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ON THE THERMODYNAMICS OF VISCOELASTIC MATERIALS OF SINGLE-INTEGRAL TYPE

By

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1. Introduction. Because of the difficulties inherent in the study of general materials with memory, there is a vast literature[1] concerning *mechanical* models which capture the essential interaction between dissipation and nonlinearity, but are sufficiently simple to allow both the characterization of real materials and the mathematical analysis of corresponding problems. In one space dimension, a large class of models is based on **single-integral laws** of the form

$$\sigma(t) = M(\gamma(t)) + \int_0^\infty m(\gamma(t), \gamma(t-\tau), \tau)\, d\tau, \tag{1.1}$$

giving the stress $\sigma(t)$ at time t when the strain $\gamma(\lambda)$ is known at all past times $\lambda \le t$.

While constitutive equations of single-integral type have also been proposed for heat conduction,[2] there is, to our knowledge, no general thermodynamical theory of single-integral laws applicable to deforming bodies under varying temperature. Our objective here is to develop such a theory.[3]

We consider a homogeneously deforming body under the influence of a spatially-uniform temperature field.[4] Then **balance of energy** takes the form[5]

$$\varepsilon^{\cdot} = \mathbf{S} \cdot \mathbf{F}^{\cdot} + r \tag{1.2}$$

with ε the **internal energy**, $\mathbf{S}$ the **stress**, $\mathbf{F}$ the **deformation gradient**, and r the **heat supply**. To this list of functions we add the (absolute) **temperature** θ and take, as

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[1]Cf., e.g., [9] and the references cited therein.

[2]Cf., e.g., [1] and the references cited therein.

[3]In this regard we are indebted to Coleman [2] for his general thermodynamical theory of materials with fading memory. In fact, much of Coleman's theory applies to single-integral laws; however, because single-integral laws have such restricted form, the corresponding set of thermodynamical restrictions is richer than the set of restrictions originally derived by Coleman.

[4]Our results extend trivially to nonhomogeneous deformations; nonhomogeneous thermal fields are discussed at the end of the Introduction.

[5]There are many interpretations of the thermodynamic variables $\mathbf{S}$ and $\mathbf{F}$ that yield an energy balance of the form (1.2). Usually $\mathbf{S}$ is the Piola-Kirchhoff stress (cf., e.g., [5] or [11]) and $\mathbf{F}$ the deformation gradient, but any choice of these variables is admissible provided $\mathbf{S} \cdot \mathbf{F}^{\cdot}$ is the associated stress power. For example, to model materials which exhibit nonlinear constitutive behavior at small strains, one might identify $\mathbf{S}$ and $\mathbf{F}$ with the Cauchy stress and (infinitesimal) displacement gradient, respectively.

constitutive equations for stress and internal energy, the single-integral laws

$$\mathbf{S}(t) = \mathbf{\Sigma}(\mathbf{F}(t), \theta(t)) + \int_0^\infty \boldsymbol{\sigma}(\mathbf{F}(t), \theta(t), \mathbf{F}(t-\tau), \theta(t-\tau), \tau)\, d\tau, \tag{1.3}$$

$$\varepsilon(t) = E(\mathbf{F}(t), \theta(t)) + \int_0^\infty e(\mathbf{F}(t), \theta(t), \mathbf{F}(t-\tau), \theta(t-\tau), \tau)\, d\tau.$$

Our main objective is to establish restrictions on these relations that are necessary and sufficient for **compatibility with the second law**. For us, compatibility means the existence of an **entropy** η, in *single-integral form*

$$\eta(t) = H(\mathbf{F}(t), \theta(t)) + \int_0^\infty h(\mathbf{F}(t), \theta(t), \mathbf{F}(t-\tau), \theta(t-\tau), \tau)\, d\tau, \tag{1.4}$$

such that the law of **entropy growth**

$$\eta^{\cdot} \geq r/\theta \tag{1.5}$$

holds in all processes. Here it is important to note that

we do not assume a priori that an entropy exists.

If we eliminate the heat supply between (1.2) and (1.5), we arrive at the inequality

$$\varepsilon^{\cdot} - \mathbf{S} \cdot \mathbf{F}^{\cdot} - \theta\eta^{\cdot} \leq 0, \tag{1.6}$$

and are therefore led to the thermodynamical problem:

Problem (T1). Find restrictions on the constitutive equations (1.3) that ensure the existence of an entropy of the form (1.4) consistent with[6] the inequality (1.6).

Suppose, for the moment, that we have an entropy (1.4) consistent with (1.6). If we define a new variable $\omega(t)$ through

$$\omega = \theta^{-1}\varepsilon - \eta, \tag{1.7}$$

then, by (1.3) and (1.4), ω is given by a constitutive equation of single-integral form:

$$\omega(t) = W(\mathbf{F}(t), \theta(t)) + \int_0^\infty w(\mathbf{F}(t), \theta(t), \mathbf{F}(t-\tau), \theta(t-\tau), \tau)\, d\tau. \tag{1.8}$$

Further, because of (1.7) the inequality (1.6) reduces to

$$\omega^{\cdot} - \theta^{-1}\mathbf{S} \cdot \mathbf{F}^{\cdot} + \theta^{-2}\varepsilon\theta^{\cdot} \leq 0. \tag{1.9}$$

We will refer to ω as a **dissipation potential.**

This new variable allows us to simplify the thermodynamical problem further. We define $\mathbf{f}$ and $\mathbf{g}$ through

$$\mathbf{f} = (\mathbf{F}, \theta), \qquad \mathbf{g} = (\theta^{-1}\mathbf{S}, -\theta^{-2}\varepsilon). \tag{1.10}$$

Then the constitutive equations (1.3) for stress and internal energy take the simple form

$$\mathbf{g}(t) = \mathscr{M}(\mathbf{f}^t) = \mathbf{M}(\mathbf{f}(t)) + \int_0^\infty \mathbf{m}(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau, \tag{1.11}$$

[6] The term "consistent with" means: given an arbitrary prescription of $\mathbf{F}(t)$ and $\theta(t)$ for all t, the corresponding functions $\mathbf{S}(t)$, $\varepsilon(t)$, and $\eta(t)$, defined by (1.3) and (1.4), satisfy (1.6) for all t.

where $\mathbf{f}^t$ is the **history** of $\mathbf{f}$ up to time t defined by $\mathbf{f}^t(\tau) = \mathbf{f}(t-\tau)$, $0 \le \tau < \infty$, and the inequality (1.9) reduces to the **dissipation inequality**

$$\omega^{\cdot} \le \mathbf{g} \cdot \mathbf{f}^{\cdot}. \tag{1.12}$$

Writing (1.8) in terms of the new independent variable $\mathbf{f}$

$$\omega(t) = W(\mathbf{f}(t)) + \int_0^\infty w(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau, \tag{1.13}$$

it is then a simple matter to show that Problem (T1) is equivalent to

Problem (T2). Find restrictions on the constitutive equation (1.11) that ensure the existence of a dissipation potential of the form (1.13) consistent with the dissipation inequality (1.12).

We will refer to the constitutive equations (1.3) (or equivalently (1.11)) as **dissipative** if there is a corresponding dissipation potential.

The general framework afforded by Problem (T2) has an added attraction: by leaving as unspecified the dimension n of the space $\mathbb{R}^n$ in which $\mathbf{f}$ and $\mathbf{g}$ have values, we have an abstract theory applicable to pure mechanics (cf. Gurtin and Hrusa [7]), to electromagnetic interactions, and to other situations of physical interest.

We proceed as follows. We begin by discussing the properties of single-integral laws. We then study dissipative single-integral laws within the general framework of Problem (T2). In particular, *we completely solve* (T2) and, in so doing, give explicit formulas for the functions w and W that describe the dissipation potential (1.13) corresponding to a dissipative single-integral law. It should be emphasized that we require the dissipation potential to be of single-integral form, and this leads to restrictions on $\mathbf{M}$ and $\mathbf{m}$ that are stronger than one would obtain for an enlarged class of dissipation potentials. Indeed, there are single-integral laws that do not satisfy our restrictions but that do have dissipation potentials; of course, such dissipation potentials cannot be expressed in single-integral form.

When $\mathbf{f}(\tau)$ is close to a given constant state $\mathbf{f}_0$ for $-\infty < \tau \le t$, the general single-integral law (1.11) has the asymptotic form

$$\mathbf{g}(t) = \mathscr{M}(\mathbf{f}_0) + \mathbf{G}(0)\mathscr{f}(t) + \int_0^\infty \mathbf{G}'(\tau)\mathscr{f}(t-\tau)\, d\tau + o(\|\mathscr{f}^t\|_\infty),$$
$$\mathscr{f}(\tau) = \mathbf{f}(\tau) - \mathbf{f}_0,$$

with $\|\mathscr{f}^t\|_\infty$ the supremum norm of the history $\mathscr{f}^t$. The function $\mathbf{G}$ is the **relaxation function for** $\mathscr{M}$ (corresponding to $\mathbf{f}_0$). Some of our main results concern this function. We show that if $\mathscr{M}$ is dissipative, then:[7]

(a) $\mathbf{G}(\tau)$ is symmetric at each $\tau \ge 0$;
(b) $\mathbf{G}'(\tau)$ is negative semi-definite at each $\tau > 0$;
(c) $\mathbf{G}''(\tau)$ is positive semi-definite at each $\tau > 0$.

[7]The results (a)–(c) are consequences of the particular structure of single-integral laws. In fact, it is clear from counterexamples of Shu and Onat [10] and Day [4] that the result (b) does not follow within Coleman's [2] framework. Coleman [3] (cf. also Gurtin and Herrera [6]) is able to show only that $\mathbf{G}(0)$ and $\mathbf{G}(\infty)$ are symmetric and the $\mathbf{G}(0) - \mathbf{G}(\infty)$ is positive semi-definite.

We next discuss the consequences of dissipativity for two particular types[8] of single-integral laws: those with noninteractive memory in which the kernel in (1.11) has the form $\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau) = \hat{\mathbf{m}}(\mathbf{f}, \tau) - \hat{\mathbf{m}}(\mathbf{p}, \tau)$, and those with relative memory in which $\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau) = \hat{\mathbf{m}}(\mathbf{f} - \mathbf{p}, \tau)$. We show that the dissipation potentials for such materials admit explicit representations which are both simple and transparent.

We close by giving the implications of our general results within the specific thermodynamical framework of Problem (T1). Within that framework there are four relaxation functions of physical interest: the stress-strain relaxation function $\mathscr{G}(\tau)$, the stress-temperature relaxation function $\mathscr{A}(\tau)$, the energy-strain relaxation function $\mathscr{D}(\tau)$, and the energy-temperature relaxation function $c(\tau)$. If the underlying constant state has temperature θ_0 and vanishing residual stress, then the general results (a)–(c) have the following consequences:[9]

(i) The stress-strain relaxation function is symmetric:

$$\mathscr{G}(\tau) = \mathscr{G}(\tau)^{\mathrm{T}} \quad \text{for all } \tau \geq 0.$$

(ii) The stress-temperature and energy-strain relaxation functions are related by

$$\mathscr{A}(\tau) = -\theta_0 \mathscr{D}(\tau) \quad \text{for all } \tau \geq 0.$$

(iii) The following definiteness conditions hold for all $\tau > 0$, all tensors $\mathbf{L}$, and all scalars λ:

$$\mathbf{L} \cdot \mathscr{G}'(\tau)\mathbf{L} + 2\lambda \mathscr{A}'(\tau) \cdot \mathbf{L} - \theta_0 c'(\tau)\lambda^2 \leq 0,$$
$$\mathbf{L} \cdot \mathscr{G}''(\tau)\mathbf{L} + 2\lambda \mathscr{A}''(\tau) \cdot \mathbf{L} - \theta_0 c''(\tau)\lambda^2 \geq 0.$$

In the presence of thermal gradients balance of energy (1.2) and growth of entropy (1.5) contain the additional terms $-\operatorname{Div}\mathbf{q}$ and $-\operatorname{Div}(\mathbf{q}/\theta)$, respectively, on their right-hand sides, with $\mathbf{q}$ the heat flux vector, and the inequality (1.6) is replaced by

$$\varepsilon^{\cdot} - \mathbf{S} \cdot \mathbf{F}^{\cdot} - \theta\eta^{\cdot} + \theta^{-1}\mathbf{q} \cdot \nabla\theta \leq 0. \tag{1.14}$$

In this situation, constitutive equations are needed for the stress, the energy, and the heat flux. Suppose that the constitutive equations for stress and energy remain unchanged, and adjoin a law of heat conduction giving the heat flux at each time when the histories, up to that time, of the deformation gradient, temperature, and temperature gradient are known:

$$\mathbf{q}(t) = \mathscr{Q}(\mathbf{F}^t, \theta^t, \nabla\theta^t). \tag{1.15}$$

Assume further that (1.15) has the property:

$$\mathbf{q}(t) = \mathbf{0} \quad \text{when } \nabla\theta(t) = \mathbf{0}. \tag{1.16}$$

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[8]Cf. the Remark in Sec. 6.

[9]Navarro [8], working within a *linear* framework, *postulates* single-integral laws consistent with (i), (ii), and a strict version of the second inequality in (iii), and then notes that these yield compatibility with the thermodynamic inequality (1.6). Interestingly, Navarro's version of the thermodynamic restrictions (i)–(iii) are important ingredients in his study of existence, uniqueness, and asymptotic stability for linear thermoviscoelasticity.

Then, since the temperature gradient does not enter the constitutive equations for stress and internal energy, it is clear, at least formally, that if we restrict the entropy to the class defined by (1.4), then our original problem (T1) remains a valid framework in which to investigate the thermodynamic compatibility of the constitutive equations (1.3) for stress and energy. But now there is an additional problem, namely that of finding suitable restrictions on the constitutive equation (1.15) for the heat flux. This problem is beyond the scope of the present paper; we remark, however, that when (1.15) has the simple form

$$\mathbf{q}(t) = \mathbf{Q}(\mathbf{F}(t), \theta(t), \nabla\theta(t)), \tag{1.17}$$

then this supplementary problem is solved by requiring that $\mathbf{Q}$ be consistent with the heat conduction inequality

$$\mathbf{Q}(\mathbf{F}, \theta, \nabla\theta) \cdot \nabla\theta \leq 0.$$

In a future paper we will discuss global existence of classical solutions for the history-value problem in one space dimension corresponding to: balance laws for momentum and energy; dissipative constitutive equations of the form (1.3); a law of heat conduction of the form (1.17).

A. General theory of single-integral laws.

2. Notation. Preliminary definitions. Throughout, n is a fixed positive integer and

$$\mathscr{U} \text{ is a simply-connected open subset of } \mathbb{R}^n.$$

Given a function $\mathbf{g}: \mathscr{U} \to \mathbb{R}^p$, we write

$$\partial\mathbf{g}(\mathbf{z}) \quad \text{or} \quad \partial_{\mathbf{z}}\mathbf{g}(\mathbf{z}) \quad \text{or} \quad \partial_{\mathbf{z}}\mathbf{g}$$

for the derivative of $\mathbf{g}$ at $\mathbf{z} \in \mathscr{U}$. $\partial\mathbf{g}(\mathbf{z})$ is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^p$; as is customary, when $p = 1$ we will identify $\partial\mathbf{g}(\mathbf{z})$ with an element of $\mathbb{R}^n$.

For functions $\mathbf{g}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_R)$ of many scalar or vector arguments, we write $\mathbf{g}_r(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_R)$ for the partial derivative with respect to $\mathbf{z}_r$:

$$\mathbf{g}_r(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_R) = \partial_{\mathbf{z}_r}\mathbf{g}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_R).$$

When $\mathbf{g}(t)$ is a function of time t, we write

$$\mathbf{g}^{\cdot}(t) = \frac{d\mathbf{g}(t)}{dt}.$$

Let $\mathbf{f}: \mathbb{R} \to \mathbb{R}^p$. Then the **history of f up to time** t is the function $\mathbf{f}^t$ on $[0, \infty)$ defined by

$$\mathbf{f}^t(\tau) = \mathbf{f}(t - \tau), \qquad 0 \leq \tau < \infty.$$

A **tame history** (with values in $\mathscr{U}$) is a function in $C^1([0, \infty); \mathscr{U})$ whose range and whose derivative's range are contained in a compact set $\mathscr{C} \subset \mathscr{U}$. A **path** (in $\mathscr{U}$) is a C^1 function $\mathbf{f}: \mathbb{R} \to \mathscr{U}$ with $\mathbf{f}^t$ a tame history for each **time** t; that is, each $t \in \mathbb{R}$.

The integral equations of interest will have kernels $\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau)$ that depend on a time-like variable $\tau \in (0, \infty)$; we will consistently use a prime to denote partial differentiation with respect to τ:

$$\mathbf{m}'(\mathbf{f}, \mathbf{p}, \tau) = \partial_\tau\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau) = \mathbf{m}_3(\mathbf{f}, \mathbf{p}, \tau).$$

The kernels will generally be smooth functions $\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau)$ on $\mathscr{U}^2 \times (0, \infty)$ subject to certain restrictions regarding their integrability in τ. Roughly speaking, for $\mathbf{f}, \mathbf{p}$ in any compact subset of $\mathscr{U}$, $\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau)$ will be dominated by an L^1 function $f(\tau)$, $\mathbf{m}'(\mathbf{f}, \mathbf{p}, \tau)$ by a function $l(\tau)$ with

$$\begin{gathered} l \in L^1(\rho, \infty) \quad \text{for all } \rho > 0, \\ \rho \int_\rho^\infty l(\tau)\, d\tau \to 0 \quad \text{as } \rho \to 0, \end{gathered} \tag{2.1}$$

so that $\mathbf{m}'(\mathbf{f}, \mathbf{p}, \tau)$ will be allowed to possess a nonintegrable singularity at $\tau = 0$.

To state these restrictions succinctly, we introduce the following terminology. Let $\mathbf{m}: \mathscr{U}^2 \times (0, \infty) \to \mathbb{R}^p$. Then

(A) $\mathbf{m}$ is **locally dominated** if, given any compact set $\mathscr{C} \subset \mathscr{U}$, there is a function $f \in L^1(0, \infty)$ such that

$$|\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau)| \le f(\tau)$$

for all $\mathbf{f}, \mathbf{p} \in \mathscr{C}$ and $\tau > 0$.

(B) $\mathbf{m}$ is **weakly locally dominated** if, given any compact set $\mathscr{C} \subset \mathscr{U}$, there is a function l on $(0, \infty)$, consistent with (2.1), such that

$$|\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau)| \le l(\tau)$$

for all $\mathbf{f}, \mathbf{p}, \in \mathscr{C}$ and $\tau > 0$.

(C) $\mathbf{m}$ is an **admissible kernel** if $\mathbf{m}$ is continuous and weakly locally dominated, and if the limit

$$\int_{0+}^\infty \mathbf{m}(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau$$

exists for every path $\mathbf{f}$ and time t. Here we use the notation

$$\int_{0+}^z = \lim_{\rho \downarrow 0} \int_\rho^z .$$

(D) $\mathbf{m}$ is **balanced** if

$$\mathbf{m}(\mathbf{f}, \mathbf{f}, \tau) = \mathbf{0} \tag{2.2}$$

for all $\mathbf{f} \in \mathscr{U}$ and $\tau > 0$.

Note that, for $\mathbf{m}$ of class C^1 on $\mathscr{U}^2 \times (0, \infty)$: if $\mathbf{m}$ and $\mathbf{m}'$ are weakly locally dominated, then

$$\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau) \to 0 \quad \text{as } \tau \to \infty \tag{2.3}$$

uniformly for $(\mathbf{f}, \mathbf{p})$ in compact sets of $\mathscr{U}^2$; if $\mathbf{m}$ is balanced, then, for all $\mathbf{f} \in \mathscr{U}$ and $\tau > 0$,

$$\mathbf{m}_1(\mathbf{f}, \mathbf{f}, \tau) = -\mathbf{m}_2(\mathbf{f}, \mathbf{f}, \tau). \tag{2.4}$$

3. Single-integral laws. In this paper we study constitutive equations

$$\mathbf{g}(t) = \mathscr{M}(\mathbf{f}^t), \tag{3.1}$$

giving the value at each time t of an ($\mathbb{R}^p$-valued) thermodynamic variable $\mathbf{g}$ when the path $\mathbf{f}$ is known. Here $\mathscr{M}$ is an $\mathbb{R}^p$-valued function on the set of tame histories.

We will refer to (3.1), or equivalently to $\mathscr{M}$ itself, as an $\mathbb{R}^p$-valued **single-integral law** if there are functions

$$\mathbf{M} \in C^1(\mathscr{U}; \mathbb{R}^p), \qquad \mathbf{m} \in C^2(\mathscr{U}^2 \times (0, \infty); \mathbb{R}^p)$$

with

$$\mathbf{m} \ \text{balanced}, \qquad \mathbf{m}, \mathbf{m}_1, \mathbf{m}_2 \ \text{locally dominated},$$
$$\mathbf{m}', \mathbf{m}_2' \ \text{weakly locally dominated},$$

such that[10]

$$\mathscr{M}(\mathbf{f}^t) = \mathbf{M}(\mathbf{f}(t)) + \int_0^\infty \mathbf{m}(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau \tag{3.2}$$

for all paths $\mathbf{f}$ and times t. We call $\mathbf{M}$ the **equilibrium response function**, $\mathbf{m}$ the **kernel**, and $(\mathbf{M}, \mathbf{m})$ the **response pair** for $\mathscr{M}$. For convenience, we will omit the phrase "$\mathbb{R}^p$-valued" when the particular value of the integer p is unimportant or obvious from the context.

The next proposition, which is a trivial consequence of Lemma A3, shows one reason for balancing the kernel.

PROPOSITION 1. The response pair of a single-integral law is unique.

Let $\mathscr{M}$, with response pair $(\mathbf{M}, \mathbf{m})$, be a single-integral law. Then $D\mathscr{M}$ defined on the set of tame histories by

$$D\mathscr{M}(\mathbf{f}^t) = \partial\mathbf{M}(\mathbf{f}(t)) + \int_0^\infty \mathbf{m}_1(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau \tag{3.3}$$

represents the derivative of $\mathscr{M}$ with respect to the "present value" $\mathbf{f}(t)$ holding the "past history" $\mathbf{f}(t-\tau)$, $\tau > 0$, fixed; we will refer to $D\mathscr{M}$ as the **instantaneous derivative** of $\mathscr{M}$. We can also define a (functional) derivative with respect to the past history $\mathbf{f}(t-\tau)$, $\tau > 0$. We will need this derivative only as it pertains to time differentiation along a path, and for that reason we introduce the functional $\delta\mathscr{M}$ defined by

$$\delta\mathscr{M}(\mathbf{f}^t) = \int_0^\infty \mathbf{m}_2(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\mathbf{f}^{\cdot}(t-\tau)\, d\tau, \tag{3.4}$$

where $\mathbf{f}^{\cdot}(t-\tau)$ is the derivative of $\mathbf{f}$ evaluated at $t-\tau$. As an immediate consequence of these definitions we have

PROPOSITION 2 (chain rule). Let $\mathscr{M}$, with response pair $(\mathbf{M}, \mathbf{m})$, be a single-integral law. Let $\mathbf{f}$ be a path and let $\mathbf{g}$ be defined on $\mathbb{R}$ through (3.1). Then $\mathbf{g}$ is differentiable and

$$\mathbf{g}^{\cdot}(t) = D\mathscr{M}(\mathbf{f}^t)\mathbf{f}^{\cdot}(t) + \delta\mathscr{M}(\mathbf{f}^t) \tag{3.5}$$

for all time t.

The next result is crucial to our analysis.

[10]The assumption that $\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau)$ be C^2 for $\tau > 0$ is made to avoid repeated hypotheses. From a physical viewpoint regularity away from $\tau = 0$ is of small importance. In this connection note that $\mathbf{m}$ is *not* required to be continuous at $\tau = 0$; our definition allows for kernels with integrable singularities at $\tau = 0$.

THEOREM 1. Let $\mathscr{M}$, with kernel $\mathbf{m}$, be a single-integral law. Then $\mathbf{m}'$ is an admissible kernel and

$$\delta\mathscr{M}(\mathbf{f}^t) = \int_{0+}^{\infty} \mathbf{m}'(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau \tag{3.6}$$

for all paths $\mathbf{f}$ and times t.

Proof. Choose a path $\mathbf{f}$ and a time t. Then, by (3.4),

$$\delta\mathscr{M}(\mathbf{f}^t) = \int_0^{\infty} \phi(\tau)\, d\tau,$$

with

$$\begin{aligned}\phi(\tau) &= \mathbf{m}_2(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\mathbf{f}^{\cdot}(t-\tau)\\ &= \mathbf{m}'(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau) - \frac{d}{d\tau}\mathbf{m}(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau).\end{aligned}$$

Moreover, by (2.3),

$$\int_\rho^{\infty} \frac{d}{d\tau}\mathbf{m}(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau = \int_\rho^{\infty} \mathbf{m}'(\mathbf{f}(t), \mathbf{f}(t-\rho), \tau)\, d\tau,$$

so that

$$\delta\mathscr{M}(\mathbf{f}^t) = \int_0^{\rho} \phi(\tau)\, d\tau - \int_\rho^{\infty} \mathbf{m}'(\mathbf{f}(t), \mathbf{f}(t-\rho), \tau)\, d\tau + \int_\rho^{\infty} \mathbf{m}'(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau. \tag{3.7}$$

Since $\mathbf{m}_2$ is locally dominated and $\mathbf{f}$ a C^1 function, the first integral on the right side of (3.7) approaches zero as $\rho \downarrow 0$. Further, for all sufficiently small $\rho > 0$, say $\rho \in (0, \rho_0)$, $\mathbf{f}(t-\rho)$ lies in a closed ball $\mathscr{C}$ in $\mathscr{U}$ centered at $\mathbf{f}(t)$. Thus, since $\mathbf{m}'$ is balanced (as $\mathbf{m}$ is balanced) and $\mathbf{m}_2'$ weakly locally dominated, there is a function l, consistent with (2.1), such that

$$|\mathbf{m}'(\mathbf{f}(t), \mathbf{f}(t-\rho), \tau)| \le l(\tau)|\mathbf{f}(t-\rho) - \mathbf{f}(t)| \quad \text{for} \quad 0 < \rho < \rho_0 \quad \text{and} \quad \tau > 0.$$

Consequently, as $\mathbf{f}$ is C^1, the second integral on the right side of (3.7) also approaches zero as $\rho \downarrow 0$. Thus the third integral on the right side of (3.7) has a limit as $\rho \downarrow 0$, and (3.6) is valid. □

LEMMA 1. Let $\mathscr{W}$, with kernel w, be a scalar-valued single-integral law. Then a necessary and sufficient condition that

$$\delta\mathscr{W}(\mathbf{f}^t) \le 0 \tag{3.8}$$

for all paths $\mathbf{f}$ and times t is that

$$w' \le 0. \tag{3.9}$$

Granted (3.9),

$$w \ge 0; \qquad w_1, w_2, w_1', w_2' \quad \text{are balanced}; \tag{3.10}$$

and

$$\begin{aligned}&\mathbf{p} \mapsto w(\mathbf{f}, \mathbf{p}, \tau) \quad \text{has a minimum at } \mathbf{p} = \mathbf{f},\\ &\mathbf{p} \mapsto w'(\mathbf{f}, \mathbf{p}, \tau) \quad \text{has a maximum at } \mathbf{p} = \mathbf{f},\end{aligned} \tag{3.11}$$

for all $\mathbf{f} \in \mathscr{U}$ and $\tau > 0$.

Proof. Sufficiency is an immediate consequence of (3.6). To prove necessity, assume that (3.8) holds for all paths. Then, since w and (hence) w' is balanced, Theorem 1 and steps analogous to those used to prove Lemma A2 yield (3.9).

Assume (3.9) holds. Integrating this inequality from τ to ∞ using (2.3) yields $w \geq 0$. Since w and w' are balanced, this inequality and (3.9) yield (3.11). Finally, (2.4) (for w and w') and (3.11) imply that w_1, w_2, w_1', and w_2' are balanced. □

4. Dissipative single-integral laws. Consider the $\mathbb{R}^n$-valued[11] single-integral law

$$\mathbf{g}(t) = \mathscr{M}(\mathbf{f}^t). \tag{4.1}$$

By a **dissipation potential** for $\mathscr{M}$ we mean a scalar-valued single-integral law $\mathscr{W}$ with the following property: given any path $\mathbf{f}$, the functions $\mathbf{g}$ and ω on $\mathbb{R}$ defined by (4.1) and

$$\omega(t) = \mathscr{W}(\mathbf{f}^t) \tag{4.2}$$

satisfy the **dissipation inequality**

$$\omega^{\cdot} \leq \mathbf{g} \cdot \mathbf{f}^{\cdot}. \tag{4.3}$$

If $\mathscr{M}$ has a dissipation potential, we will call $\mathscr{M}$ **dissipative.**

THEOREM 2. Let $\mathscr{M}$ and $\mathscr{W}$ be single-integral laws with $\mathscr{M}$ $\mathbb{R}^n$-valued and $\mathscr{W}$ scalar-valued. Then the following are equivalent:[12]

(a) $\mathscr{W}$ is a dissipation potential for $\mathscr{M}$;

(b) for every path $\mathbf{f}$ and time t,

$$\mathscr{M}(\mathbf{f}^t) = D\mathscr{W}(\mathbf{f}^t), \qquad \delta\mathscr{W}(\mathbf{f}^t) \leq 0; \tag{4.4}$$

(c) the response pairs $(\mathbf{M}, \mathbf{m})$ and (W, w) for $\mathscr{M}$ and $\mathscr{W}$ satisfy

$$\mathbf{M} = \partial W, \qquad \mathbf{m} = w_1, \qquad w' \leq 0. \tag{4.5}$$

Proof. By (4.1), (4.2), (4.3), and the chain-rule (3.5) (for $\mathscr{W}$), (a) holds if and only if

$$\{D\mathscr{W}(\mathbf{f}^t) - \mathscr{M}(\mathbf{f}^t)\} \cdot \mathbf{f}^{\cdot}(t) + \delta\mathscr{W}(\mathbf{f}^t) \leq 0 \tag{4.6}$$

for every path $\mathbf{f}$ and time t. Trivially, (4.4) implies (4.6). On the other hand, assume that (4.6) holds. Choose a path $\mathbf{f}$ and a time t. If we apply (4.6) to the sequence $\{\mathbf{f}_n\}$ of Lemma A4, and let $n \to \infty$, we find, using (3.3) (for $\mathscr{M}$ and $\mathscr{W}$) and (3.4) (for $\mathscr{W}$) in conjunction with the properties of the kernels $\mathbf{m}$ and w, that (4.6) holds with $\mathbf{f}^{\cdot}(t)$ replaced by an *arbitrary* vector $\mathbf{a} \in \mathbb{R}^n$; this implies (4.4). Thus (a) and (b) are equivalent.

Assume that (b) is satisfied. Because of Lemma 1, (4.4) implies $(4.5)_3$. Thus (3.10) holds and w_1 is balanced. Since $\mathbf{m}$ is also balanced, (4.4), (3.2), (3.3) applied to $\mathscr{W}$, and Lemma A3 yield the remainder of (4.5). Thus (b) implies (c). The converse is obvious, since (4.5), (3.3) applied to $\mathscr{W}$, and Lemma 1 imply (4.4). □

We are now in a position to give conditions which ensure that a single-integral law be dissipative.

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[11] It is essential that $\mathscr{M}$ be $\mathbb{R}^n$-valued, so that $\mathbf{m}(t)$ and $\mathbf{f}(t)$ belong to the same space.

[12] The equivalence of (a) and (b) is due to Coleman [2].

THEOREM 3 (existence of a dissipation potential). Let $\mathscr{M}$, with response pair $(\mathbf{M}, \mathbf{m})$, be an $\mathbb{R}^n$-valued single-integral law. Then a necessary and sufficient condition that $\mathscr{M}$ be dissipative is that

$$\partial\mathbf{M} \quad \text{and} \quad \mathbf{m}_1 \quad \text{are symmetric} \tag{4.7}$$

and, for all $\mathbf{f}, \mathbf{p} \in \mathscr{U}$ and $\tau > 0$,

$$\int_{\mathbf{p}}^{\mathbf{f}} \mathbf{m}'(\mathbf{a}, \mathbf{p}, \tau) \cdot d\mathbf{a} \leq 0 \tag{4.8}$$

((4.7) insures path-independence of the integral). Granted (4.7) and (4.8), the dissipation potential $\mathscr{W}$ for $\mathscr{M}$ is unique up to an additive constant; in fact, the kernel w for $\mathscr{W}$ is (unique and) given by

$$w(\mathbf{f}, \mathbf{p}, \tau) = \int_{\mathbf{p}}^{\mathbf{f}} \mathbf{m}(\mathbf{a}, \mathbf{p}, \tau) \cdot d\mathbf{a} \tag{4.9}$$

for all $\mathbf{f}, \mathbf{p} \in \mathscr{U}$ and $\tau > 0$; the equilibrium response function W for $\mathscr{W}$ is any solution of $\partial W = \mathbf{M}$.

Proof. Let $\mathscr{W}$, with response pair (W, w), be a dissipation potential for $\mathscr{M}$. Then (4.5) holds and (4.7) is satisfied. Further, by $(4.5)_2$ and the fact that w is balanced, w and $\mathbf{m}$ are related through (4.9). Thus

$$w'(\mathbf{f}, \mathbf{p}, \tau) = \int_{\mathbf{p}}^{\mathbf{f}} \mathbf{m}'(\mathbf{a}, \mathbf{p}, \tau) \cdot d\mathbf{a} \tag{4.10}$$

for all $\mathbf{f}, \mathbf{p} \in \mathscr{U}$ and $\tau > 0$, and $(4.5)_3$ implies (4.8).

Conversely, assume that (4.7) and (4.8) are satisfied. Define w on $\mathscr{U}^2 \times (0, \infty)$ through (4.9), and let W be any solution of $\partial W = \mathbf{M}$ ((4.7) ensures that w and W are well defined, although W is obviously not unique). Then w is balanced. In fact, it is not difficult to verify that (W, w) is the response pair of a single-integral law $\mathscr{W}$. Trivially, the first two relations in (4.5) are satisfied. Further, (4.9) implies (4.10), and, in view of (4.8), this implies $(4.5)_3$. Thus $\mathscr{W}$ is a dissipation potential for $\mathscr{M}$.

The last assertion of the theorem is immediate. □

In view of (3.3) and (4.7), Theorem 3 has the following

COROLLARY. Let $\mathscr{M}$ be a dissipative $\mathbb{R}^n$-valued single-integral law. Then, for every path $\mathbf{f}$ and time t,

$$D\mathscr{M}(\mathbf{f}^t) \quad \text{is symmetric.} \tag{4.11}$$

5. Convexity and symmetry of the relaxation function. Of crucial importance is the behavior of an $\mathbb{R}^n$-valued single-integral law

$$\mathbf{g}(t) = \mathscr{M}(\mathbf{f}^t) \tag{5.1}$$

when the path $\mathbf{f}$ is close to a given point $\mathbf{f}_0$ of $\mathscr{U}$. Here the term "close" is in the sense of the supremum norm $\|\cdot\|_\infty$, so that

$$\|\mathbf{f}^t - \mathbf{f}_0\|_\infty \quad \text{is small.} \tag{5.2}$$

(We use the symbol $\mathbf{f}_0$ both for the point of $\mathscr{U}$ and for the constant history with value $\mathbf{f}_0$.)

Let $(\mathbf{M}, \mathbf{m})$ denote the response pair for $\mathscr{M}$. Then the Frechet derivative of $\mathscr{M}$ at $\mathbf{f}_0$ is the linear functional defined by

$$D\mathscr{M}(\mathbf{f}_0)\mathbf{h}(0) + \int_0^\infty \mathbf{m}_2(\mathbf{f}_0, \mathbf{f}_0, \tau)\mathbf{h}(\tau)\,d\tau$$

for every tame history $\mathbf{h}$. We define a function $\mathbf{G}$, with domain $[0, \infty)$ and codomain the space of linear transformations from $\mathbb{R}^n$ into itself, through

$$\mathbf{G}'(\tau) = \mathbf{m}_2(\mathbf{f}_0, \mathbf{f}_0, \tau), \qquad \mathbf{G}(0) = D\mathscr{M}(\mathbf{f}_0). \tag{5.3}$$

Then, letting $\not{f}(t)$ denote the *perturbation*

$$\not{f}(t) = \mathbf{f}(t) - \mathbf{f}_0,$$

the relation (5.1) has the asymptotic form

$$\mathbf{g}(t) = \mathscr{M}(\mathbf{f}_0) + \mathbf{G}(0)\not{f}(t) + \int_0^\infty \mathbf{G}'(\tau)\not{f}(t-\tau)\,d\tau + o(\|\not{f}^t\|_\infty). \tag{5.4}$$

We will refer to $\mathbf{G}$ as the **relaxation function** for $\mathscr{M}$ (corresponding to $\mathbf{f}_0$).

THEOREM 4 (properties of the relaxation function). Let $\mathbf{G}$ be the relaxation function for an $\mathbb{R}^n$-valued single-integral law $\mathscr{M}$ with C^3 kernel. If $\mathscr{M}$ is dissipative, then

(a) $\mathbf{G}(\tau)$ is symmetric at each $\tau \geq 0$;
(b) $\mathbf{G}'(\tau)$ is negative semi-definite at each $\tau > 0$;
(c) $\mathbf{G}''(\tau)$ is positive semi-definite at each $\tau > 0$.

Proof. By (4.7) and (2.4), $\mathbf{m}_2(\mathbf{f}_0, \mathbf{f}_0, \tau)$ is symmetric, and (a) follows from (4.11) and (5.3). Let w be the kernel of a dissipation potential for $\mathscr{M}$. Then (4.9) and the assumed smoothness of $\mathbf{m}$ imply that w is C^3. Since w satisfies (3.11),

$$\begin{aligned} &w_{22}(\mathbf{f}_0, \mathbf{f}_0, \tau) \quad \text{is positive semi-definite},\\ &w_{22}'(\mathbf{f}_0, \mathbf{f}_0, \tau) \quad \text{is negative semi-definite.}\end{aligned} \tag{5.5}$$

Further, since w_2 is balanced, (2.4) (for w_2) and $(4.5)_2$ imply that

$$\mathbf{m}_2(\mathbf{f}_0, \mathbf{f}_0, \tau) = w_{12}(\mathbf{f}_0, \mathbf{f}_0, \tau) = -w_{22}(\mathbf{f}_0, \mathbf{f}_0, \tau);$$

in view of (5.3) and (5.5), this yields (b) and (c). □

Note that, by (2.4) and (5.3),

$$\mathbf{G}(\infty) = \partial\mathbf{M}(\mathbf{f}_0),$$

and so $\mathbf{G}$ is generally *not* integrable on $(0, \infty)$. Note also that, by (b), $\mathbf{G}(0) - \mathbf{G}(\infty)$ is positive semi-definite.

6. Noninteractive memory. Relative memory. An important class of single-integral laws are those for which instantaneous changes are independent of the past history. Precisely, a single-integral law $\mathscr{M}$ has **noninteractive memory** if, given any two paths $\mathbf{f}, \mathbf{g}$ and any time t,

$$\mathbf{f}(t) = \mathbf{g}(t) \quad \text{implies} \quad D\mathscr{M}(\mathbf{f}^t) = D\mathscr{M}(\mathbf{g}^t). \tag{6.1}$$

The following definition helps to describe such laws. Let $\mathbf{m}: \mathscr{U}^2 \times (0, \infty) \to \mathbb{R}^p$ be C^2 and balanced. Then $\mathbf{m}$ **splits** if $\mathbf{m}_{12}(\mathbf{f}, \mathbf{p}, \tau) = \mathbf{0}$ for all $\mathbf{f}, \mathbf{p} \in \mathscr{U}$ and $\tau > 0$. In view of (2.2) (and the connectivity properties of $\mathscr{U} \times \mathscr{U}$), each such $\mathbf{m}$ admits the representation

$$\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau) = \hat{\mathbf{m}}(\mathbf{f}, \tau) - \hat{\mathbf{m}}(\mathbf{p}, \tau); \tag{6.2}$$

to ensure that $\hat{\mathbf{m}}$ be uniquely determined by $\mathbf{m}$, we require the *normalization*

$$\hat{\mathbf{m}}(\mathbf{f}_0, \tau) = \mathbf{0} \tag{6.3}$$

for all $\tau > 0$, where $\mathbf{f}_0$ is a prescribed point of $\mathscr{U}$. When $\mathbf{m}$ is the kernel of a single-integral law, we will call $\hat{\mathbf{m}}$ the **essential kernel** of $\mathscr{M}$.

PROPOSITION 3. A single-integral law has noninteractive memory if and only if its kernel splits.

Proof. Let $\mathscr{M}$ have noninteractive memory. If we apply (6.1) to an arbitrary path $\mathbf{f}$ and to the constant path $\mathbf{g}$ with value $\mathbf{f}(t)$, and use (3.3), we are lead to the conclusion that the kernel $\mathbf{m}$ of $\mathscr{M}$ satisfies

$$\int_0^\infty \{\mathbf{m}_1(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau) - \mathbf{m}_1(\mathbf{f}(t), \mathbf{f}(t), \tau)\}\, d\tau = 0$$

for all paths $\mathbf{f}$ and times t; in view of Lemma A2, this yields $\mathbf{m}_1(\mathbf{f}, \mathbf{p}, \tau) = \mathbf{m}_1(\mathbf{f}, \mathbf{f}, \tau)$ for all $\mathbf{f}, \mathbf{p} \in \mathscr{U}$ and $\tau > 0$, so that $\mathbf{m}_{12} = \mathbf{0}$ and $\mathbf{m}$ splits.

The converse assertion is immediate. □

Consider now an $\mathbb{R}^n$-valued single-integral law

$$\mathbf{g}(t) = \mathscr{M}(\mathbf{f}^t)$$

with noninteractive memory. Let $\mathbf{M}$ denote the equilibrium response function and $\mathbf{m}$ the *essential* kernel (we write $\mathbf{m}$ in place of $\hat{\mathbf{m}}$), so that

$$\mathscr{M}(\mathbf{f}^t) = \mathbf{M}(\mathbf{f}(t)) + \int_0^\infty \{\mathbf{m}(\mathbf{f}(t), \tau) - \mathbf{m}(\mathbf{f}(t-\tau), \tau)\}\, d\tau \tag{6.4}$$

for all paths $\mathbf{f}$ and times t. If

$$\partial \mathbf{M}(\mathbf{f}) \quad \text{and} \quad \mathbf{m}_1(\mathbf{f}, \tau) \quad \text{are symmetric} \tag{6.5}$$

for all $\mathbf{f} \in \mathscr{U}$ and $\tau > 0$, then there are *unique* scalar-valued functions $W: \mathscr{U} \to \mathbb{R}$ and $\Psi: \mathscr{U} \times (0, \infty) \to \mathbb{R}$ such that

$$\begin{gathered} \mathbf{M}(\mathbf{f}) = \partial W(\mathbf{f}), \qquad \mathbf{m}(\mathbf{f}, \tau) = \Psi_1(\mathbf{f}, \tau), \\ W(\mathbf{f}_0) = \Psi(\mathbf{f}_0, \tau) = 0 \end{gathered} \tag{6.6}$$

for all $\mathbf{f} \in \mathscr{U}$ and $\tau > 0$. In this case we call W and Ψ the **integrals** of $\mathbf{M}$ and $\mathbf{m}$. A fairly simple consequence of Theorem 3 is

THEOREM 5. Let $\mathscr{M}$ be an $\mathbb{R}^n$-valued single-integral law with noninteractive memory. Let $\mathbf{M}$ and $\mathbf{m}$, respectively, denote the equilibrium response function and the essential kernel of $\mathscr{M}$. Then necessary and sufficient that $\mathscr{M}$ have a dissipation potential $\mathscr{W}$ is that (6.5) hold and the integral Ψ of $\mathbf{m}$ have

$$\mathbf{f} \mapsto \Psi'(\mathbf{f}, \tau) \quad \text{concave on } \mathscr{U} \tag{6.7}$$

for all $\tau > 0$. Granted these conditions:

(i) the equilibrium response function for $\mathscr{W}$ is the integral W of $\mathbf{M}$;

(ii) the kernel w for $\mathscr{W}$ is given by

$$w(\mathbf{f}, \mathbf{p}, \tau) = \Psi(\mathbf{f}, \tau) - \Psi(\mathbf{p}, \tau) - \mathbf{m}(\mathbf{p}, \tau) \cdot (\mathbf{f} - \mathbf{p}) \tag{6.8}$$

for all $\mathbf{f}, \mathbf{p} \in \mathscr{U}$ and $\tau > 0$;

(iii) for each $\tau > 0$,

$$\mathbf{f} \mapsto \Psi(\mathbf{f}, \tau) \quad \text{is convex on } \mathscr{U}. \tag{6.9}$$

We now discuss a different type of memory. Let $\mathscr{M}$, with response pair $(\mathbf{m}, \mathbf{M})$, be a single-integral law, and let

$$\hat{\mathscr{U}} = \{\mathbf{f} - \mathbf{p} : \mathbf{f}, \mathbf{p} \in \mathscr{U}\}.$$

Then $\mathscr{M}$ has **relative memory** if there is a function $\hat{\mathbf{m}}$, called the **essential kernel** of $\mathscr{M}$, such that

$$\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau) = \hat{\mathbf{m}}(\mathbf{f} - \mathbf{p}, \tau), \qquad \hat{\mathbf{m}}(\mathbf{0}, \tau) = \mathbf{0}$$

for all $\mathbf{f}, \mathbf{p} \in \mathscr{U}$ and $\tau > 0$. In this case, writing $\mathbf{m}$ in place of $\hat{\mathbf{m}}$, $\mathbf{m}$ has $\hat{\mathscr{U}} \times (0, \infty)$ as domain, and

$$\mathscr{M}(\mathbf{f}^t) = \mathbf{M}(\mathbf{f}(t)) + \int_0^\infty \mathbf{m}(\mathbf{f}(t) - \mathbf{f}(t-\tau), \tau)\, d\tau$$

for all paths $\mathbf{f}$ and times t.

If $\mathscr{M}$ is $\mathbb{R}^n$-valued with

$$\partial \mathbf{M}(\mathbf{f}) \quad \text{and} \quad \mathbf{m}_1(\mathbf{r}, \tau) \quad \text{symmetric} \tag{6.10}$$

for all $\mathbf{f} \in \mathscr{U}$, $\mathbf{r} \in \hat{\mathscr{U}}$, and $\tau > 0$, then there are functions $W : \mathscr{U} \to \mathbb{R}$ and $w : \hat{\mathscr{U}} \times (0, \infty) \to \mathbb{R}$, with w *unique*, such that

$$\mathbf{M}(\mathbf{f}) = \partial W(\mathbf{f}), \qquad \mathbf{m}(\mathbf{r}, \tau) = w_1(\mathbf{r}, \tau), \qquad w(\mathbf{0}, \tau) = 0$$

for all $\mathbf{f} \in \mathscr{U}$, $\mathbf{r} \in \hat{\mathscr{U}}$, and $\tau > 0$. In this case we call W and w **integrals** of $\mathbf{M}$ and $\mathbf{m}$.

THEOREM 6. Let $\mathscr{M}$ be an $\mathbb{R}^n$-valued single-integral law with relative memory. Let $\mathbf{M}$ and $\mathbf{m}$, respectively, denote the equilibrium response function and the essential kernel of $\mathscr{M}$. Then necessary and sufficient that $\mathscr{M}$ have a dissipation potential $\mathscr{W}$ is that (6.10) hold and the integral w of $\mathbf{m}$ satisfy

$$w'(\mathbf{r}, \tau) \leq 0$$

for all $\mathbf{r} \in \hat{\mathscr{U}}$ and $\tau > 0$. Granted these conditions, the dissipation potential $\mathscr{W}$ for $\mathscr{M}$ necessarily has relative memory, and the equilibrium response function for $\mathscr{W}$ is an integral W of $\mathbf{M}$; the essential kernel for $\mathscr{W}$ is w; and $w \geq 0$.

The proof of this theorem can safely be omitted.

REMARK. Within our original framework, with $\mathbf{S}$ the Piola-Kirchhoff stress and $\mathbf{F}$ the deformation gradient, noninteractive memory and relative memory in the constitutive equation for stress are generally ruled out by material frame-indifference.

However, there are three-dimensional frame-indifferent relations of single-integral type that have noninteractive memory for longitudinal motions. For example, letting $\mathbf{B}$ and $\mathbf{C}_r$, respectively, denote the left Cauchy-Green strain and the relative right Cauchy-Green strain,[13] the constitutive equation

$$\mathbf{T}(t) = \int_0^\infty \mathbf{J}(\mathbf{B}(t), \mathbf{C}_r(t-\tau, t), \tau)\, d\tau$$

for the Cauchy stress $\mathbf{T}$ has this property when

$$\mathbf{J}(\mathbf{B}, \mathbf{C}_r, \tau) = \alpha(\tau)[\mathbf{J}_0(\mathbf{B}) + \mathbf{J}_1(\mathbf{B}\mathbf{C}_r)] \qquad (\mathbf{J}_0, \mathbf{J}_1 \text{ isotropic}).$$

Similarly, the choice

$$\mathbf{J}(\mathbf{B}, \mathbf{C}_r, \tau) = \alpha(\tau)[\mathbf{J}_0(\mathbf{B}) + \mathbf{J}_1(\mathbf{C}_r)] \qquad (\mathbf{J}_0, \mathbf{J}_1 \text{ isotropic})$$

yields relative memory in simple shear.

B. Thermoviscoelastic materials of single-integral type.

7. Notation. We now apply the abstract theory to thermoviscoelastic materials of single-integral type. The underlying "physical space" is $\mathbb{R}^p$ with $p = 1, 2$, or 3. We use the term "tensor" as a synonym for "linear transformation from $\mathbb{R}^p$ into itself," and write Lin for the space of all tensors. We follow the terminology of Sec. 2 with $n = p^2 + 1$ and

$$\mathscr{U} \quad \text{a simply-connected open subset of } \mathrm{Lin} \times (0, \infty).$$

Let $\mathscr{L}$ be a linear transformation from $\mathrm{Lin} \times \mathbb{R}$ into itself. Then $\mathscr{L}$ has "components" $\mathbf{L}, \mathbf{A}, \mathbf{B}, \alpha$ (with $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$, $\alpha \in \mathbb{R}$, $\mathbf{L}$: Lin $\to$ Lin a linear transformation) such that, for all $(\mathbf{H}, h) \in \mathrm{Lin} \times \mathbb{R}$,

$$\mathscr{L}(\mathbf{H}, h) = (\mathbf{L}\mathbf{H} + \mathbf{A}h, \mathbf{B} \cdot \mathbf{H} + \alpha h);$$

it is convenient to arrange these components as an **array**

$$\begin{bmatrix} \mathbf{L} & \mathbf{A} \\ \mathbf{B} & \alpha \end{bmatrix}.$$

Note that $\mathscr{L}$ is symmetric if and only if $\mathbf{L} = \mathbf{L}^\top$ and $\mathbf{A} = \mathbf{B}$. Note also that, given a smooth map $(\mathbf{F}, \theta) \mapsto \mathscr{G}(\mathbf{F}, \theta) = (\mathbf{G}(\mathbf{F}, \theta), g(\mathbf{F}, \theta))$ from $\mathscr{U}$ into $\mathrm{Lin} \times \mathbb{R}$, the derivative $\partial \mathscr{G}$ corresponds to an array

$$\begin{bmatrix} \partial_{\mathbf{F}} \mathbf{G} & \partial_\theta \mathbf{G} \\ \partial_{\mathbf{F}} g & \partial_\theta g \end{bmatrix}.$$

8. Thermodynamic restrictions. **A thermoviscoelastic material of single-integral type** is defined by two single-integral laws: an $\mathbb{R}^{p^2}$-valued law

$$\mathbf{S}(t) = \mathscr{S}(\mathbf{F}^t, \theta^t) \tag{8.1}$$

for the stress, and a scalar-valued law

$$\varepsilon(t) = \mathscr{E}(\mathbf{F}^t, \theta^t) \tag{8.2}$$

[13] Cf. [11], Sec. 23.

for the internal energy. We write $(\boldsymbol{\Sigma}, \boldsymbol{\sigma})$ and $(\boldsymbol{\Phi}, \varphi)$, respectively, for the response pairs of $\mathscr{S}$ and $\mathscr{E}$, so that

$$\begin{aligned} \mathbf{S}(t) &= \boldsymbol{\Sigma}(\mathbf{F}(t), \theta(t)) + \int_0^\infty \boldsymbol{\sigma}(\mathbf{F}(t), \theta(t), \mathbf{F}(t-\tau), \theta(t-\tau), \tau)\, d\tau, \\ \varepsilon(t) &= \boldsymbol{\Phi}(\mathbf{F}(t), \theta(t)) + \int_0^\infty \phi(\mathbf{F}(t), \theta(t), \mathbf{F}(t-\tau), \theta(t-\tau), \tau)\, d\tau. \end{aligned} \tag{8.3}$$

The instantaneous derivative $D\mathscr{S}(\mathbf{F}^t, \theta^t)$ then has "components"

$$\begin{aligned} D_{\mathbf{F}}\mathscr{S}(\mathbf{F}^t, \theta^t) &= \boldsymbol{\Sigma}_1(\mathbf{F}(t), \theta(t)) + \int_0^\infty \boldsymbol{\sigma}_1(\mathbf{F}(t), \theta(t), \mathbf{F}(t-\tau), \theta(t-\tau), \tau)\, d\tau, \\ D_\theta\mathscr{S}(\mathbf{F}^t, \theta^t) &= \boldsymbol{\Sigma}_2(\mathbf{F}(t), \theta(t)) + \int_0^\infty \boldsymbol{\sigma}_2(\mathbf{F}(t), \theta(t), \mathbf{F}(t-\tau), \theta(t-\tau), \tau)\, d\tau, \end{aligned} \tag{8.4}$$

which represent instantaneous derivatives with respect to the current values of $\mathbf{F}$ and θ, respectively; similar interpretations apply to $D_{\mathbf{F}}\mathscr{E}(\mathbf{F}^t, \theta^t)$ and $D_\theta\mathscr{E}(\mathbf{F}^t, \theta^t)$. The functions

$$c(\mathbf{F}^t, \theta^t) = D_\theta\mathscr{E}(\mathbf{F}^t, \theta^t), \qquad \mathbf{E}(\mathbf{F}^t, \theta^t) = D_{\mathbf{F}}\mathscr{S}(\mathbf{F}^t, \theta^t),$$
$$\mathbf{A}(\mathbf{F}^t, \theta^t) = D_\theta\mathscr{S}(\mathbf{F}^t, \theta^t) \tag{8.5}$$

represent the **instantaneous specific heat**, the **instantaneous elasticity**, and the **instantaneous stress-temperature modulus**.

An **entropy** for the material is a *single-integral law* $\mathscr{H}$ with the property that: given any path $(\mathbf{F}, \theta)$, the functions

$$\mathbf{S}(t) = \mathscr{S}(\mathbf{F}^t, \theta^t), \qquad \varepsilon(t) = \mathscr{E}(\mathbf{F}^t, \theta^t), \qquad \eta(t) = \mathscr{H}(\mathbf{F}^t, \theta^t)$$

are consistent with the inequality

$$\varepsilon^{\cdot} - \mathbf{S} \cdot \mathbf{F}^{\cdot} - \theta\eta^{\cdot} \le 0 \tag{8.6}$$

for all time; if an entropy exists, the material is **compatible with thermodynamics**.

It is clear from the introduction that the material is compatible with thermodynamics if and only if the $\mathbb{R}^n$-valued single-integral law $\mathscr{M}$ defined by

$$\mathscr{M}(\mathbf{F}^t, \theta^t) = (\theta^{-1}(t)\mathscr{S}(\mathbf{F}^t, \theta^t), -\theta^{-2}(t)\mathscr{E}(\mathbf{F}^t, \theta^t)), \tag{8.7}$$

is dissipative (cf. (1.10)). Thus, in view of Theorem 3, a necessary and sufficient condition for such compatibility is that

$$\begin{gathered} \boldsymbol{\Sigma}_1(\mathbf{F}, \theta) = \boldsymbol{\Sigma}_1(\mathbf{F}, \theta)^\top, \qquad \boldsymbol{\Sigma}(\mathbf{F}, \theta) - \theta\boldsymbol{\Sigma}_2(\mathbf{F}, \theta) = \boldsymbol{\Phi}_1(\mathbf{F}, \theta), \\ \boldsymbol{\sigma}_1(\mathbf{F}, \theta, \mathbf{P}, \vartheta, \tau) = \boldsymbol{\sigma}_1(\mathbf{F}, \theta, \mathbf{P}, \vartheta, \tau)^\top, \\ \boldsymbol{\sigma}(\mathbf{F}, \theta, \mathbf{P}, \vartheta, \tau) - \theta\boldsymbol{\sigma}_2(\mathbf{F}, \theta, \mathbf{P}, \vartheta, \tau) = \phi_1(\mathbf{F}, \theta, \mathbf{P}, \vartheta, \tau), \\ \int_{(\mathbf{P}, \vartheta)}^{(\mathbf{F}, \theta)} \{\alpha^{-1}\boldsymbol{\sigma}_5(\mathbf{R}, \alpha, \mathbf{P}, \vartheta, \tau) \cdot d\mathbf{R} - \alpha^{-2}\phi_5(\mathbf{R}, \alpha, \mathbf{P}, \vartheta, \tau)\, d\alpha\} \le 0, \end{gathered} \tag{8.8}$$

for all $(\mathbf{F}, \theta), (\mathbf{P}, \vartheta) \in \mathscr{U}$ and $\tau > 0$, the integral being along any path in $\mathscr{U}$ from $(\mathbf{P}, \vartheta)$ to $(\mathbf{F}, \theta)$. While the restrictions (8.8) are not in themselves illuminating, they do represent requirements that must be verified to ensure that a particular model (of single-integral type) be compatible with thermodynamics.

To avoid repeated assumptions, we henceforth assume that (8.8) are satisfied. Then as a consequence of (4.11), (8.7), and (8.5),

$$\mathbf{E}(\mathbf{F}^t, \theta^t) = \mathbf{E}(\mathbf{F}^t, \theta^t)^\top,$$
$$D_{\mathbf{F}}\mathscr{E}(\mathbf{F}^t, \theta^t) = \mathscr{S}(\mathbf{F}^t, \theta^t) - \theta(t)\mathbf{A}(\mathbf{F}^t, \theta^t).$$

This latter result has an important consequence. Consider a path $(\mathbf{F}, \theta)$ and let $\mathbf{S}(t)$ and $\varepsilon(t)$ be the corresponding stress and internal energy as defined by (8.1) and (8.2). Then, in view of (3.5) (applied to $\mathscr{E}$) and (8.5),

$$\varepsilon^{\cdot}(t) = c(\mathbf{F}^t, \theta^t)\theta^{\cdot}(t) + \{\mathbf{S}(t) - \theta(t)\mathbf{A}(\mathbf{F}^t, \theta^t)\} \cdot \mathbf{F}^{\cdot}(t) + \delta\mathscr{E}(\mathbf{F}^t, \theta^t);$$

this relation is an essential ingredient in the derivation of the partial differential equation describing balance of energy.

9. Relaxation functions. We assume there is a temperature θ_0 at which the reference configuration is both stress-free and devoid of internal energy (the latter involves no loss in generality, as the internal energy may be adjusted by an additive constant without affecting the theory). Thus, since the undeformed body corresponds to a deformation gradient $\mathbf{F} = \mathbf{1}$,

$$\mathscr{S}(\mathbf{1}, \theta_0) = \mathbf{0}, \qquad \mathscr{E}(\mathbf{1}, \theta_0) = 0. \tag{9.1}$$

We assume further that the kernels $\boldsymbol{\sigma}$ and ϕ are C^3.

There are four relevant relaxation functions: the **stress-strain relaxation function** $\mathscr{G}$, the **stress-temperature relaxation function** $\mathscr{A}$, the **energy-strain relaxation function** $\mathscr{D}$, and the **energy-temperature relaxation function** $\mathscr{c}$; these functions are defined through the relations

$$\mathscr{G}'(\tau) = \boldsymbol{\sigma}_3(\mathbf{1}, \theta_0, \mathbf{1}, \theta_0, \tau), \qquad \mathscr{G}(0) = D_{\mathbf{F}}\mathscr{S}(\mathbf{1}, \theta_0),$$
$$\mathscr{A}'(\tau) = \boldsymbol{\sigma}_4(\mathbf{1}, \theta_0, \mathbf{1}, \theta_0, \tau), \qquad \mathscr{A}(0) = D_\theta\mathscr{S}(\mathbf{1}, \theta_0),$$
$$\mathscr{D}'(\tau) = \phi_3(\mathbf{1}, \theta_0, \mathbf{1}, \theta_0, \tau), \qquad \mathscr{D}(0) = D_{\mathbf{F}}\mathscr{E}(\mathbf{1}, \theta_0),$$
$$\mathscr{c}'(\tau) = \phi_4(\mathbf{1}, \theta_0, \mathbf{1}, \theta_0, \tau), \qquad \mathscr{c}(0) = D_\theta\mathscr{E}(\mathbf{1}, \theta_0).$$

If we consider a path $(\mathbf{F}, \theta)$ which is close to $(\mathbf{1}, \theta_0)$, then, writing

$$\mathbf{H} = \mathbf{F} - \mathbf{1}$$

(for the displacement gradient) the constitutive equations (8.1) and (8.2) have the following asymptotic form at each time t:

$$\mathbf{S}(t) = \mathscr{G}(0)\mathbf{H}(t) + \int_0^\infty \mathscr{G}'(\tau)\mathbf{H}(t-\tau)\,d\tau + \mathscr{A}(0)\theta(t) + \int_0^\infty \mathscr{A}'(\tau)\theta(t-\tau)\,d\tau + o(\delta),$$
$$\varepsilon(t) = \mathscr{D}(0)\cdot\mathbf{H}(t) + \int_0^\infty \mathscr{D}'(\tau)\cdot\mathbf{H}(t-\tau)\,d\tau + \mathscr{c}(0)\theta(t)$$
$$+ \int_0^\infty \mathscr{c}'(\tau)\theta(t-\tau)\,d\tau + o(\delta),$$

with

$$\delta = \|\mathbf{F}^t - \mathbf{1}\|_\infty + \|\theta^t - \theta_0\|_\infty.$$

If $\mathbf{G}$ is the relaxation function corresponding to the single-integral law (8.7), then, in view (9.1) and the remarks of Sec. 7, for each $\tau > 0$, the array of components of $\theta_0\mathbf{G}(\tau)$ is

$$\begin{bmatrix} \mathscr{G}(\tau) & \mathscr{A}(\tau) \\ -\theta_0^{-1}\mathscr{D}(\tau) & -\theta_0^{-1}c(\tau) \end{bmatrix};$$

and we have the following direct consequences of Theorem 4:

(i) The stress-strain relaxation function is symmetric:

$$\mathscr{G}(\tau) = \mathscr{G}(\tau)^\top \quad \text{for all } \tau \geq 0.$$

(ii) The stress-temperature and energy-strain relaxation functions are related by

$$\mathscr{D}(\tau) = -\theta_0\mathscr{A}(\tau) \quad \text{for all } \tau \geq 0.$$

(iii) The following definiteness conditions hold for all $\tau > 0$, $\mathbf{L} \in \mathrm{Lin}$, and $\lambda \in \mathbb{R}$:

$$\mathbf{L}\cdot\mathscr{G}'(\tau)\mathbf{L} + 2\lambda\mathscr{A}'(\tau)\cdot\mathbf{L} - \theta_0 c'(\tau)\lambda^2 \leq 0,$$
$$\mathbf{L}\cdot\mathscr{G}''(\tau)\mathbf{L} + 2\lambda\mathscr{A}''(\tau)\cdot\mathbf{L} - \theta_0 c''(\tau)\lambda^2 \geq 0.$$

Appendix. Some useful lemmas.

LEMMA A1. Suppose that $\mathbf{m}: \mathscr{U}^2 \times (0, \infty) \to \mathbb{R}^p$ is an admissible kernel. Let $\mathbf{c}: \mathbb{R} \to \mathscr{U}$ be piecewise constant with a finite set $\mathscr{D}$ of discontinuities. Then there is a sequence $\{\mathbf{f}_n\}$ of paths such that, given any time $t \notin \mathscr{D}$,

$$\lim_{n\to\infty}\int_{0+}^{\infty} \mathbf{m}(\mathbf{f}_n(t), \mathbf{f}_n(t-\tau), \tau)\,d\tau = \int_{0+}^{\infty} \mathbf{m}(\mathbf{c}(t), \mathbf{c}(t-\tau), \tau)\,d\tau. \qquad \text{(A1)}$$

Proof. There is a sequence $\{\mathbf{f}_n\}$ of paths such that, for all sufficiently large n (say $n > N_0$), if we consider the intervals $(d - n^{-1}, d + n^{-1})$, $d \in \mathscr{D}$, then $\mathbf{f}_n = \mathbf{c}$ outside these intervals, while there is a fixed compact set $K \subset \mathscr{U}$ that contains the ranges of all of the $\mathbf{f}_n$. Call the union of these intervals B_n. Choose $t \notin \mathscr{D}$. Then there exists a $\rho_0 > 0$ such that $[t - \rho_0, t]$ lies outside of B_n for all $n > N_0$; for $\rho \in (0, \rho_0)$,

$$\int_{\rho}^{\infty} \mathbf{m}(\mathbf{f}_n(t), \mathbf{f}_n(t-\tau), \tau)\,d\tau$$
$$= \int_{\rho}^{\rho_0} \mathbf{m}(\mathbf{c}(t), \mathbf{c}(t-\tau), \tau)\,d\tau + \int_{\rho_0}^{\infty} \mathbf{m}(\mathbf{f}_n(t), \mathbf{f}_n(t-\tau), \tau)\,d\tau. \qquad \text{(A2)}$$

Since $\mathbf{m}$ is an admissible kernel, if we let $\rho \downarrow 0$, the first integral on the right converges; on the other hand, if we let $n \to \infty$, the second integral converges to the analogous integral of $\mathbf{m}(\mathbf{c}(t), \mathbf{c}(t-\tau), \tau)$. □

LEMMA A2. Let $\mathbf{m}: \mathscr{U}^2 \times (0, \infty) \to \mathbb{R}^p$ be an admissible kernel and suppose that

$$\int_{0+}^{\infty} \mathbf{m}(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\,d\tau = \int_{0+}^{\infty} \mathbf{m}(\mathbf{f}(t), \mathbf{f}(t), \tau)\,d\tau \qquad \text{(A3)}$$

for all paths $\mathbf{f}$ and times t. Then

$$\mathbf{m}(\mathbf{f}, \mathbf{p}, \tau) = \mathbf{m}(\mathbf{f}, \mathbf{f}, \tau) \tag{A4}$$

for all $\mathbf{f}, \mathbf{p} \in \mathscr{U}$ and $\tau > 0$.

Proof. Choose $\mathbf{f}_0, \mathbf{p} \in \mathscr{U}$ and $t_0, h > 0$. By Lemma A1, we may apply (A3) at $t = 0$ with $\mathbf{f}: \mathbb{R} \to \mathscr{U}$ defined by

$$\mathbf{f}(\lambda) = \begin{cases} \mathbf{p}, & \lambda \in [-t_0 - h, -t_0], \\ \mathbf{f}_0, & \text{otherwise}; \end{cases} \tag{A5}$$

the result is

$$\int_{t_0}^{t_0+h} \{\mathbf{m}(\mathbf{f}_0, \mathbf{p}, \tau) - \mathbf{m}(\mathbf{f}_0, \mathbf{f}_0, \tau)\}\, d\tau = 0,$$

and this clearly yields the validity of (A4). □

LEMMA A3. Let $\mathbf{M}, \mathbf{N}: \mathscr{U} \to \mathbb{R}^p$, $\mathbf{m}, \mathbf{n}: \mathscr{U}^2 \times (0, \infty) \to \mathbb{R}^p$ with $\mathbf{m}, \mathbf{n}$ continuous, balanced, and locally dominated. Assume that

$$M(\mathbf{f}(t)) + \int_0^\infty \mathbf{m}(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau = \mathbf{N}(\mathbf{f}(t)) + \int_0^\infty \mathbf{n}(\mathbf{f}(t), \mathbf{f}(t-\tau), \tau)\, d\tau \tag{A6}$$

for all paths $\mathbf{f}$ and times t. Then

$$\mathbf{M} = \mathbf{N}, \qquad \mathbf{m} = \mathbf{n}.$$

Proof. Since

$$\mathbf{m} \quad \text{and} \quad \mathbf{n} \quad \text{are balanced}, \tag{A7}$$

an arbitrary *constant* path in (A6) yields $\mathbf{M} = \mathbf{N}$; and this result, (A7), and Lemma A2 applied to $\mathbf{m} - \mathbf{n}$ implies $\mathbf{m} = \mathbf{n}$. □

The proof of the next lemma can safely be omitted.

LEMMA A4. Let $\mathbf{f}$ be a path, t a time, and $\mathbf{a} \in \mathbb{R}^n$. Then there is a sequence $\{\mathbf{f}_n\}$ of paths such that

$$\begin{aligned} &\mathbf{f}_n \to \mathbf{f} \quad \text{as} \quad n \to \infty \text{ uniformly on } (-\infty, t), \\ &\mathbf{f}_n^{\cdot} \to \mathbf{f}^{\cdot} \quad \text{as } n \to \infty \text{ pointwise on } (-\infty, t), \\ &|\mathbf{f}_n^{\cdot}| \quad \text{is bounded on } (-\infty, t), \text{ uniformly in } n, \end{aligned}$$

and such that, for each n,

$$\mathbf{f}_n(t) = \mathbf{f}(t), \qquad \mathbf{f}_n^{\cdot}(t) = \mathbf{a}.$$

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