seen that the region of type D maximizes the determinant of B subject to the conditions of size and unbiasedness. Hence it minimizes the area of our infinitesimal ellipse subject to these same conditions.

4. Theorems concerning regions of type D in the two-parameter theory. Having defined regions of type D, we now wish to obtain a theorem which will characterize the structure of such regions for us. We shall assume the following fundamental condition is satisfied:

There exists a joint density function $p(E \mid \theta_1, \theta_2)$ for any point (θ_1, θ_2) in the parameter space Ω; and for any fixed region w in the sample space the integral $\int_w p(E \mid \theta_1, \theta_2)\, dE$ has second partial derivatives with respect to θ_1 and θ_2 in a *neighborhood of* $(\theta_1, \theta_2) = (0, 0)$ *which are continuous at* $(0, 0)$, *and the integral can be differentiated twice under the integral sign with respect to* θ_1 *and* θ_2 *at* $(0, 0)$.

The derivatives of the above types taken at $(\theta_1, \theta_2) = (0, 0)$ will be denoted simply as follows:

$$\frac{\partial}{\partial \theta_i} \int_w p(E \mid \theta_1, \theta_2)\, dE \,\Big|_{\theta_1=\theta_2=0} = \int_w p_i\, dE = \beta_i(w), \qquad i = 1, 2, \tag{4.1}$$

$$\frac{\partial^2}{\partial \theta_i\, \partial \theta_j} \int_w p(E \mid \theta_1, \theta_2)\, dE \,\Big|_{\theta_1=\theta_2=0} = \int_w p_{ij}\, dE = \beta_{ij}(w), \qquad i, j = 1, 2, \tag{4.2}$$

where

$$p_i = \frac{\partial p(E \mid \theta_1, \theta_2)}{\partial \theta_i}\Bigg|_{\theta_1=\theta_2=0}, \qquad p_{ij} = \frac{\partial^2 p(E \mid \theta_1, \theta_2)}{\partial \theta_i\, \partial \theta_j}\Bigg|_{\theta_1=\theta_2=0}. \tag{4.3}$$

We also write $p(E \mid 0, 0) = p$.

We seek a theorem which will tell us how to characterize the structure of a region w_0 such that $\int_w g_{11}\, dE \int_w g_{22}\, dE - \left(\int_w g_{12}\, dE\right)^2$ is a maximum, subject to the side conditions that $\int_w f_i\, dE = c_i$, $i = 1, \cdots, m$, where the g_{ij} and the f_i are given integrable functions and the c_i are given constants. If we have such a theorem, then by taking $m = 3$, $g_{11} = p_{11}$, $g_{12} = p_{12}$, $g_{22} = p_{22}$, $f_1 = p$, $f_2 = p_1$, $f_3 = p_2$, $c_1 = \alpha$, $c_2 = 0$, $c_3 = 0$, we will be able to use the theorem to characterize the structure of a region of type D, since in terms of the p's our conditions on a type D region are:

I′. $\int_{w_0} p\, dE = \alpha$;

II′. $\int_{w_0} p_i\, dE = 0, \qquad i = 1, 2;$

III′. *The matrix* $P_{w_0} = \left(\int_{w_0} p_{ij}\, dE\right)$, $i, j = 1, 2$, *is positive definite*;

IV′. *det* $P_{w_0} \geq$ *det* P_w *for any other region* w *satisfying* I′–III′.

We will now state the Neyman-Pearson fundamental lemma, which is used in the one-parameter theory to find regions of type A, in order to indicate the type of theorem we are seeking and also because we shall use this lemma in proving our theorem.

THE NEYMAN-PEARSON FUNDAMENTAL LEMMA. *Suppose* $m + 1$ *given integrable functions* $f_0, f_1, \cdots, f_m$ *are defined in an n-dimensional space. Consider the set of all regions* w *for which the following conditions are fulfilled*:

$$\int_w f_i\, dE = c_i, \qquad i = 1, \cdots, m, \tag{4.4}$$

where the c_i are m given constants. If w_0 is a region which satisfies the m conditions (4.4) *and if*

$$f_0 \geq \sum_{i=1}^{m} k_i f_i \quad in \quad w_0, \tag{4.5}$$
$$f_0 \leq \sum_{i=1}^{m} k_i f_i \quad outside \quad w_0,$$

for m suitably chosen constants k_i, then w_0 has the property that

$$\int_{w_0} f_0 \, dE \geq \int_w f_0 \, dE \tag{4.6}$$

for any region w which satisfies (4.4).

We proceed to state and prove a lemma which will tell us how to characterize a region w_0 maximizing $\int_w g_{11} \, dE \int_w g_{22} \, dE$ subject to integral side conditions, and then to use the lemma and a corollary to it to prove a theorem which will characterize the structure of a region w_0 which maximizes $\int_w g_{11} \, dE \int_w g_{22} \, dE - \left(\int_w g_{12} \, dE\right)^2$ subject to integral side conditions.

LEMMA 1. *Suppose $m + 2$ given integrable functions $g_{11}, g_{22}, f_1, \cdots, f_m$ are defined in an n-dimensional space. Consider the set of all regions w for which the following conditions are fulfilled:*

$$\int_w f_i \, dE = c_i, \qquad i = 1, \cdots, m, \tag{4.7}$$

$$\int_w g_{jj} \, dE > 0, \qquad j = 1, 2, \tag{4.8}$$

where the c_i are m given constants. If w_0 is a region which satisfies conditions (4.7) *and* (4.8), *and if*

$$\sum_{i=1}^{2} k_{ii} g_{ii} \geqq \sum_{i=1}^{m} k_i f_i \quad in \quad w_0, \tag{4.9}$$
$$\sum_{i=1}^{2} k_{ii} g_{ii} \leqq \sum_{i=1}^{m} k_i f_i \quad outside \quad w_0,$$

where $k_{11} = \int_{w_0} g_{22} \, dE$, $k_{22} = \int_{w_0} g_{11} \, dE$, and the k_i are m suitably chosen constants, then w_0 has the property that

$$\prod_{j=1}^{2} \int_{w_0} g_{jj} \, dE \geqq \prod_{j=1}^{2} \int_w g_{jj} \, dE \tag{4.10}$$

for any region w which satisfies (4.7) *and* (4.8).

We note that we must know our region w_0 in advance so that we can calculate k_{11} and k_{22} and thus verify whether w_0 has the structure required by the lemma or not.

PROOF. We apply the Neyman-Pearson fundamental lemma to the function

$$f_0 = \sum_{i=1}^{2} k_{ii} g_{ii} = g_{11} \int_{w_0} g_{22}\, dE + g_{22} \int_{w_0} g_{11}\, dE.$$

From (4.6) we obtain

$$\int_{w} g_{11}\, dE \int_{w_0} g_{22}\, dE + \int_{w} g_{22}\, dE \int_{w_0} g_{11}\, dE \leq 2 \int_{w_0} g_{11}\, dE \int_{w_0} g_{22}\, dE \tag{4.11}$$

for any region w satisfying (4.7). Knowing (4.11) we must prove that

$$\int_{w} g_{11}\, dE \int_{w} g_{22}\, dE \leq \int_{w_0} g_{11}\, dE \int_{w_0} g_{22}\, dE \tag{4.12}$$

for any region w satisfying (4.7) and (4.8).

Let

$$x_j = \frac{\int_{w} g_{jj}\, dE}{\int_{w_0} g_{jj}\, dE}, \qquad j = 1, 2. \tag{4.13}$$

Since the integrals $\int_{w} g_{jj}\, dE$, $\int_{w_0} g_{jj}\, dE$, $j = 1, 2$, are all positive by (4.8) we may rewrite (4.11) and (4.12) in terms of the x_j's as follows:

$$x_1 + x_2 \leq 2, \tag{4.14}$$

$$x_1 x_2 \leq 1. \tag{4.15}$$

Thus we must prove that $x_1 x_2 \leq 1$ whenever $\frac{1}{2}(x_1 + x_2) \leq 1$, where x_1 and x_2 are positive real numbers. But this follows immediately from the well known inequality between the arithmetic and geometric mean, and hence our lemma is proved.

COROLLARY. *If a region w_0 satisfies conditions* (4.7), (4.8), *and* (4.9) *of the lemma, and if g_{12} is a given integrable function for which $\int_{w_0} g_{12}\, dE = 0$, then*

$$\int_{w_0} g_{11}\, dE \int_{w_0} g_{22}\, dE - \left(\int_{w_0} g_{12}\, dE\right)^2 \geq \int_{w} g_{11}\, dE \int_{w} g_{22}\, dE - \left(\int_{w} g_{12}\, dE\right)^2 \tag{4.16}$$

for any region w satisfying conditions (4.7) *and* (4.8) *of the lemma.*

We now use the lemma and the corollary to it to prove the following theorem:

THEOREM 1. *Suppose the elements g_{ij} of a symmetric 2×2 matrix*

$$G \doteq \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

are given integrable functions defined in an n-dimensional space; and that $f_1, \cdots, f_m$ are m other given integrable functions defined in this space. For any region w, let

$$G_w = \begin{vmatrix} \int_w g_{11}\, dE & \int_w g_{12}\, dE \\ \int_w g_{21}\, dE & \int_w g_{22}\, dE \end{vmatrix}.$$

Consider the set of all regions w for which the following conditions are fulfilled:

$$\int_w f_i\, dE = c_i, \qquad i = 1, \cdots, m, \tag{4.17}$$

$$G_w \text{ is positive definite,} \tag{4.18}$$

where the c_i are m given constants. If w_0 is a region which satisfies the conditions (4.17) *and* (4.18), *and if*

$$\begin{aligned} \sum_{i,j=1}^{2} k_{ij} g_{ij} &\geq \sum_{i=1}^{m} k_i f_i \quad \text{in} \quad w_0, \\ \sum_{i,j=1}^{2} k_{ij} g_{ij} &\leq \sum_{i=1}^{m} k_i f_i \quad \text{outside} \quad w_0, \end{aligned} \tag{4.19}$$

where $k_{11} = \int_{w_0} g_{22}\, dE$, $k_{22} = \int_{w_0} g_{11}\, dE$, $k_{12} = k_{21} = -\int_{w_0} g_{12}\, dE$, and the k_i are m suitably chosen constants, then w_0 has the property that

$$\det G_{w_0} \geq \det G_w \tag{4.20}$$

for any region w which satisfies (4.17) *and* (4.18).

PROOF. We know there exists an orthogonal matrix H of constants which diagonalizes G_{w_0}; that is, $H'G_{w_0}H$ is a diagonal matrix, and $H'H = I$. Apply this transformation to

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

and let

$$G^* = \begin{pmatrix} g_{11}^* & g_{12}^* \\ g_{21}^* & g_{22}^* \end{pmatrix} = H'GH.$$

We note that

$$G^*_{w_0} = \begin{Vmatrix} \int_{w_0} g^*_{11}\, dE & 0 \\ 0 & \int_{w_0} g^*_{22}\, dE \end{Vmatrix} = H'G_{w_0}H. \tag{4.21}$$

Since H is orthogonal, we know that $\det G_{w_0} = \det G^*_{w_0}$, and also $\det G_w = \det G^*_w$, where

$$G^*_w = \begin{Vmatrix} \int_w g^*_{11}\, dE & \int_w g^*_{12}\, dE \\ \int_w g^*_{21}\, dE & \int_w g^*_{22}\, dE \end{Vmatrix} = H'G_wH.$$

Thus we see that if $\det G^*_{w_0} \geq \det G^*_w$ for any region w satisfying (4.17) and (4.18), then $\det G_{w_0} \geq \det G_w$ for any such region, and this is what we seek to prove. But since G_{w_0} and G_w are positive definite, we know that $G^*_{w_0}$ and G^*_w are positive definite; hence their diagonal elements are positive and they satisfy condition (4.8); then by our lemma and its corollary we know that $\det G^*_{w_0} \geq \det G^*_w$ for any region w satisfying (4.17) and (4.18) (and hence (4.8)), if w_0 satisfies

$$\begin{aligned} g^*_{11} \int_{w_0} g^*_{22}\, dE + g^*_{22} \int_{w_0} g^*_{11}\, dE &\geq \sum_{i=1}^{m} k_i f_i \quad \text{in} \quad w_0, \\ g^*_{11} \int_{w_0} g^*_{22}\, dE + g^*_{22} \int_{w_0} g^*_{11}\, dE &\leq \sum_{i=1}^{m} k_i f_i \quad \text{outside} \quad w_0. \end{aligned} \tag{4.22}$$

It now remains only to prove that the conditions (4.22) are implied by (4.19). To do this we shall prove that

$$g^*_{11} \int_{w_0} g^*_{22}\, dE + g^*_{22} \int_{w_0} g^*_{11}\, dE = g_{11} \int_{w_0} g_{22}\, dE + g_{22} \int_{w_0} g_{11}\, dE - 2g_{12} \int_{w_0} g_{12}\, dE.$$

Denote the adjoint of a matrix A by adj A. Then $(\text{adj } G^*_{w_0})G^* = H'(\text{adj } G_{w_0})HH'GH = H'(\text{adj } G_{w_0})GH$, since H is orthogonal. But

$$g_{11} \int_{w_0} g_{22}\, dE + g_{22} \int_{w_0} g_{11}\, dE - 2g_{12} \int_{w_0} g_{12}\, dE$$

is the trace of $(\text{adj } G_{w_0})G$ and similarly

$$g^*_{11} \int_{w_0} g^*_{22}\, dE + g^*_{22} \int_{w_0} g^*_{11}\, dE$$

is the trace of $(\text{adj } G^*_{w_0})G^*$. Hence our two expressions are equal, as we know that the trace of a matrix is invariant under an orthogonal transformation, and we have just seen that $(\text{adj } G^*_{w_0})G^*$ is obtained from $(\text{adj } G_{w_0})G$ by such a transformation. This completes the proof.

We shall now prove a result we mentioned earlier; namely, the invariance of regions of type D under transformations of the parameters.

THEOREM 2. *If the transformation* $\theta_i = T_i(\Theta_1, \Theta_2)$, $i = 1, 2$, *is such that the first and second partial derivatives* $\partial\theta_s/\partial\Theta_i$ *and* $\partial^2\theta_s/\partial\Theta_i\partial\Theta_j$ *exist and are continuous at* $(\Theta_1, \Theta_2) = (0, 0)$, $i, j, s = 1, 2$, *the Jacobian* $\partial(\theta_1, \theta_2)/\partial(\Theta_1, \Theta_2)$ *differs from zero at* $(\Theta_1, \Theta_2) = (0, 0)$, *and* $(0, 0)$ *maps into* $(0, 0)$; *then a region* w_0, *which is an unbiased critical region of type D for testing* $(\theta_1, \theta_2) = (0, 0)$ *against the set of alternative hypotheses specifying the values of the parameters* θ_1 *and* θ_2, *will remain an unbiased critical region of type D for testing* $(\Theta_1, \Theta_2) = (0, 0)$ *against the set of transformed hypotheses specifying the values of the new parameters* Θ_1 *and* Θ_2.

PROOF. We adopt the following notation:

$$\text{(4.23)} \qquad \left.\frac{\partial\theta_1}{\partial\Theta_1}\right|_{\Theta_1=\Theta_2=0} = K, \qquad \left.\frac{\partial\theta_1}{\partial\Theta_2}\right|_{\Theta_1=\Theta_2=0} = L,$$
$$\left.\frac{\partial\theta_2}{\partial\Theta_1}\right|_{\Theta_1=\Theta_2=0} = M, \qquad \left.\frac{\partial\theta_2}{\partial\Theta_2}\right|_{\Theta_1=\Theta_2=0} = N.$$

By hypothesis the determinant of

$$J = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$$

is not equal to zero. We denote by $\beta_{(i)}$ and $\beta_{(ij)}$ the partial derivatives of the power function with respect to Θ_i and Θ_j *evaluated at* $(\Theta_1, \Theta_2) = (0, 0)$. Also let

$$B_{(w)} = \begin{pmatrix} \beta_{(11)}(w) & \beta_{(12)}(w) \\ \beta_{(12)}(w) & \beta_{(22)}(w) \end{pmatrix}.$$

Then we can write

$$\text{(4.24)} \qquad \beta_{(1)} = \beta_1 K + \beta_2 M,$$
$$\beta_{(2)} = \beta_1 L + \beta_2 N.$$

The condition that $\beta(0, 0 \mid w_0) = \alpha$ is unchanged by the transformation of parameters. Since we know that for an unbiased region $\beta_1 = 0$ and $\beta_2 = 0$, we obtain from (4.24) that $\beta_{(1)} = 0$ and $\beta_{(2)} = 0$. Thus, since the partial derivatives of the transformation are continuous, our property of unbiasedness is retained. Also since $\beta_1 = 0$ and $\beta_2 = 0$, it is easily seen that

$$\text{(4.25)} \qquad B_{(w)} = J' B_w J.$$

Since J is nonsingular by hypothesis, we know that $B_{(w)}$ is positive definite since B_w is a positive definite matrix. Also we have that

$$\text{(4.26)} \qquad \det B_{(w)} = (\det J)^2 \det B_w ;$$

and since $\det J \neq 0$, it follows that if $\det B_{w_0} \geq \det B_w$, then $\det B_{(w_0)} \geq$

det $B_{(w)}$. Thus we have seen that w_0 satisfies all the conditions of a region of type D for testing $(\Theta_1, \Theta_2) = (0, 0)$, and our proof is completed.

The inequalities which must hold within and outside the unbiased critical regions of type D can frequently be simplified if we express them in terms of the derivatives of $\log p(E \mid \theta_1, \theta_2)$. We write

$$\phi_t = \left.\frac{\partial \log p(E \mid \theta_1, \theta_2)}{\partial \theta_t}\right|_{(\theta_1, \theta_2) = (0,0)}, \tag{4.27}$$
$$\phi_{ts} = \left.\frac{\partial^2 \log p(E \mid \theta_1, \theta_2)}{\partial \theta_t \partial \theta_s}\right|_{(\theta_1, \theta_2) = (0,0)},$$

where $t, s = 1, 2$. In particular, the simplification will be considerable if

$$\phi_{ts} = A_{ts} + B_{ts}\phi_1 + C_{ts}\phi_2, \qquad t, s = 1, 2, \tag{4.28}$$

where A_{ts}, B_{ts}, C_{ts} are independent of the sample point E but may depend on (θ_1, θ_2). If (4.28) is true, it will be seen that

$$p_1 = \phi_1 p, \qquad p_2 = \phi_2 p, \tag{4.29}$$

$$p_{ts} = (\phi_t\phi_s + A_{ts} + B_{ts}\phi_1 + C_{ts}\phi_2)p. \tag{4.30}$$

Consequently, the type of inequalities (4.19) occurring among the sufficient conditions of Theorem 1 for a region of type D will reduce to the following for points where $p > 0$ (assuming that $W_+(\theta_1, \theta_2) = \{E \mid p(E \mid \theta_1, \theta_2) > 0\}$ is independent of (θ_1, θ_2)):

$$\left(\int_{w_0} p_{11}\, dE\right)\phi_2^2 + \left(\int_{w_0} p_{22}\, dE\right)\phi_1^2 - 2\left(\int_{w_0} p_{12}\, dE\right)\phi_1\phi_2 \gtrless k_1' + k_2'\phi_1 + k_3'\phi_2, \tag{4.31}$$

where the k_i' are new constants easily expressible in terms of the k_i, $\int_{w_0} p_{ij}\, dE$, and the coefficients in (4.28). The k_i' must be determined so as to satisfy $\int_{w_0} p\, dE = \alpha$, $\int_{w_0} p_1\, dE = 0$, $\int_{w_0} p_2\, dE = 0$, which, owing to (4.29), reduce to $\int_{w_0} p\, dE = \alpha$, $\int_{w_0} \phi_1 p\, dE = 0$, $\int_{w_0} \phi_2 p\, dE = 0$, respectively. Using these relationships, the inequality (4.31) will further simplify to

$$\left(\int_{w_0} \phi_1^2 p\, dE + A_{11}\alpha\right)\phi_2^2 + \left(\int_{w_0} \phi_2^2 p\, dE + A_{22}\alpha\right)\phi_1^2 - 2\left(\int_{w_0} \phi_1\phi_2 p\, dE + A_{12}\alpha\right)\phi_1\phi_2 \gtrless k_1' + k_2'\phi_1 + k_3'\phi_2. \tag{4.32}$$

Here the sign $\geq$ applies in w_0 and $\leq$ outside w_0. The region described by this inequality is obviously the region outside an ellipse in the ϕ_1, ϕ_2-plane.

5. Generalization of the theory to the k-parameter case. We shall now indicate how to generalize the theory of Sections 3 and 4 to the case where we have k parameters, where $k \geq 2$. Our main task here will be to obtain a generalization of Lemma 1 and Theorem 1 of Section 4.

The power function is now designated by $\beta(\theta_1, \theta_2, \cdots, \theta_k \mid w)$, and we are testing the hypothesis that $(\theta_1, \theta_2, \cdots, \theta_k) = (0, 0, \cdots, 0)$. For brevity we write $\theta = (\theta_1, \theta_2, \cdots, \theta_k)$, so that $\beta(\theta \mid w)$ now will symbolize the power function and the hypothesis is $\theta = \mathbf{0} = (0, 0, \cdots, 0)$. $\beta_i(w)$ and $\beta_{ij}(w)$ will again denote the partial derivatives of $\beta(\theta \mid w)$ evaluated at $\theta = 0$, where now i and j run from 1 to k.

We now define the generalized Gaussian curvature of $\beta(\theta \mid w)$ at $\theta = 0$ as follows:

$$K = \frac{\begin{vmatrix} \beta_{11}(w) & \cdots & \beta_{1k}(w) \\ \vdots & & \vdots \\ \beta_{k1}(w) & \cdots & \beta_{kk}(w) \end{vmatrix}}{\left(1 + \sum_{j=1}^{k} \beta_j^2(w)\right)^2} = \frac{\det B_w}{\left(1 + \sum_{j=1}^{k} \beta_j^2(w)\right)^2}, \tag{5.1}$$

where

$$B_w = \begin{pmatrix} \beta_{11}(w) & \cdots & \beta_{1k}(w) \\ \vdots & & \vdots \\ \beta_{k1}(w) & \cdots & \beta_{kk}(w) \end{pmatrix}.$$

The generalized Gaussian curvature is invariant under translation and rotation of the coordinate axes in the $(k + 1)$-dimensional space of $(\beta, \theta_1, \theta_2, \cdots, \theta_k)$. Imposing our condition of unbiasedness on $\beta(\theta \mid w)$ at $\theta = 0$ gives us $\beta_j(w) = 0$, $j = 1, \cdots, k$; and hence we have for $\beta(\theta \mid w)$ at $\theta = 0$

$$K = \det B_w. \tag{5.2}$$

In view of this discussion, our definition of a region of type D in Section 3 immediately generalizes to k parameters.

Geometrically, the region of type D may be regarded as minimizing the volume of a certain infinitesimal k-dimensional ellipsoid $\sum_{i,j=1}^{k} \beta_{ij}\theta_i\theta_j = \delta$, as explained in detail in Section 3 for the case $k = 2$.

We again assume the fundamental condition at the beginning of Section 4 is satisfied. We use the notation of (4.1), (4.2), and (4.3); i.e., $\beta_i(w) = \int_w p_i \, dE$, and $\beta_{ij}(w) = \int_w p_{ij} \, dE$, where now $i, j = 1, \cdots, k$, and we let $p(E \mid 0) = p$. In terms of these p's our conditions on a type D region are expressed by I′–IV′ of Section 4, with i and j running from 1 to k.

We thus see that in the k-parameter theory we need a theorem which tells how to characterize the structure of a region which maximizes the determinant of a symmetric positive definite $k \times k$ matrix, whose elements are integrals over the region, subject to integral side conditions. To this end, we obtain generalizations of Lemma 1 and Theorem 1 of Section 4.

We generalize the statement of Lemma 1 by replacing 2 by k whenever a 2 occurs in the statement. Relation (4.9) is replaced by

$$\begin{aligned} \sum_{i=1}^{k} k_{i i} g_{i i} &\geq \sum_{i=1}^{m} k_{i} f_{i} \quad \text{in} \quad w_0, \\ \sum_{i=1}^{k} k_{i i} g_{i i} &\leq \sum_{i=1}^{m} k_{i} f_{i} \quad \text{outside} \quad w_0, \end{aligned} \tag{5.3}$$

where

$$k_{ii} = \prod_{\substack{j=1 \\ j \neq i}}^{k} \int_{w_0} g_{jj} \, dE.$$

The proof of the lemma then proceeds exactly as it does in the case $k = 2$.

The corollary to Lemma 1 is now given in the following form:

COROLLARY. *Consider a symmetric matrix*

$$G = \begin{pmatrix} g_{11} & \cdots & g_{1k} \\ \vdots & & \vdots \\ g_{k1} & \cdots & g_{kk} \end{pmatrix}$$

whose elements are given integrable functions defined in an n-dimensional space. For any region w in this space, let

$$G_w = \begin{pmatrix} \int_w g_{11} \, dE & \cdots & \int_w g_{1k} \, dE \\ \vdots & & \vdots \\ \int_w g_{k1} \, dE & \cdots & \int_w g_{kk} \, dE \end{pmatrix}.$$

Now if w_0 is a region that satisfies the conditions (4.7), (4.8), *and* (5.3) *of the lemma, and if furthermore* $\int_{w_0} g_{ij} \, dE = 0$ *when* $i \neq j$, *then* $\det G_{w_0} \geq \det G_w$, *where w is any region in the space for which G_w is positive definite and the conditions* (4.7) *are satisfied.*

PROOF.

$$\det G_{w_0} = \prod_{j=1}^{k} \int_{w_0} g_{jj} \, dE \geq \prod_{j=1}^{k} \int_{w} g_{jj} \, dE \geq \det G_w, \tag{5.4}$$

where the first inequality follows from the lemma and the second is a well known inequality for positive definite matrices (see Cramér [4], p. 116).

Proceeding to Theorem 1, we generalize the statement by once again replacing 2 by k whenever a 2 occurs in the statement. Relation (4.19) is replaced by

$$(5.5)\qquad \begin{aligned} \sum_{i,j=1}^{k} k_{ij} g_{ij} &\geq \sum_{i=1}^{m} k_i f_i \quad \text{in} \quad w_0, \\ \sum_{i,j=1}^{k} k_{ij} g_{ij} &\leq \sum_{i=1}^{m} k_i f_i \quad \text{outside} \quad w_0, \end{aligned}$$

where k_{ij} is the (i, j) element in the adjoint matrix of G_{w_0}. We note that $\sum_{i,j=1}^{k} k_{ij}g_{ij}$ = trace $[(\text{adj } G_{w_0})G]$, where adj G_{w_0} denotes the adjoint matrix of G_{w_0}. The proof of the theorem then proceeds exactly as it does in the case $k = 2$.

Regions of type D in the k-parameter theory remain invariant under transformations of the parameter space which are locally one-to-one and twice differentiable with continuous partial derivatives. This result is obtained by a direct and immediate generalization of Theorem 2 in Section 4.

As in the two-parameter theory, the inequalities which must hold within and outside the unbiased critical region of type D can frequently be simplified if we express them in terms of the derivatives of $\log p(E \mid \theta)$. We write:

$$(5.6)\qquad \begin{aligned} \phi_t &= \left.\frac{\partial \log p(E \mid \theta)}{\partial \theta_t}\right|_{\theta=0}, \\ \phi_{ts} &= \left.\frac{\partial^2 \log p(E \mid \theta)}{\partial \theta_t \, \partial \theta_s}\right|_{\theta=0}, \end{aligned}$$

where $t, s = 1, 2, \cdots, k$. In particular the simplification will be considerable if

$$(5.7)\qquad \phi_{ts} = A_{ts} + \sum_{j=1}^{k} B_{tsj} \phi_j, \qquad t, s = 1, 2, \cdots, k,$$

where A_{ts} and the B_{tsj} are independent of the sample point E but may depend on θ. An unbiased critical region of type D found by application of Theorem 1 will then be the outside of an ellipsoid in the space of the ϕ_t, $t = 1, \cdots, k$.

6. Examples. Suppose that the joint density functions specified by the admissible hypotheses are all of the form

$$(6.1)\qquad \begin{aligned} &p(E \mid \mu_1, \mu_2) \\ &= \frac{1}{(2\pi)^{\frac{1}{2}(n_1+n_2)} \sigma_1^{n_1} \sigma_2^{n_2}} \exp\left[-\frac{1}{2} \left\{ \frac{1}{\sigma_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 + \frac{1}{\sigma_2^2} \sum_{i=n_1+1}^{n_1+n_2} (x_i - \mu_2)^2 \right\} \right], \end{aligned}$$

with known σ_1 and σ_2, for $-\infty < x_i < \infty$, $i = 1, 2, \cdots, (n_1 + n_2)$. Thus it is assumed that the observations represent two samples of n_1 and n_2 individuals respectively, randomly and independently drawn from two normal populations with *known* standard deviations σ_1 and σ_2 respectively and with *unknown* means equal respectively to μ_1 and μ_2. The simple hypothesis H_0 to be tested is that $(\mu_1, \mu_2) = (0, 0)$. We shall find the unbiased critical region of type D for testing H_0.

The joint density function (6.1), as is well known, satisfies all the conditions required in the present theory. Making some simple calculations and substituting into (4.32), we obtain

$$f(\bar{x}_1, \bar{x}_2) = \left(\int_{w_0} \left(\frac{n_1 \bar{x}_1}{\sigma_1^2}\right)^2 p\, dE - \frac{n_1 \alpha}{\sigma_1^2}\right)\left(\frac{n_2 \bar{x}_2}{\sigma_2^2}\right)^2$$

$$(6.2) \qquad + \left(\int_{w_0} \left(\frac{n_2 \bar{x}_2}{\sigma_2^2}\right)^2 p\, dE - \frac{n_2 \alpha}{\sigma_2^2}\right)\left(\frac{n_1 \bar{x}_1}{\sigma_1^2}\right)^2$$

$$- 2\left(\int_{w_0} \frac{n_1 n_2 \bar{x}_1 \bar{x}_2}{\sigma_1^2 \sigma_2^2} p\, dE\right)\left(\frac{n_1 n_2 \bar{x}_1 \bar{x}_2}{\sigma_1^2 \sigma_2^2}\right) - k_1' - k_2' \frac{n_1 \bar{x}_1}{\sigma_1^2} - k_3' \frac{n_2 \bar{x}_2}{\sigma_2^2} \geq 0$$

as the inequality defining the critical region w_0 we seek, providing it can be found by our methods, where $\bar{x}_1$ and $\bar{x}_2$ denote the means of the two samples. It is seen that w_0 is bounded by a surface corresponding to the equation $f(\bar{x}_1, \bar{x}_2) =$ constant, and that, therefore, the conditions $\int_{w_0} p\, dE = \alpha$, $\int_{w_0} \phi_1 p\, dE = 0$, $\int_{w_0} \phi_2 p\, dE = 0$, which the critical region has to satisfy, and also the integrals involved in (4.32) can be expressed by means of integrals taken over a region w_0' in the plane of $\bar{x}_1$ and $\bar{x}_2$, determined by the same inequality (6.2). Of course, instead of the original joint density function $p(E \mid 0, 0)$, we shall have that of $\bar{x}_1$ and $\bar{x}_2$. We further simplify our notation by introducing, instead of $\bar{x}_1$ and $\bar{x}_2$, the variables

$$(6.3) \qquad u = \sqrt{n_1}\bar{x}_1/\sigma_1, \qquad v = \sqrt{n_2}\bar{x}_2/\sigma_2 .$$

Our problem will now be to guess a region w_0'' in the u, v-plane and then see if we can determine the constants k_i' so that the plane region determined by the inequality

$$(6.4) \qquad \frac{n_1 n_2}{\sigma_1^2 \sigma_2^2}\left[\left(\iint_{w_0''} u^2 p(u, v)\, du\, dv - \alpha\right) v^2 + \left(\iint_{w_0''} v^2 p(u, v) du\, dv - \alpha\right) u^2 - 2\left(\iint_{w_0''} uvp(u, v) du\, dv\right) uv\right] \geqq k_1' + k_2' \frac{\sqrt{n_1}}{\sigma_1} u + k_3' \frac{\sqrt{n_2}}{\sigma_2} v$$

will be the region w_0'', where w_0'' satisfies the following conditions:

$$(6.5) \qquad \iint_{w_0''} p(u, v)\, du\, dv = \alpha;$$

$$(6.6) \qquad \iint_{w_0''} up(u, v)\, du\, dv = 0, \qquad \iint_{w_0''} vp(u, v)\, du\, dv = 0;$$

$$(6.7) \qquad \begin{pmatrix} \iint_{w_0''} u^2 p(u, v)\, du\, dv - \alpha & \iint_{w_0''} uvp(u, v)\, du\, dv \\ \iint_{w_0''} uvp(u, v)\, du\, dv & \iint_{w_0''} v^2 p(u, v)\, du\, dv - \alpha \end{pmatrix}$$

is positive definite, where

$$p(u, v) = (2\pi)^{-1} \exp [-\tfrac{1}{2}(u^2 + v^2)]. \tag{6.8}$$

(6.5) is the condition of size; and (6.6), (6.7) are the conditions of unbiasedness. If we have such a region, then by Theorem 1, w_0'' is an unbiased critical region of type D for testing $(\mu_1, \mu_2) = (0, 0)$. In the u, v-plane, the likelihood ratio test indicates the region $u^2 + v^2 \geq a^2$ for testing H_0, where a^2 is determined so as to give size α to the test. Since u^2 and v^2 are each independently distributed as χ^2 with one degree of freedom, $u^2 + v^2$ is distributed as χ^2 with two degrees of freedom and so a^2 can be obtained from a χ^2-table. We shall take $u^2 + v^2 \geq a^2$ as the region w_0'' and shall verify that k_1', k_2', k_3' in (6.4) can indeed be determined so as to give rise to this region. We will also see that (6.7) is satisfied for this region. Then since $u^2 + v^2 \geq a^2$ obviously satisfies the condition (6.6) by symmetry considerations, and a^2 has been determined so as to satisfy (6.5), this will prove that $u^2 + v^2 \geq a^2$ is an unbiased critical region of type D for testing H_0.

One can easily verify that

$$\iint_{u^2+v^2 \geq a^2} u^2 p(u, v)\, du\, dv = \iint_{u^2+v^2\geq a^2} v^2 p(u, v)\, du\, dv = \alpha(1 + \tfrac{1}{2}a^2); \tag{6.9}$$

and since $p(u, v)$ is an even function,

$$\iint_{u^2+v^2\geq a^2} uvp(u, v)\, du\, dv = 0. \tag{6.10}$$

In view of these relations, we see that the matrix in (6.7) is

$$\begin{pmatrix} \alpha a^2/2 & 0 \\ 0 & \alpha a^2/2 \end{pmatrix},$$

which is obviously positive definite. Also, (6.4) can now be written as

$$\frac{n_1 n_2}{\sigma_1^2 \sigma_2^2}\left[\frac{\alpha a^2}{2}(u^2 + v^2)\right] \geq k_1' + k_2' \frac{\sqrt{n_1}}{\sigma_1} u + k_3' \frac{\sqrt{n_2}}{\sigma_2} v. \tag{6.11}$$

If we choose $k_1' = n_1 n_2 \alpha a^4/(2\sigma_1^2\sigma_2^2)$, $k_2' = 0$, $k_3' = 0$, the inequality (6.11) becomes

$$u^2 + v^2 \geqq a^2, \tag{6.12}$$

and this proves our result.

This result can in turn be used to find an unbiased critical region of type D for testing a simple hypothesis about the means of a bivariate normal population with known covariance matrix, since it is possible by an orthogonal transformation of variables to transform this problem into the one we have solved.

The result of (6.12) can also be immediately extended to find an unbiased critical region of type D for testing a simple hypothesis about the means of k independent normal populations with known variances; and then this latter result can be used to find a type D region for testing a simple hypothesis about the means of a k-variate normal distribution with known covariance matrix. The type D regions in these cases turn out to be the likelihood ratio tests.

My attempts to find an unbiased critical region of type D for testing a simple hypothesis about the mean and variance of a univariate normal distribution on the basis of a random sample of size n were unsuccessful because I was unable to evaluate the integrals occurring on the left side of our basic inequality (4.19) over the conjectured region; there were also other difficulties involved. One can, however, use the result of (6.12) for large sample sizes to approximate a type D region for testing the simple hypothesis $(\mu, \sigma^2) = (\mu_0, \sigma_0^2)$. Since $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ and $(s')^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$ are joint sufficient statistics for μ and σ^2, just as we reduced the problem of testing a simple hypothesis about the means of two normal populations to a problem in the $\bar{x}_1, \bar{x}_2$-plane by use of (6.2), so we can reduce the problem of testing $(\mu, \sigma^2) = (\mu_0, \sigma_0^2)$ to a problem in the $\bar{x}$, $(s')^2$-plane. The density function of $\bar{x}$ is normal with mean μ_0 and variance σ_0^2/n under the null hypothesis, and the density function of $(n-1)(s')^2/\sigma_0^2$ is that of χ^2 with $(n-1)$ degrees of freedom under the null hypothesis; since $\bar{x}$ and $(s')^2$ are independently distributed in a normal population, we can use these two density functions immediately to obtain the joint density function of $\bar{x}$ and $(s')^2$. The problem of finding a type D region in the $\bar{x}$, $(s')^2$-plane is, however, the one I was unable to solve. But we know that $(n-1)(s')^2/\sigma_0^2$ has a χ^2 distribution with mean $(n-1)$ and variance $2(n-1)$ and we also know that a χ^2 distribution with m degrees of freedom is asymptotically normal with mean m and variance $2m$; hence we know that $(s')^2$ is asymptotically normally distributed with mean σ_0^2 and variance $2\sigma_0^4/(n-1)$ under the null hypothesis. If we approximate the density function of $(s')^2$ by a normal density function with mean σ_0^2 and variance $2\sigma_0^4/(n-1)$, and let

$$(6.13) \qquad u = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma_0}, \qquad v = \frac{\sqrt{n-1}((s')^2 - \sigma_0^2)}{\sqrt{2}\sigma_0^2},$$

then with this approximation our problem becomes that of finding a region w_0' in the u, v-plane satisfying (6.4), subject to conditions (6.5)–(6.7), where $p(u, v)$ is given by (6.8), and in (6.4) $n_1 = n$, $n_2 = (n-1)$, $\sigma_1 = \sigma_0$, $\sigma_2 = \sqrt{2}\sigma_0^2$. For this problem we have seen that the solution is given by (6.12). In the $\bar{x}$, $(s')^2$-plane this gives the region

$$(6.14) \qquad \frac{n(\bar{x} - \mu_0)^2}{\sigma_0^2} + \frac{(n-1)((s')^2 - \sigma_0^2)^2}{2\sigma_0^4} \geq a^2,$$

where a^2 is determined from a χ^2-table with two degrees of freedom. For large sample sizes this region should be a good approximation to an unbiased critical region of type D for testing $(\mu, \sigma^2) = (\mu_0, \sigma_0^2)$.

7. Remarks on the theory of testing composite hypotheses with two or more constraints. A composite hypothesis with k constraints is a hypothesis which specifies the values of k parameters out of a total of s parameters, where $k < s$. At present the theory of composite hypotheses with two or more constraints is in much less satisfactory shape than the theory of composite hypotheses with one constraint. (For the latter see Scheffé [5] and Lehmann [6].) We can define an unbiased critical region of type E for testing a composite hypothesis with k constraints ($k \geq 2$, $k < s$) as follows:

DEFINITION. *Let* $\Theta = (\theta_1, \theta_2, \cdots, \theta_k, \theta_{k+1}, \cdots, \theta_s) = (\theta_1, \theta_2, \cdots, \theta_k, \tau)$ *denote the parameter point in the parameter space* Ω *which is a subset of an s-dimensional Euclidean space, where* $\tau = (\theta_{k+1}, \cdots, \theta_s)$ *denotes the nuisance parameters (i.e., the parameters unspecified by the hypothesis). The hypothesis* H_0 *states* Θ *lies in the k-dimensional subspace* ω *of* Ω *defined by* $\theta = \theta_0$*, where* $\theta = (\theta_1, \theta_2, \cdots, \theta_k)$ *and* $\theta_0 = (\theta_{10}, \theta_{20}, \cdots, \theta_{k0})$*. Then* w_0 *is said to be an unbiased critical region of type E for testing* H_0 *if for all* θ *in* ω *(i.e., all* (θ_0, τ)*)*:

I. $\beta(\theta_0, \tau \mid w_0) = \alpha$, *where* α *is independent of* τ;

II. $\beta_i(\theta_0, \tau \mid w_0) = 0$ *for* $i = 1, \cdots, k$;

III. $\begin{pmatrix} \beta_{11}(\theta_0, \tau \mid w_0) & \cdots & \beta_{1k}(\theta_0, \tau \mid w_0) \\ \vdots & & \vdots \\ \beta_{k1}(\theta_0, \tau \mid w_0) & \cdots & \beta_{kk}(\theta_0, \tau \mid w_0) \end{pmatrix} = B_{w_0}$ *is positive definite*;

IV. *det* $B_{w_0} \geq$ *det* B_w *for any region* w *satisfying* I–III.

These regions of type E should prove useful in the further development of the theory of composite hypotheses with two or more constraints.

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