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ON THE THREE DIMENSIONAL MINIMAL MODEL PROGRAM IN POSITIVE CHARACTERISTIC

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1. Introduction

The minimal model program (MMP) is one of the main tools in the classification of higher dimensional algebraic varieties. It aims to generalize to dimension ≥ 3 the results obtained by the Italian school of algebraic geometry at the beginning of the 20-th century.

In characteristic 0, much progress has been made towards establishing the minimal model program. In particular the minimal model program is true in dimension ≤ 3, and in higher dimensions, it is known that the canonical ring is finitely generated, flips and divisorial contractions exist, minimal models exist for varieties of general type, and we have termination of flips for the minimal model program with scaling on varieties of general type (see [BCHM10] and the references contained therein). The fundamental tool used in establishing these results is Nadel-Kawamata-Viehweg vanishing (a powerful generalization of Kodaira vanishing).

Unluckily, vanishing theorems are known to fail for varieties in characteristic $p > 0$ and so very little is known about the minimal model program in characteristic $p > 0$. Another serious difficulty is that a resolution of singularities is not yet known in characteristic $p > 0$ and dimension >3. The situation is as follows: in dimension 2, the full minimal model program holds (see [KK, Tanaka12a] and references therein). In dimension 3, the resolution of singularities is known (see [Abhyankar98, Cutkosky04, CP08, CP09]). Partial results towards the existence of divisorial and flipping contractions are proven in [Keel99]. Termination of flips for terminal pairs, holds by the usual counting argument and Kawamata has shown the existence of relative minimal models for semi-stable families when $p > 3$ [Kawamata94]. Thus the main remaining questions are the base point free theorem, the existence of flips and abundance. In this paper we prove the following.

Theorem 1.1. *Let $f : (X, B) \to Z$ be an extremal flipping contraction of a $\mathbb{Q}$-factorial dlt threefold defined over an algebraically closed field of characteristic $p > 5$ such that the coefficients of $\{B\}$ belong to the standard set $\{1 - \frac{1}{n} | n \in \mathbb{N}\}$. Then the flip exists.*

We have the following result on the existence of minimal models. We thank one of the referees pointing out the cases of non-general type in (2).

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Theorem 1.2. *Let (X,Δ) be a $\mathbb{Q}$-factorial projective three dimensional canonical pair over an algebraically closed field k of characteristic $p>5$. Assume all coefficients of Δ are in the standard set $\{1-\frac{1}{n}|n\in\mathbb{N}\}$ and $N_\sigma(K_X+\Delta)\wedge\Delta=0$. If $K_X+\Delta$ is pseudo-effective, then*

(1) *there exists a minimal model (X^m,Δ^m) of (X,Δ), and*
(2) *if, moreover, $k=\overline{\mathbb{F}}_p$, then (X^m,Δ^m) can be obtained by running the usual $(K_X+\Delta)$-MMP. Furthermore, the log canonical ring*

$$R(K_X+\Delta)=\bigoplus_{m\geq 0}H^0(X,\mathcal{O}_X(m(K_X+\Delta)))$$

is finitely generated.

We remark that in general the minimal model in (1) is not obtained by running the MMP in the usual sense unless $k=\overline{\mathbb{F}}_p$. See Section 5 for more details.

1.1. **Sketch of the proof.** After Shokurov's work, it has been known in characteristic 0 that the existence of flips can be reduced to a special case, called *pl-flips* (see [Shokurov92, Fujino07]). We apply the same idea in characteristic p. The key result of this paper is the proof of the existence of pl-flips cf. (4.12). Since the base point free theorem has not yet been established in full generality for threefolds in characteristic $p>0$, instead of running the MMP in the usual sense, we run a variant of the MMP (which we call a *generalized MMP*), which yields a minimal model (of terminal threefolds, see Section 5).

Our strategy to show the existence of pl-flips will follow closely ideas of Shokurov as explained in [Corti07, Chapter 2]. It is well known that if $f:(X,S+B)\to Z$ is a pl-flipping contraction (so that X is $\mathbb{Q}$-factorial, $(X,S+B)$ is plt and $-(K_X+S+B)$ and $-S$ are ample over Z), then the existence of the pl-flip is equivalent to the finite generation of the restricted algebra $R_{S/Z}(K_X+S+B)$. When Z is affine, then the m-th graded piece of this algebra is given by the image of the restriction map

$$H^0(X,\mathcal{O}_X(m(K_X+S+B)))\to H^0(S,\mathcal{O}_S(m(K_S+B_S)))$$

(where S is normal and B_S is defined by adjunction $(K_X+S+B)|_S=K_S+B_S$). For any proper birational morphism $g:Y\to X$ we can consider the corresponding plt pair $K_Y+S'+B_Y$ where S' is the strict transform of S. The moving part of the image of the restriction of $H^0(Y,\mathcal{O}_Y(m(K_Y+S'+B_Y)))$ to S' gives a mobile b-divisor $\mathbf{M}_{m,S'}$ corresponding to the m-th graded piece of $R_{S/Z}(K_X+S+B)$. Dividing by m, we obtain a sequence of non-decreasing $\mathbb{Q}$-b-divisor $\frac{1}{m}\mathbf{M}_{m,S'}$. The required finite generation is equivalent to showing that this non-decreasing sequence eventually stabilizes (so that $\frac{1}{m}\mathbf{M}_{m,S'}$ is fixed for all $m>0$ sufficiently divisible and in particular these divisors descend to a fixed birational model of S).

In characteristic 0, the proof has three steps: first, we show that this sequence of b-divisors descends to a fixed model (in fact they descend to $\bar{S}$ the terminalization of (S,B_S)); next we show that the limiting divisor is a $\mathbb{Q}$-divisor (instead of an arbitrary $\mathbb{R}$-divisor); finally we show that for sufficiently large degree, the sequence stabilizes. All of these steps rely heavily on the use of vanishing theorems.

In characteristic $p>0$, the main difficulty is (of course) that the Kawamata-Viehweg vanishing theorem fails. However, after [HH90], techniques involving the Frobenius map have been developed to recover many of the results which are traditionally deduced from vanishing theorems (see Section 2.3). In this paper, we will use these results for lifting sections from the divisor S to the ambient variety X.

However, one of the difficulties we encounter is that we can only lift the sections in $S^0(\bullet) \subset H^0(\bullet)$, which is roughly speaking the subspace given by the images of the maps induced by iterations of the Frobenius. Thus it is necessary to understand under what conditions the inclusion $S^0(\bullet) \subset H^0(\bullet)$ is actually an equality. It is easy to see that if (C, Δ) is a one dimensional klt pair and L is a sufficiently ample line bundle, then $H^0(C, L) = S^0(C, \sigma(C, \Delta) \otimes L)$ (cf. (2.4)). Unluckily, in higher dimensions, this frequently fails.

It turns out that the first two steps in the proof of characteristic 0 can be achieved by lifting sections from curves (corresponding to general divisors in $|\mathbf{M}_{m,S'}|$). So after a suitable modification, the argument works in general (an added difficulty is that since Bertini's theorem fails in a positive characteristic, these curves may be singular). Unluckily, the third step uses a lifting result in the surface case. This lifting result is subtler, however when the coefficients of B are in the standard set $\{1 - \frac{1}{n} | n \in \mathbb{N}\}$ and the characteristic is $p > 5$, we are able to prove that $S^0(\bullet) = H^0(\bullet)$ even in the surface case (see (4.13)). This is achieved by a careful study of the geometry of a relative weak log Fano surface over a birational base, using Shokurov's idea of finding a complement (see Section 3).

2. Preliminaries

2.1. **Notation and conventions.** We work over an algebraically closed field k of characteristic $p > 0$. We will use the standard notation in [KM98]. In particular, see [KM98, 0.4, 2.34, 2.37] for the definitions of *terminal, klt, plt* and *dlt* singularities.

We use the notation *b-divisor* as defined in [Corti07, Subsection 2.3.2]. In particular, see [Corti07, 2.3.12] for examples of b-divisors.

Let $(X, S+B)$ be a plt pair such that $\lfloor S+B \rfloor = S$ is irreducible and $f : X \to Z$ a birational contraction. Then f is a *pl-flipping contraction* if f is small, $\rho(X/Z) = 1$, $-(K_X+S+B)$ is f-ample and $-S$ is f-ample. The *pl-flip* of f, if it exists, is a small birational morphism $f^+ : X^+ \to Z$ such that $\rho(X^+/Z) = 1$ and $K_{X^+} + S^+ + B^+$ is f^+-ample, where $S^+ + B^+$ denotes the strict transform of $S + B$. It is well known that $X^+ = \mathrm{Proj} R(X/Z, K_X + S + B)$ and thus the pl-flip f^+ exists if and only if $R(X/Z, K_X + S + B) = \bigoplus_{i \geq 0} f_* \mathcal{O}_X(i(K_X + S + B))$ is a finitely generated $\mathcal{O}_Z$-algebra.

We refer the reader to [BCHM10] for the definitions of *minimal model* (which in [BCHM10] is called a log terminal model), $(K_X + \Delta)$-*non-positive* and $(K_X + \Delta)$-*negative* maps. Also see [BCHM10] for Nakayama's definition of $N_\sigma(D)$ for a pseudo-effective divisor D.

Let $f : X \to Z$ be a proper morphism, and $\mathcal{F}$ be a coherent sheaf, then $\mathcal{F}$ is *relatively globally generated* if $f^* f_* \mathcal{F} \to \mathcal{F}$ is surjective. If $\mathcal{F} \cong \mathcal{O}_X(M)$ for some divisor M on Y, then M is *a relatively free divisor* if $\mathcal{F}$ is relatively globally generated. Let C be a $\mathbb{Q}$-divisor, such that $\lceil C \rceil \geq 0$. Then a divisor D is *relatively C-saturated* if the natural injection

$$f_* \mathcal{O}_X(D) \to f_* \mathcal{O}_X(\lceil D + C \rceil)$$

is an isomorphism. Let L be a divisor on X, the *relative mobile b-divisor* $\mathbf{Mob}_Z(L) = \mathbf{N}$, is the unique b-divisor $\mathbf{N}$ such that for any birational morphism $g : Y \to X$, we have $\mathbf{N}_Y \leq g^* L$, $\mathbf{N}_Y$ is mobile over Z and the natural morphism $(f \circ g)_* \mathbf{N}_Y \to f_* L$ is an isomorphism.

For a variety X defined over a field k of characteristic $p > 0$, we always denote by $F : X \to X$ the absolute Frobenius.

2.2. **Resolution of singularities.**

Theorem 2.1. *Let X be a quasi-projective variety of dimension 3 over an algebraically closed field k of characteristic $p > 0$. Then there exists a non-singular quasi-projective variety Y and a proper birational morphism $f : Y \to X$ which is an isomorphism above the non-singular locus $X \backslash \Sigma$. We may assume that $f^{-1}(\Sigma)$ is a divisor with simple normal crossings. Moreover, if $\mathcal{I} \subset \mathcal{O}_X$ is an ideal and we replace Σ by its union with the support of $\mathcal{O}_X/\mathcal{I}$, then we may assume that $\mathcal{I} \cdot \mathcal{O}_Y$ is locally principal and that f is given by a sequence of blow ups along smooth centers lying over Σ.*

Proof. See [Cutkosky04], [CP08] and [CP09]. □

2.3. ***F*-singularities.** Test ideals and the theory of tight closure were introduced by Hochster-Huneke [HH90]. Since then it has become increasingly clear that there is a deep connection between the classes of singularities defined in terms of Frobenius splitting properties and the ones appearing in the minimal model program. This has led to exciting progress in both commutative algebra and birational geometry (see for example the survey paper [ST11] and the references therein). The definitions and results in this subsection are well-known to the experts; we include them for the reader's convenience.

Let (X, Δ) be a pair whose index is not divisible by p. Choose $e > 0$ so that $(p^e - 1)(K_X + \Delta)$ is Cartier and let $\mathcal{L}_{e,\Delta} = \mathcal{O}_X((1 - p^e)(K_X + \Delta))$. Then there is a canonically determined (up to multiplication by a unit) map

$$\phi_\Delta : F^e_* \mathcal{L}_{e,\Delta} \to \mathcal{O}_X.$$

The *non-F-pure ideal* $\sigma(X, \Delta)$ is the unique biggest ideal (respectively *the test ideal* $\tau(X, \Delta)$ is the unique smallest non-zero ideal) $J \subset \mathcal{O}_X$ such that

$$(\phi_\Delta \circ F^e_*)(J \cdot \mathcal{L}_{e,\Delta}) = J$$

for any $e > 0$. If $\sigma(X, \Delta) = \mathcal{O}_X$, we say that (X, Δ) is *sharply F-pure.* If $\tau(X, \Delta) = \mathcal{O}_X$, we say that (X, Δ) is *strongly F-regular.* The relationship between strongly F-regular and sharply F-pure is similar to the difference between klt and log canonical singularities. For a pair (X, Δ), we can define F-pure centers as in [Schwede10]. See [Schwede09, 9.5] for more background. In this note, we will only need to discuss non-trivial F-pure centers in the case of a simple normal crossing pair. We have the following.

Lemma 2.2. *Assume that $(X, \Delta = \sum \delta_i \Delta_i)$ is a simple normal crossing pair whose index is not divisible by p and that $0 \leq \delta_i \leq 1$. Then $\sigma(X, \Delta) = \mathcal{O}_X$ and each strata of $\lfloor \Delta \rfloor$ is an F-pure center.*

Proof. The first statement is [FST11, 15.1] (also see [ST10, 6.18]). For the rest, see the main result of [HSZ10] and [Schwede10, 3.8, 3.9] (see also [ST11, 3.5]). □

Assume (X, Δ) is a proper pair. For any line bundle M we define

$$\begin{aligned} &S^0(X, \sigma(X, \Delta) \otimes M) \\ &\quad = \bigcap_{n > 0} \operatorname{Im}\big(H^0(X, F^{ne}_*(\sigma(X, \Delta) \otimes \mathcal{L}_{ne,\Delta} \otimes M^{p^{ne}})) \to H^0(X, \sigma(X, \Delta) \otimes M)\big). \end{aligned}$$

Since $H^0(X, M)$ is a finitely dimensional vector space, we have

$$S^0(X, \sigma(X, \Delta) \otimes M) = \mathrm{Im}\big(H^0(X, F_*^{ne}(\mathcal{L}_{ne,\Delta} \otimes M^{p^{ne}})) \to H^0(X, M)\big)$$

for all $n \gg 0$. Recall the following result (cf. [Schwede11, 5.1, 5.3]).

Proposition 2.3. Fix X a normal projective variety and suppose that (X, Δ) is a pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by $p > 0$. Suppose that $S \subset X$ is any union of F-pure centers of (X, Δ) and that M is a Cartier divisor such that $M - K_X - \Delta$ is ample. Then there is a natural surjective map:

$$S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(M)) \to S^0(S, \sigma(S, \phi_S^\Delta) \otimes \mathcal{O}_S(M)),$$

where ϕ_S^Δ is defined as in [Schwede11, 5.1].

We note that in this paper, we will apply (2.3) mostly when S is normal. In this case $\phi_S^\Delta = \Delta_S$ where Δ_S is an effective $\mathbb{Q}$-divisor such that $(K_X + \Delta)|_S \sim_{\mathbb{Q}} K_S + \Delta_S$ (cf. [Schwede11, 5.1]).

We need the following result.

Lemma 2.4. *Let (X, Δ) be a sharply F-pure pair and L an ample Cartier divisor. Assume the index of $K_X + \Delta$ is not divisible by p. Then there is an integer $m_0 > 0$ such that for any nef Cartier divisor P and any integer $m \geq m_0$, we have*

$$S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(mL + P)) = H^0(X, \mathcal{O}_X(mL + P)).$$

Proof. See [Patakfalvi12, 2.23]. $\square$

Corollary 2.5. Let (X, Δ) be an snc pair and L an ample Cartier divisor. Assume the index of $K_X + \Delta$ is not divisible by p. Then there is an integer $m_0 > 0$ such that for any nef Cartier divisor P and any integer $m \geq m_0$, we have

$$S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(mL + P)) = H^0(X, \mathcal{O}_X(mL + P - \lfloor (1 - \epsilon)\Delta \rfloor))$$

for $0 < \epsilon \ll 1$.

Proof. Let $M = \lfloor (1 - \epsilon)\Delta \rfloor$. Note that for $0 < \epsilon \ll 1$, M is independent of ϵ and the coefficients of $\Delta - M$ are contained in $[0, 1]$. Thus $\sigma(X, \Delta - M) = \mathcal{O}_X$, i.e., $(X, \Delta - M)$ is a sharply F-pure pair.

It is easy to see, using the projection formula, that

$$S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(mL + P)) \supset S^0(X, \sigma(X, \Delta - M) \otimes \mathcal{O}_X(mL + P - M)).$$

It then follows from (2.4) above that

$$S^0(X, \sigma(X, \Delta - M) \otimes \mathcal{O}_X(mL + P - M)) = H^0(X, \mathcal{O}_X(mL + P - M)).$$

On the other hand, we have $(p^e - 1)\Delta \geq p^e M$ for $e \gg 0$, thus by the projection formula again, we easily see that for any Cartier divisor N we have

$$S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(N)) \subset S^0(X, \sigma(X, 0) \otimes \mathcal{O}_X(N - M)),$$

and the reverse inclusion immediately follows. $\square$

Next we introduce a global version of strongly F-regular singularities.

Definition 2.6 (cf. [SS10, 3.1, 3.8])**.** Let (X, Δ) be a pair with a proper morphism $f : X \to T$ between normal varieties over an algebraically closed field of characteristic $p > 0$. Assume X is normal and Δ is an effective $\mathbb{Q}$-divisor on X. The pair

(X, Δ) is *globally F-regular over T* if for every effective divisor D, there exists some $e > 0$ such that the natural map

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

splits locally over T (i.e. there exists an open cover $T = \cup T_i$ such that if $X_i = f^{-1}(T_i)$, then for any i, the homomorphism

$$\mathcal{O}_{X_i} \to F^e_* \mathcal{O}_{X_i}(\lceil (p^e - 1)\Delta|_{X_i} \rceil + D|_{X_i})$$

splits for some $e > 0$).

When $T = X$, this definition coincides with the original definition of (X, Δ) being *strongly F-regular* and when T is affine, it coincides with the definition of a *globally F-regular* pair (cf. [SS10]).

The next result shows that the global F-regularity is a very restrictive condition.

Theorem 2.7 (Schwede-Smith)**.** *If $f : X \to T$ is a projective morphism of normal quasi-projective varieties over an algebraically closed field of characteristic $p > 0$ and (X, Δ) is globally F-regular over T, then there is a $\mathbb{Q}$-divisor $\Delta' \geq \Delta$ such that the pair (X, Δ') is globally F-regular over T, $-(K_X + \Delta')$ is ample over T and the index of $K_X + \Delta'$ is not divisible by p.*

Proof. See [SS10, 4.3] and its proof. □

We will need the following.

Lemma 2.8. *Let (X, Δ) be a globally F-regular (over T) pair and D an effective divisor. Then there exists a rational number $\epsilon > 0$ such that $(X, \Delta + \epsilon D)$ is globally F-regular.*

Proof. Since (X, Δ) is globally F-regular, then the map

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

splits for some $e > 0$ and so $(X, \Delta + 2\epsilon D)$ is globally sharply F-split where $2\epsilon = \frac{1}{p^e - 1}$ (cf. [SS10, 3.1]). The claim is now immediate from [SS10, 3.9] (applied with $C = \epsilon D$). □

As in the definition of sharp F-purity, in this note, we will mostly work with the dual version of the above definition.

Lemma 2.9. *Let X be a normal variety. Let E be an integral divisor. For any $e \in \mathbb{Z}_{\geq 0}$, then there is an isomorphism*

$$F^e_* H^0(X, \mathcal{O}_X((1 - p^e)K_X - E)) \cong \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X(E), \mathcal{O}_X).$$

Proof. Let $L = \mathcal{O}_X(E)$. We have the following equalities of sheaves

$$\mathcal{H}om_{\mathcal{O}_X}(F^e_* L, \mathcal{O}_X) = F^e_* \mathcal{H}om_{\mathcal{O}_X}(L, (1 - p^e)K_X) = F^e_* \mathcal{O}_X((1 - p^e)K_X - E).$$

In fact, as X is normal and the above sheaves satisfy Serre's condition S_2, it suffices to check this along the smooth locus X_{sm}, where it follows easily from Grothendieck duality and the projection formula that

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}_{X_{\mathrm{sm}}}}(F^e_* L, \mathcal{O}_{X_{\mathrm{sm}}}) &= \mathcal{H}om_{\mathcal{O}_{X_{\mathrm{sm}}}}(F^e_*(L \otimes \omega^{p^e}_{X_{\mathrm{sm}}}), \omega_{X_{\mathrm{sm}}}) \\ &= F^e_* \mathcal{H}om_{\mathcal{O}_{X_{\mathrm{sm}}}}(L \otimes \omega^{p^e}_{X_{\mathrm{sm}}}, \omega_{X_{\mathrm{sm}}}) \\ &= F^e_* \mathcal{H}om_{\mathcal{O}_{X_{\mathrm{sm}}}}(L, \omega^{\otimes(1-p^e)}_{X_{\mathrm{sm}}}). \end{aligned}$$

Taking global sections, we obtain the claim. □

Applying the above lemma to $E = \lceil (p^e - 1)\Delta \rceil + D$, it immediately follows that:

Proposition 2.10. Let $T = \mathrm{Spec}(A)$ for some finitely generated k-algebra A and (X, Δ) be a pair such that X is projective over T. Then (X, Δ) is globally F-regular over T if and only if for any effective divisor D, there is an integer $e \in \mathbb{N}$ and a surjection

$$H^0(X, \mathcal{O}_X(\lfloor (1-p^e)(K_X + \Delta) \rfloor - D)) \to H^0(X, \mathcal{O}_X).$$

Proposition 2.11. Let (X, Δ) be a globally F-regular pair (over T) and $f : X' \to X$ a proper birational morphism between normal varieties such that $f^*(K_X + \Delta) = K_{X'} + \Delta'$ where $\Delta' \geq 0$. Then (X', Δ') is globally F-regular over T.

Proof. Assume that the index of $K_X + \Delta$ is not divisible by $p > 0$. Let D' be an effective divisor on X' and pick D a Cartier divisor on X such that $f^*D \geq D'$. Since (X, Δ) is globally F-regular, we have a surjection

$$H^0(X, \mathcal{O}_X((1-p^e)(K_X + \Delta) - D)) \to H^0(X, \mathcal{O}_X)$$

for $e > 0$ sufficiently divisible. By the projection formula, this is equivalent to the surjection

$$H^0(X', \mathcal{O}_{X'}((1-p^e)(K_{X'} + \Delta') - f^*D)) \to H^0(X', \mathcal{O}_{X'})$$

which factors through $H^0(X', \mathcal{O}_{X'}((1-p^e)(K_{X'} + \Delta') - D'))$. Thus (X', Δ') is globally F-regular.

To see the general case, note that we may work locally over T and hence we may assume that there is a $\mathbb{Q}$-divisor $\Delta_1 \geq \Delta$ such that (X, Δ_1) is a globally F-regular pair (over T) with an index not divisible by p. It then follows that $f^*(K_X + \Delta_1) = K_{X'} + \Delta'_1$ where $\Delta'_1 \geq 0$ and (X', Δ'_1) is globally F-regular. But then (X', Δ') is globally F-regular since $\Delta' \leq \Delta'_1$. ☐

We will need the following easy consequence (cf. [MR85]).

Lemma 2.12. *Let $f : X \to T$ be a proper birational morphism between normal varieties such that (X, Δ) is globally F-regular over T. Then $(T, \Delta_T = f_*\Delta)$ is strongly F-regular.*

Proof. We may assume that T is affine. Let D be an effective divisor on T. Pick $D' \geq D$ such that D' is Cartier. Since (X, Δ) is globally F-regular over T, we have a surjection

$$H^0(X, F^e_*\mathcal{O}_X(\lfloor (1-p^e)(K_X + \Delta) \rfloor - f^*D')) \to H^0(X, \mathcal{O}_X).$$

Since $H^0(X, F^e_*\mathcal{O}_X(\lfloor (1-p^e)(K_X + \Delta) \rfloor - f^*D')) = H^0(T, f_*F^e_*\mathcal{O}_X(\lfloor (1-p^e)(K_X + \Delta) \rfloor - f^*D'))$ is contained in $H^0(T, F^e_*\mathcal{O}_T(\lfloor (1-p^e)(K_T + \Delta_T) \rfloor - D'))$, we have surjections

$$F^e_*\mathcal{O}_T(\lfloor (1-p^e)(K_T + \Delta_T) \rfloor - D') \to \mathcal{O}_T.$$

Since the above map factors through $F^e_*\mathcal{O}_T(\lfloor (1-p^e)(K_T + \Delta_T) \rfloor - D)$, $(T, \Delta_T = f_*\Delta)$ is strongly F-regular. ☐

We will also need the following well known perturbation lemma (cf. [Patakfalvi12, 3.15] and [SS10, 3.12]).

Lemma 2.13. *Let (X, Δ) be a log pair, $E \geq 0$ a divisor such that $E - K_X$ is Cartier and $H = \mathrm{Supp}(E + \Delta)$. Then for any $\epsilon > 0$, we can find an effective $\mathbb{Q}$-Cartier divisor $D \leq \epsilon H$ such that the $\mathbb{Q}$-Cartier index of $K_X + \Delta + D$ is not divisible by p.*

Proof. Assume that $mp^{e_0}(K_X+\Delta)$ is Cartier, where $p \nmid m \in \mathbb{N}$. Pick $e > e_0$ such that $\frac{1}{p^e-1}(\Delta+E) \le \epsilon H$ and let $D = \frac{1}{p^e-1}(\Delta+E)$. Then

$$\begin{aligned} m(p^e-1)(K_X+\Delta+D) &= m(p^e-1)\left(K_X+\frac{p^e}{p^e-1}\Delta+\frac{1}{p^e-1}E\right) \\ &= mp^e\Delta + m(p^e-1)K_X + mE \\ &\sim mp^e\Delta + m(p^eK_X+E-K_X) \end{aligned}$$

is Cartier. □

2.4. **Stabilization of S^0 in the relative case.** The results of this section were communicated to us by Karl Schwede (cf. [Schwede13]). We thank him for allowing us to include them in this note.

Suppose that $(X, \Delta \ge 0)$ is a pair such that $L_{g,\Delta} = (1-p^g)(K_X+\Delta)$ is Cartier for some $g > 0$, $f : X \to Y$ is a projective morphism with Y normal and M is a Cartier divisor on X.

Definition 2.14. With the above notation, $S^0 f_*(\sigma(X,\Delta)\otimes\mathcal{O}_X(M))$ is defined to be the intersection:

$$\bigcap_{e\ge 0} \operatorname{Image}\left(\operatorname{Tr}^{eg} : F^{eg}_* f_*\mathcal{O}_X((1-p^{eg})(K_X+\Delta)+p^{eg}M) \to f_*\mathcal{O}_X(M)\right)$$

which is more compactly written as

$$\bigcap_{e\ge 0} \operatorname{Image}\left(\operatorname{Tr}^{eg} : F^{eg}_* f_*\mathcal{O}_X(L_{eg,\Delta}+p^{eg}M) \to f_*\mathcal{O}_X(M)\right).$$

Here Tr^{eg} is defined by the pushing forward via f_* of the sequence

$$F^{eg}_*\mathcal{O}_X((1-p^{eg})(K_X+\Delta)+p^{eg}M) \to F^{eg}_*\mathcal{O}_X((1-p^{eg})K_X+p^{eg}M) \to \mathcal{O}_X(M),$$

which is obtained tensoring the sequence in [Schwede11, Section 2] by $\mathcal{O}_X(M)$.

This intersection is a descending intersection, so a priori it need not stabilize. Therefore, it is unclear if it is a coherent sheaf. At least when $M-K_X-\Delta$ is ample, we show now that this is the case.

Proposition 2.15. If $M-K_X-\Delta$ is f-ample, then

$$S^0 f_*(\sigma(X,\Delta)\otimes\mathcal{O}_X(M)) = \operatorname{Image}\left(\operatorname{Tr}^{eg} : F^{eg}_* f_*\mathcal{O}_X(L_{eg,\Delta}+p^{eg}M) \to f_*\mathcal{O}_X(M)\right)$$

for $e \gg 0$. In particular, it is a coherent sheaf.

Proof. It is easy to see that $S^0 f_*(\sigma(X,\Delta)\otimes\mathcal{O}_X(M))$ is equal to

$$\bigcap_{e\ge 0} \operatorname{Image}\left(\operatorname{Tr}^{eg} : F^{eg}_* f_*(\sigma(X,\Delta)\otimes\mathcal{O}_X(L_{eg,\Delta}+p^{eg}M)) \to f_*(\sigma(X,\Delta)\otimes\mathcal{O}_X(M))\right).$$

Let $\mathcal{K}$ denote the kernel of the surjective map

$$F^g_*(\sigma(X,\Delta)\otimes\mathcal{O}_X(L_{g,\Delta})) \to \sigma(X,\Delta).$$

By Serre vanishing there is an integer $e_0 > 0$ such that

$$R^1 f_*(\mathcal{K}\otimes\mathcal{O}_X(p^{eg}M+L_{eg,\Delta})) = 0$$

for all $e \ge e_0$. It follows that the map

$$\begin{aligned} &F^{(1+e)g}_* f_*(\sigma(X,\Delta)\otimes\mathcal{O}_X(p^{(1+e)g}M+L_{(1+e)g,\Delta})) \\ &\quad\to F^{eg}_* f_*(\sigma(X,\Delta)\otimes\mathcal{O}_X(p^{eg}M+L_{eg,\Delta})) \end{aligned}$$

is surjective for all $e \geq e_0$ and hence so are the maps

$$\begin{aligned} &F_*^{(d+e)g} f_*(\sigma(X,\Delta) \otimes \mathcal{O}_X(p^{(d+e)g}M + L_{(d+e)g,\Delta})) \\ &\to F_*^{eg} f_*(\sigma(X,\Delta) \otimes \mathcal{O}_X(p^{eg}M + L_{eg,\Delta})) \end{aligned}$$

for all $e \geq e_0$ and $d \geq 1$. Therefore, these maps have the same images under the trace. In other words,

$$\begin{aligned} &\text{Image}\,(\text{Tr}^{eg} : F_*^{eg} f_* \mathcal{O}_X(L_{eg,\Delta} + p^{eg}M) \to f_*\mathcal{O}_X(M)) \\ &= \text{Image}\,(\text{Tr}^{e_0 g} : F_*^{e_0 g} f_* \mathcal{O}_X(L_{e_0 g,\Delta} + p^{e_0 g}M) \to f_*\mathcal{O}_X(M)) \end{aligned}$$

for all $e \geq e_0$ and the proposition is proven. ☐

Remark 2.16. If Y is affine, then we can identify

$$F_*^{eg} f_* \mathcal{O}_X((1-p^{eg})(K_X+\Delta) + p^{eg}M)$$

with

$$F_*^{eg} H^0(X, \mathcal{O}_X((1-p^{eg})(K_X+\Delta) + p^{eg}M))$$

and hence $S^0 f_*(\sigma(X,\Delta) \otimes \mathcal{O}_X(M))$ with $S^0(X, \sigma(X,\Delta) \otimes \mathcal{O}_X(M))$. By (2.10), it follows that if (X,Δ) is globally F-regular over Y, then $S^0 f_*(\sigma(X,\Delta)) = f_*\mathcal{O}_X$ (the stabilization in this case is automatic and does not require that $(p^g-1)(K_X+\Delta)$ is Cartier for some $g > 0$.

Remark 2.17. It is easy to see that if $M - K_X - \Delta$ is f-ample, then the results of Section 2.3 easily translate to corresponding results about the sheaves $S^0 f_*(\sigma(X,\Delta) \otimes \mathcal{O}_X(M))$. We will explicitly state only the results that will be frequently used in what follows.

Proposition 2.18. With the above notation, assume that p does not divide the index of $K_X + \Delta$, M is a Cartier divisor, $M - (K_X+\Delta)$ is f-ample and $S \subset X$ is any union of F-pure centers of (X,Δ). Then there is a natural surjective map

$$S^0 f_*(\sigma(X,\Delta) \otimes \mathcal{O}_X(M)) \to S^0(f|_S)_*(\sigma(S, \phi_S^\Delta) \otimes \mathcal{O}_S(M)).$$

Proof. The proof is immediate from the arguments in the proof of [Schwede11, 5.3]. ☐

Lemma 2.19. *Let L be an f-ample Cartier divisor and (X,Δ) a sharply F-pure pair such that p does not divide the index of $K_X+\Delta$. Then there exists an integer $m_0 > 0$ such that for any f-nef Cartier divisor P and any integer $m \geq m_0$, we have*

$$S^0 f_*(\sigma(X,\Delta) \otimes \mathcal{O}_X(mL+P)) = f_*\mathcal{O}_X(mL+P).$$

Proof. The proof is immediate from the arguments in the proof of [Patakfalvi12, 2.23]. ☐

Lemma 2.20. *Let $f : X \to Y$ be a projective morphism of normal quasi-projective varieties such that (X,Δ) is globally F-regular over Y. If A is a sufficiently ample divisor on Y, then*

$$S^0(X, \sigma(X,\Delta) \otimes \mathcal{O}_X(f^*A)) = H^0(X, \mathcal{O}_X(f^*A)).$$

Proof. Since (X,Δ) is globally F-regular over Y, by (2.7), we may pick $\Delta' \geq \Delta$ such that (X,Δ') is also globally F-regular over Y, the index of $K_X+\Delta'$ is not divisible by p and $-(K_X+\Delta')$ is ample over Y. Since (X,Δ') is globally F-regular over Y, (X, Δ') is strongly F-regular so that $\sigma(X, \Delta') = \mathcal{O}_X$. Fix $g > 0$ such that $(p^g - 1)(K_X + \Delta')$ is Cartier, and

$$F_*^{eg}\mathcal{L}_{eg,\Delta'} \to \mathcal{O}_X, \text{ and } \qquad f_*F_*^{eg}\mathcal{L}_{eg,\Delta'} \to f_*\mathcal{O}_X$$

are surjective for any $e > 0$. Let $\mathcal{K}$ be the kernel of the map

$$F_*^{g}\mathcal{L}_{g,\Delta'} \to \mathcal{O}_X.$$

Since $(F_*^g\mathcal{L}_{g,\Delta'}) \otimes \mathcal{L}_{(e-1)g,\Delta'} = F_*^g\mathcal{L}_{eg,\Delta'}$, twisting by $\mathcal{L}_{(e-1)g,\Delta'}$ and pushing forward by $F_*^{(e-1)g}$ we obtain the short exact sequence

$$0 \to F_*^{(e-1)g}(\mathcal{K} \otimes \mathcal{L}_{(e-1)g,\Delta'}) \to F_*^{eg}\mathcal{L}_{eg,\Delta'} \to F_*^{(e-1)g}\mathcal{L}_{(e-1)g,\Delta'} \to 0.$$

Since $-(K_X+\Delta')$ is f-ample, there exists $e_0 > 0$ such that $R^1 f_*(\mathcal{K}\otimes\mathcal{L}_{(e-1)g,\Delta'}) = 0$ for all $e \geq e_0$. Since (X, Δ') is F-regular over Y, pushing forward via f we obtain the short exact sequences

$$0 \to F_*^{(e-1)g}f_*(\mathcal{K} \otimes \mathcal{L}_{(e-1)g,\Delta'}) \to F_*^{eg}f_*\mathcal{L}_{eg,\Delta'} \to F_*^{(e-1)g}f_*\mathcal{L}_{(e-1)g,\Delta'} \to 0.$$

If $e \gg 0$ and A is sufficiently ample, then $\mathcal{L}_{(e-1)g,\Delta'} + f^*p^{(e-1)g}A$ is sufficiently ample so that by Fujita vanishing (see [Keeler03]), we have that

$$\begin{aligned} &H^1(Y, F_*^{(e-1)g}f_*(\mathcal{K} \otimes \mathcal{L}_{(e-1)g,\Delta'}) \otimes \mathcal{O}_Y(A)) \\ &\quad = H^1(Y, f_*(\mathcal{K} \otimes \mathcal{L}_{(e-1)g,\Delta'} \otimes \mathcal{O}_X(p^{(e-1)g}f^*A))) \\ &\quad = H^1(X, \mathcal{K} \otimes \mathcal{L}_{(e-1)g,\Delta'} \otimes f^*\mathcal{O}_Y(p^{(e-1)g}A)) = 0. \end{aligned}$$

We may also assume that

$$H^0(Y, f_*F_*^{e_0g}\mathcal{L}_{e_0g,\Delta'} \otimes \mathcal{O}_Y(A)) \to H^0(Y, f_*\mathcal{O}_X \otimes \mathcal{O}_Y(A))$$

is surjective. But then since

$$\begin{aligned} &H^0(Y, f_*F_*^{eg}(\mathcal{L}_{eg,\Delta'} \otimes \mathcal{O}_X(p^{eg}f^*A))) \\ &\quad \to H^0(Y, f_*F_*^{(e-1)g}(\mathcal{L}_{(e-1)g,\Delta'} \otimes \mathcal{O}_X(p^{(e-1)g}f^*A))) \end{aligned}$$

is surjective for all $e > e_0$, it follows that

$$H^0(Y, f_*F_*^{eg}(\mathcal{L}_{eg,\Delta'} \otimes \mathcal{O}_X(p^{eg}f^*A))) \to H^0(Y, f_*\mathcal{O}_X \otimes \mathcal{O}_Y(A))$$

is surjective for any $e \geq e_0$ and hence

$$S^0(X, \sigma(X, \Delta') \otimes \mathcal{O}_X(f^*A)) = H^0(Y, \mathcal{O}_Y(A)).$$

Since $S^0(X, \sigma(X, \Delta') \otimes \mathcal{O}_X(f^*A)) \subset S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(f^*A))$ the lemma is proven. □

2.5. **Surfaces.** In this subsection, we collect the results in MMP theory for surfaces (in characteristic $p > 0$) that we will need later.

Proposition 2.21. Let $f : S \to R$ be a projective morphism from a normal surface, and B_S be a $\mathbb{Q}$-divisor on S such that (S, B_S) is a relative weak log Fano surface, i.e., (S, B_S) is klt and $-(K_S + B_S)$ is f-nef and f-big. Then we have the following

(1) any relatively nef divisor is semi-ample over R,
(2) the nef cone of S (over R) is finitely generated.

Proof. See [Tanaka12a, 15.2] and [Tanaka12b, 3.2]. □

Lemma 2.22. *Let (S, B_S) be a klt pair. There exists a unique birational morphism $\nu : \bar{S} \to S$ such that*

(1) $K_{\bar{S}} + B_{\bar{S}} = \nu^*(K_S + B_S)$, *where* $B_{\bar{S}} \geq 0$, *and*
(2) $(\bar{S}, B_{\bar{S}})$ *is terminal.*

$(\bar{S}, B_{\bar{S}})$ is the terminalization of (S, B_S). In particular $\bar{S}$ is smooth and $\mathrm{mult}_p(B_{\bar{S}}) < 1$ *for all $p \in \bar{S}$.*

Proof. We reproduce the following well known proof. For any log resolution of $g_{S'} : S' \to S$, we write

$$g_{S'}^*(K_S + B_S) + E = K_{S'} + B_{S'},$$

where E, $B_{S'}$ are effective and have no common components. Passing to a higher resolution, we can assume $(S', B_{S'})$ is terminal.

We then run the minimal model program for $(S', B_{S'})$ over S and we obtain a relative minimal model $\mu : S' \to \bar{S}$ with a morphism $\nu : \bar{S} \to S$ (see [KK], [Tanaka12a]). Note that $\mu_* E \sim_{S,\mathbb{Q}} K_{\bar{S}} + B_{\bar{S}}$ is nef where $B_{\bar{S}} = \mu_* B_{S'}$. Since $\mu_* E$ is an exceptional curve, its self-intersection is non-positive and if $\mu_* E \neq 0$ the self-intersection is negative. Thus $\mu_* E = 0$ and hence $K_{\bar{S}} + B_{\bar{S}} \sim_{\mathbb{Q}} \nu^*(K_S + B_S)$. As $(S', B_{S'})$ is terminal, so is $(\bar{S}, B_{\bar{S}})$. Uniqueness is also well known, and we omit the proof. □

Even though the Kawamata-Viehweg vanishing theorem does not hold for surfaces, it is still true for a birational morphism between surfaces.

Lemma 2.23 ([KK, 2.2.5])**.** *Let $h : S' \to S$ be a proper birational morphism between normal surfaces, such that $(S', B_{S'})$ is a klt pair. Let L be a Cartier divisor on S', and N an h-nef $\mathbb{Q}$-divisor such that $L \equiv K_{S'} + B_{S'} + N$. Then $R^1 h_* \mathcal{O}_{S'}(L) = 0$.*

Proof. [KK, 2.25] proves the case when S' is smooth. In general, we can take the terminalization of (S, B_S) and use a simple spectral sequence argument. □

3. On the F-regularity of weak log del Pezzo surfaces

Hara proved that a klt surface S is strongly F-regular if the characteristic is larger than 5 (see [Hara98]). The aim of this section is to generalize Hara's result to establish the global F-regularity for relative weak log del Pezzo surfaces of birational type with standard coefficients $f : (S, B) \to T$ when the characteristic is larger than 5. We will use Shokurov's theory of *complements* (see [Shokurov00, Prokhorov99]), which fits in this context very well.

More precisely, our main theorem of this section is (3.1). To prove it, we proceed as follows. By a result of Shokurov (cf. (3.2)) we can find a $\mathbb{Q}$-divisor $B^c \geq B$ such that $N(K_S + B^c) \sim 0$ and $N \in \{1, 2, 3, 4, 6\}$ where (S, B^c) is lc but not klt. We then consider a smooth dlt model $(\tilde{S}, B^c_{\tilde{S}}) \to (S, B^c)$ with $C = \lfloor B^c_{\tilde{S}} \rfloor \neq 0$. By a careful study of the pair $(\tilde{S}, B^c_{\tilde{S}})$, we define a $\mathbb{Q}$-divisor $B^*_{\tilde{S}}$ such that $B^c_{\tilde{S}} \geq B^*_{\tilde{S}} \geq B_{\tilde{S}} + C$, $-(K_{\tilde{S}} + B^*_{\tilde{S}})$ is relatively nef and $(\tilde{S}, B^*_{\tilde{S}})$ is plt. By the numerical properties that the coefficients of $B^*_{\tilde{S}}$ satisfy, we can apply Fedder's criteria to check that if $p > 5$, then $(C, \mathrm{Diff}_C(B^*_{\tilde{S}}))$ is globally F-regular. By the relative Kawamata-Viehweg theorem for surfaces, it follows that (S, B) is globally F-regular over T (in a neighborhood of $f(C)$).

Theorem 3.1. *Assume the ground field k is algebraically closed of characteristic $p > 5$. Let (S, B) be a pair with a birational proper morphism $f : S \to T$ on to a normal surface germ $(T, 0)$ such that*

(1) *(S, B) is klt,*
(2) *$-(K_S + B)$ is f-nef, and*
(3) *the coefficients of B are in the standard set $\{\frac{n-1}{n} | n \in \mathbb{N}\}$.*

Then (S, B) is globally F-regular over T.

We will need the following result on complements due to Shokurov.

Theorem 3.2. *Notation as in* (3.1). *There exists a divisor $B^c \geq B$ and an integer $N \in \mathcal{R}N_2 = \{1, 2, 3, 4, 6\}$, such that $N(K_S + B^c) \sim_T 0$ and (S, B^c) is log canonical but not klt. Let $\nu : \tilde{S} \to S$ be a dlt modification, $K_{\tilde{S}} + B^c_{\tilde{S}} = \nu^*(K_S + B^c)$ and $C = \lfloor B^c_{\tilde{S}} \rfloor$. Then*

(1) *If $(\tilde{S}, B^c_{\tilde{S}})$ is plt, then we may assume that $(C, \mathrm{Diff}_C(B^c_{\tilde{S}} - C))$ belongs to one of the case listed in* [Prokhorov99, 4.1.12].
(2) *If $(\tilde{S}, B^c_{\tilde{S}})$ is not plt, then we may assume that $N \in \{1, 2\}$ and $(C, \mathrm{Diff}_C(B^c_{\tilde{S}} - C))$ is of the type $(2, 2, \infty)$ or (∞, ∞).*

Proof.

Claim 3.3. There exists an effective $\mathbb{Q}$-divisor Δ on S such that $(S, B + \Delta)$ is log canonical but not klt, $K_S + B + \Delta \sim_{T,\mathbb{Q}} 0$ and there is a unique valuation C such that $a(C; S, B_S + \Delta) = -1$.

Proof. $-(K_S + B)$ is semi-ample over T. Let $\phi : S \to \bar{S}$ be the corresponding morphism so that $-(K_S + B) = \phi^* H_{\bar{S}}$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H_{\bar{S}}$ on $\bar{S}$ which is ample over T. We may pick a log resolution $\mu : S' \to S$ of (S, B) such that there is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor F where $-F$ is ample over $\bar{S}$. Let $K_{S'} + B_{S'} = \mu^*(K_S + B)$, then $(S', B_{S'})$ is sub klt and $B_{S'}$ has simple normal crossings support. For $0 < \epsilon \ll 1$ we have that $\mu^* \phi^* H_{\bar{S}} - \epsilon F$ is ample over T and $(S', B_{S'} + \epsilon F)$ is sub klt. Let $H_{S'}$ be a general $\mathbb{Q}$-divisor such that $H_{S'} \sim_{\mathbb{Q},T} \mu^* \phi^* H_{\bar{S}} - \epsilon F$. Then $(S', B_{S'} + \epsilon F + H_{S'})$ is sub klt. Let $H_S = \mu_* H_{S'}$, then $H_S + \epsilon \mu_* F \sim_{\mathbb{Q},T} -(K_S + B)$ and $(S, B + H_S + \epsilon \mu_* F)$ is klt. Let G be a general $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on T whose support contains 0 and let τ be the log canonical threshold of $(S, B + H_S + \epsilon \mu_* F)$ with respect to $f^* G$ (over a neighborhood of $0 \in T$). By the usual tie breaking argument, we may assume that there is a unique valuation C such that $a(C; S, B + H_S + \epsilon \mu_* F + \tau f^* G) = -1$. We let $\Delta = B + H_S + \epsilon \mu_* F + \tau f^* G$. $\square$

Let $g : S' \to S$ be a dlt modification of $(S, B_S + \Delta)$ i.e. proper birational morphism that only extracts C (if C is a divisor on S, then we let g be the identity on S). It follows that $(S', g_*^{-1} B_S + C)$ is plt. Let $c = -a(C; S, B_S) < 1$. We write

$$g^*(K_S + B) = K_{S'} + B_1 + cC \qquad \text{and} \qquad g^*(K_S + B + \Delta) = K_{S'} + B_2 + C$$

where $B_1 \wedge C = 0$ and $B_2 \wedge C = 0$. Then, since $K_{S'} + B_2 + C \sim_{T,\mathbb{Q}} 0$, we have

$$K_{S'} + B_1 + C \sim_{T,\mathbb{Q}} -(B_2 - B_1) \leq 0.$$

Let $G = B_2 - B_1$. Since $(S', B_2 + C + \epsilon G)$ is plt for $0 < \epsilon \ll 1$ and

$$K_{S'} + B_2 + C + \epsilon G \sim_{T,\mathbb{Q}} \epsilon G,$$

we can run the G-MMP over T to get a G-minimal model $h : S' \to S''$. Since C is not contained in the support of G, this MMP does not contract C. We denote by $B_3 = h_*B_1$ and $C'' = h_*(C)$.

Since $K_{S''} + h_*(B_2 + C + \epsilon G)$ is plt, $h_*C = C'' \neq 0$ and $B_3 + C'' \leq h_*(B_2 + C)$, it follows that $K_{S''} + B_3 + C''$ is plt. Since the coefficients of B_1 are contained in $\{\frac{n-1}{n} | n \in \mathbb{N}\}$, the same holds for the coefficients of B_3. Thus

$$(K_{S''} + B_3 + C'')|_{C''} = K_{\mathbb{P}^1} + \sum_i a_i P_i$$

where each a_i is of the form $\frac{n_i - 1}{n_i}$ for some positive integer n_i. Note that as $K_{S''} + B_3 + C'' \sim_{\mathbb{Q},T} -h_*G$, it follows that $-(K_{S''} + B_3 + C'')$ is nef over T. Therefore, the divisor B_3 belongs to one of the cases listed in [Prokhorov99, 4.1.11, 4.1.12]. These cases are

(1) $\sum a_i < 2$: (n_1, n_2), $(2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. One easily sees that they are 1, 2, 3, 4 and 6 complementary with complements (∞, ∞), $(2, 2, \infty)$, $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ respectively (here $(2, 2, \infty)$ denotes a divisor with coefficients $1 - \frac{1}{2}$, $1 - \frac{1}{2}$ and 1).
(2) $\sum a_i = 2$: $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ which are 2, 3, 4 and 6 complementary (and in fact equal their complement).

Since $-(K_{S''} + B_3 + C'')$ is nef over T, by [Prokhorov99, 4.4.1] there is a N-complement $(B'')^c$ for $C'' + B_3$ (where $N \in \{1, 2, 3, 4, 6\}$). Thus $N(K_{S''} + (B'')^c) \sim_T 0$, $K_{S''} + (B'')^c$ is lc and $N(B'')^c \geq nC'' + \lfloor (N + 1)B_3 \rfloor$ (cf. [Prokhorov99, 4.1.3]). Since the coefficients of B_3 belong to the set $\{1 - \frac{1}{n} | n \in \mathbb{N}\}$ it follows easily that

$$(B'')^c \geq C'' + B_3.$$

Since h is $(K_{S'} + B_1 + C)$-positive, it follows that

$$(K_{S'} + B_1 + C) - h^*(K_{S''} + B_3 + C'') \leq 0.$$

Let $K_{S'} + (B')^c = h^*(K_{S''} + (B'')^c)$, then $(B')^c \geq C + B_1$, $N(K_{S'} + (B')^c) \sim_T 0$ and $(S', (B')^c)$ is lc. Finally, let $B^c = g_*(B')^c$, then $B^c \geq g_*(C + B_1) = B$, $N(K_S + B^c) \sim_T 0$ and (S, B^c) is lc. □

Proof of (3.1). We will assume that N is minimal as above. Let $\nu : \tilde{S} \to S$ be a smooth dlt modification of (S, B^c) (so that $(\tilde{S}, B^c_{\tilde{S}})$ has dlt singularities) and write

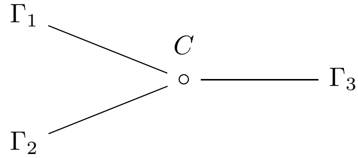
$$K_{\tilde{S}} + B^c_{\tilde{S}} = \nu^*(K_S + B^c) \qquad \text{and} \qquad K_{\tilde{S}} + B_{\tilde{S}} = \nu^*(K_S + B).$$

In particular $B^c_{\tilde{S}} \geq 0$. Note however that $B_{\tilde{S}}$ is possibly not effective.

We assume first that we are in the *exceptional case* that is $(\tilde{S}, B^c_{\tilde{S}})$ is plt. We let $C = \lfloor B^c_{\tilde{S}} \rfloor$. Note that by Kollár-Shokurov Connectedness, C is connected and hence irreducible.

Consider the extended dual graph G (see [Kollár13, 2.26] for the definition) corresponding to the exceptional curves for $\mu = f \circ \nu$ and the strict transform of $B^c_T = f_*B^c$. Let G_C be the subgraph constructed by first removing all vertices such that the corresponding curve appears in $B^c_{\tilde{S}} - B_{\tilde{S}}$ with coefficient 0 and then discarding all connected components not containing the vertex corresponding to C. Note that by the proof of (3.2), the curve C is exceptional over T and as $\tilde{S}$ dominates the minimal resolution and all exceptional curves are smooth rational curves meeting transversely.

When $I = 3$, the graph looks like



where each Γ_i is a connected tree. (If $|I| = 4$, we define Γ_i analogously.)

Let Ψ_i (resp. Ψ_i^c) be the subdivisor of $B_{\tilde{S}}$ (resp. $B_{\tilde{S}}^c$) consisting of components in $B_{\tilde{S}}$ contained in the support of Γ_i. Then

$$B_{\tilde{S}} = cC + \sum_{i=1}^{|I|} \Psi_i + \Lambda_{\tilde{S}}, \text{ and } \qquad B_{\tilde{S}}^c = C + \sum_{i=1}^{|I|} \Psi_i^c + \Lambda_{\tilde{S}}^c$$

where $c < 1$, $\Lambda_{\tilde{S}}$ and $\Lambda_{\tilde{S}}^c$ are the remaining components (whose support is contained in the components corresponding to $G - G_C$). Note that $\Psi_i^c \geq 0$ but the Ψ_i are not necessarily effective. We say that a chain is *non-exceptional* if it has at least one component which is not μ-exceptional.

Lemma 3.4. *There exists an $i = i_0 \in I$, and a non-exceptional chain $\Gamma_i^0 \subset \Gamma_i$ intersecting C such that* $\mathrm{Supp}(\Psi_i^c - \Psi_i) \supset \Gamma_i^0$.

Proof. Let D be the connected component of $(1-c)C + \sum(\Psi_i^c - \Psi_i)$ containing C. Note that the support of D contains C and so D is non-zero. If D is exceptional, then it is easy to see that

$$(B_{\tilde{S}}^c - B_{\tilde{S}}) \cdot D = D^2 < 0.$$

This is impossible as

$$B_{\tilde{S}}^c - B_{\tilde{S}} = K_{\tilde{S}} + B_{\tilde{S}}^c - (K_{\tilde{S}} + B_{\tilde{S}}) = \nu^*(K_S + B_S^c - K_S - B_S) \sim_{\mathbb{Q},T} -\nu^*(K_S + B_S)$$

is nef over T. $\square$

We assume that $i_0 = 1$ and let Γ_1^0 be the subchain of Γ_1 defined in (3.4). We assume that the first vertex corresponds to a curve intersecting C and the last vertex corresponds to a non-exceptional curve in the support of $B_{\tilde{S}}^c - B_{\tilde{S}}$. We define the divisor

$$B_{\tilde{S}}^* = C + \Psi_1^* + \sum_{i=2}^{m} \Psi_i^c + \Lambda_{\tilde{S}}^c,$$

where Ψ_1^* is defined by replacing each coefficient $\frac{p}{q}$ of a component of $\Psi_1^c \wedge \Gamma_1^0$ with $q = N \in \{1, 2, 3, 4, 6\}$ by the coefficient $\frac{p-1}{q-1}$ and we leave the remaining coefficients unchanged.

Lemma 3.5. $-(K_{\tilde{S}} + B_{\tilde{S}}^*)$ *is nef over T.*

Proof. Since $B_{\tilde{S}}^* \leq B_{\tilde{S}}^c$, it is clear that $(K_{\tilde{S}} + B_{\tilde{S}}^*) \cdot C \leq 0$. If D is an exceptional curve not in Γ_1^0, then by the same argument, it is clear that $(K_{\tilde{S}} + B_{\tilde{S}}^*) \cdot D \leq 0$.

For any exceptional curve D contained in Γ_1^0 which does not meet any non-exceptional component of Γ_1^0, we know that its intersection number with the adjacent components in Γ_1^0 are all 1. Thus the equation $(K_{\tilde{S}} + B_{\tilde{S}}^c) \cdot D = 0$ implies that

$$\frac{p_{j-1}}{q} + \frac{p_{j+1}}{q} + \frac{p_j}{q} D^2 + \frac{r}{q} - 2 - D^2 = 0 \qquad \text{or}$$
$$p_{j-1} + p_{j+1} + r - 2q + (p_j - q)D^2 = 0.$$

(We have used the fact that by adjunction one has $K_{\tilde{S}} \cdot D = -2 - D^2$.) Here we denote by p_j/q the multiplicity of D in $B^c_{\tilde{S}}$ and by p_{j-1}/q (resp. p_{j+1}/q) the multiplicity of $B^c_{\tilde{S}}$ along the previous (resp. the following) curve in Γ^i_0. If D is the curve in Γ^0_1 that intersects C, then we let $j-1=0$ and $p_0 = q$ as the multiplicity of $B^c_{\tilde{S}}$ along C is 1. r/q denotes the intersection of all other components of $B^c_{\tilde{S}}$ with D. To obtain $B^*_{\tilde{S}}$, we replace $(p_{j-1}, p_j, p_{j+1}, q)$ by $(p_{j-1}-1, p_j-1, p_{j+1}-1, q-1)$; note that $p_0 = q$ and hence $(p_0-1)/(q-1) = 1$ so that the multiplicity along C is unchanged. The above equality implies

$$\frac{p_{j-1}-1}{q-1} + \frac{p_{j+1}-1}{q-1} + \frac{p_j-1}{q-1} D^2 + \frac{r}{q} - 2 - D^2 \le 0$$

which says that $(K_{\tilde{S}} + B^*_{\tilde{S}}) \cdot D \le 0$.

For an exceptional curve $D \in \Gamma^0_1$ which meets a non-exceptional component G of Γ^0_i, the same calculation implies that $(K_{\tilde{S}} + B^*_{\tilde{S}}) \cdot D = 0$ if the intersection number is $D \cdot G = 1$, and otherwise $(K_{\tilde{S}} + B^*_{\tilde{S}}) \cdot D < 0$. ☐

Lemma 3.6. *We have $B^*_{\tilde{S}} \ge B_{\tilde{S}}$.*

Proof. Since $B_{\tilde{S}} - B^*_{\tilde{S}} = K_{\tilde{S}} + B_{\tilde{S}} - (K_{\tilde{S}} + B^*_{\tilde{S}})$ is nef over S, by the negativity lemma it suffices to show that $\nu_* B^*_{\tilde{S}} \ge B$. This follows by (3.7) below, since $B^c - B \ge 0$, the support of $B^c - B$ contains Γ^0_i and the coefficients of B are of the form $\frac{n-1}{n}$. ☐

Lemma 3.7. *Let p, q, j be natural numbers such that $\frac{j-1}{j} < \frac{p}{q}$ and $q \ge 2$, then $\frac{j-1}{j} \le \frac{p-1}{q-1}$.*

Proof. Obvious. ☐

Lemma 3.8. *We have that $(C, \mathrm{Diff}_C(B^*_{\tilde{S}}))$ is globally F-regular.*

Proof. We know that $N \in \{1,2,3,4,6\}$. If follows from (3.1) that the coefficients of $\mathrm{Diff}_C(B^c_{\tilde{S}})$ are of the form $\frac{n-1}{n}$. If $N = 6$, then we have $(C, \mathrm{Diff}_C(B^c_{\tilde{S}})) = (\mathbb{P}^1, \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3)$. But then $(C, \mathrm{Diff}_C(B^*_{\tilde{S}})) = (\mathbb{P}^1, aP_1 + bP_2 + cP_3)$ where

$$(a,b,c) \in \{(2/5, 2/3, 5/6), (1/2, 3/5, 5/6), (1/2, 2/3, 4/5)\}.$$

If $N = 4$, then $(C, \mathrm{Diff}_C(B^c_{\tilde{S}})) = (\mathbb{P}^1, \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{3}{4}P_3)$. But then $(C, \mathrm{Diff}_C(B^*_{\tilde{S}})) = (\mathbb{P}^1, aP_1 + bP_2 + cP_3)$ where

$$(a,b,c) \in \{(1/3, 3/4, 3/4), (1/2, 2/3, 3/4)\}.$$

If $N = 3$, then $(C, \mathrm{Diff}_C(B^c_{\tilde{S}})) = (\mathbb{P}^1, \frac{2}{3}P_1 + \frac{2}{3}P_2 + \frac{2}{3}P_3)$. But then $(C, \mathrm{Diff}_C(B^*_{\tilde{S}})) = (\mathbb{P}^1, \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{2}{3}P_3)$.

If $N = 2$, then $(C, \mathrm{Diff}_C(B^c_{\tilde{S}})) = (\mathbb{P}^1, \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + \frac{1}{2}P_4)$. But then $(C, \mathrm{Diff}_C(B^*_{\tilde{S}})) = (\mathbb{P}^1, \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3)$.

All of these cases are globally F-regular by (3.10). ☐

Proposition 3.9. Notation and assumptions as above. We assume that there is an effective $\mathbb{Q}$-divisor $B^*_{\tilde{S}}$ on $\tilde{S}$ such that

(1) $-(K_{\tilde{S}} + B^*_{\tilde{S}})$ is nef over T, $B^c_{\tilde{S}} \geq B^*_{\tilde{S}} \geq B_{\tilde{S}}$,
(2) $(\tilde{S}, B^*_{\tilde{S}})$ is plt,
(3) $(C, \mathrm{Diff}_C(B^*_{\tilde{S}}))$ is globally F-regular.

Then (S, B) is globally F-regular over T.

Proof. The argument is a generalization of [Hara98, 4.3].

It follows from our assumption that $-(K_{\tilde{S}} + B^*_{\tilde{S}})$ is semi-ample over T. We denote by

$$\hat{S} := \mathrm{Proj} R(\tilde{S}/T, -(K_{\tilde{S}} + B^*_{\tilde{S}})).$$

Let F be an exceptional relatively anti-ample divisor for $\tilde{S}$ over $\hat{S}$. We note that by (3), $-(K_{\tilde{S}} + B^*_{\tilde{S}}) \cdot C > 0$ and hence $\tilde{S} \to \hat{S}$ does not contract C.

We choose $0 < \epsilon \ll 1$ a rational number such that $(\tilde{S}, B^*_{\tilde{S}} + \epsilon F)$ is plt and $(C, \mathrm{Diff}_C(B^*_{\tilde{S}} + \epsilon F))$ is globally F-regular. For e sufficiently divisible, we may assume that p does not divide the index of $K_{\tilde{S}} + B^*_{\tilde{S}} + \epsilon F$. Let E be any effective integral divisor over T. We write $E = n_0 C + E'$ where the support of E' does not contain C. We have the following commutative diagram.

$$\begin{tikzcd}
H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}((1-p^e)(K_{\tilde{S}} + B^*_{\tilde{S}} + \epsilon F) - E')) \arrow[r, "\alpha"'] \arrow[d, "\gamma"'] & H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) \arrow[d, "\beta"] \\
H^0(C, \mathcal{O}_C((1-p^e)(K_C + \mathrm{Diff}_C(B^*_{\tilde{S}} + \epsilon F)) - E'|_C)) \arrow[r, "\xi"] & H^0(C, \mathcal{O}_C)
\end{tikzcd}$$

Note that commutativity follows since S is Cartier and hence the different $\mathrm{Diff}_C(B^*_{\tilde{S}} + \epsilon F)$ agrees with the F-different (this follows for example from [Schwede09, Section 7]). By assumption $(C, \mathrm{Diff}_C(B^*_{\tilde{S}} + \epsilon F))$ is globally F-regular, hence ξ is a surjection for $e \gg 0$ sufficiently divisible. Since, for $0 < \epsilon \ll 1$, $-(K_{\tilde{S}} + B^*_{\tilde{S}} + \epsilon F)$ is ample over T, it follows that for any sufficiently divisible positive e, the homomorphism γ is also a surjection by the Kawamata-Viehweg vanishing theorem (2.23). Thus $\beta \circ \alpha$ is a surjection and hence so is α by Nakayama's lemma (since β is given by $A \to A/m$ where $A \cong H^0(\tilde{S}, \mathcal{O}_{\tilde{S}})$ and m is the maximal ideal of the image of C in T).

Finally, since $1 = \mathrm{mult}_C(B^*_{\tilde{S}}) > \mathrm{mult}_C(B_{\tilde{S}})$, then for any $e \gg 0$, the image of α is contained in the image of

$$H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}((1-p^e)(K_{\tilde{S}} + B_{\tilde{S}}) - E)) \to H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}).$$

$\square$

We now consider the case that (S, B^c) is not plt and hence $N \in \{1, 2\}$. The proof is similar but easier than the plt case and so we only indicate the necessary changes to the above arguments. Notice that since $(T, B_T := f_*B)$ is klt, then $B^c_T := f_*B^c \neq B_T$. Let $\nu : \tilde{S} \to S$ be a smooth dlt modification, then by the Kollár-Shokurov connectedness theorem, which follows from the Kawamata-Viehweg vanishing theorem in this case (as in the well known characteristic 0 case), we know that the components in $B^c_{\tilde{S}}$ with coefficient 1 form a connected graph. Let $\Gamma^\bullet$ be the graph corresponding to all exceptional divisors (over T) with coefficient

1. This graph is a chain $\Gamma^\bullet$, with the property that only the two end points of $\Gamma^\bullet$ can be connected to components in $\mathrm{Supp}(B^c_{\tilde{S}})\backslash\Gamma^\bullet$.

Let C be any component in $\Gamma^\bullet$. By the same argument as (3.4), there is a non-exceptional chain $\Gamma'' \subset (\Gamma\backslash C)$ intersecting C such that $\mathrm{Supp}(B^c_{\tilde{S}} - B_{\tilde{S}}) \supset \Gamma''$. Thus, $\Gamma^\bullet \cup \Gamma''$ is a chain, with a non-exceptional component at one end. The other end is exceptional, and hence it is in $\Gamma^\bullet$. We rechoose C to coincide with this end. We define $\Gamma' = (\Gamma'' \cup \Gamma^\bullet)\backslash\{C\}$, which is a chain with the property that $\Gamma' \subset \mathrm{Supp}(B^c_{\tilde{S}} - B_{\tilde{S}})$.

Write $B^c_{\tilde{S}} = B' + B''$, where B' consists of the components contained in the support of Γ'. For all exceptional curves $C_i \neq C$ in Γ', consider the equations

$$\left(K_{\tilde{S}} + B'' + \sum a_i C_i\right) \cdot C_i = 0.$$

Since the intersection matrix $C_i \cdot C_j$ is negative definite, we have a unique solution. We easily see that

$$\mathrm{Supp}\left(B' - \sum a_i C_i\right) = \mathrm{Supp}(\Gamma').$$

Then we first define an effective $\mathbb{Q}$-divisor $B^\sharp_{\tilde{S}}$ as follows: for a component not in Γ', its coefficient in $B^\sharp_{\tilde{S}}$ is the same as in $B^c_{\tilde{S}}$; for a component in Γ', its coefficient is the same as in $\sum a_i C_i$. It easily follows that

$$(K_{\tilde{S}} + B^\sharp_{\tilde{S}}) \cdot D \leq 0,$$

for any exceptional curve D.

Now we define $B^*_{\tilde{S}} = (1-\epsilon)B^c_{\tilde{S}} + \epsilon B^\sharp_{\tilde{S}}$. Since $B^\sharp_{\tilde{S}} \geq 0$, if we choose ϵ sufficiently small, we see that $B^*_{\tilde{S}} \geq B_{\tilde{S}} \vee 0$ and $(\tilde{S}, B^*_{\tilde{S}})$ is plt.

By (3.1), we know that $(K_{\tilde{S}} + B^c_{\tilde{S}})|_C = K_{\mathbb{P}^1} + \sum_{i=1}^j b_i P_i$ and $j = 3$ where $(b_1, b_2, b_3) = (1, \frac{1}{2}, \frac{1}{2})$ or $j = 2$ where $(b_1, b_2) = (1, 1)$. Thus

$$(K_{\tilde{S}} + B^*_{\tilde{S}})|_C = K_{\mathbb{P}^1} + a_1 P_1 + a_2 P_1 + a_3 P_3 \text{ or } (K_{\tilde{S}} + B^*_{\tilde{S}})|_C = K_{\mathbb{P}^1} + a_1 P_1 + a_2 P_1,$$

where in the first case we have $(a_1, a_2) \leq (\frac{1}{2}, \frac{1}{2})$ and $a_3 < 1$ and in the second case $a_1, a_2 < 1$. By (3.9), (S, B) is globally F-regular over T. $\square$

Proposition 3.10. Let k be an algebraically closed field of characteristic $p > 5$, $P_1, P_2, P_3 \in \mathbb{P}^1$ three distinct points and $D = \frac{2}{5}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3$, $D = \frac{1}{3}P_1 + \frac{3}{4}P_2 + \frac{3}{4}P_3$, $D = \frac{1}{2}P_1 + \frac{3}{5}P_2 + \frac{5}{6}P_3$ or $D = \sum_{i=1}^r \frac{d_i - 1}{d_i} P_i$, where $r \leq 2$ or $r = 3$ and

$$(d_1, d_2, d_3) \in \{(2,2,d), (2,3,3), (2,3,4), (2,3,5)\},$$

then $(\mathbb{P}^1, D)$ is globally F-regular.

Proof. By [Watanabe91, 4.2], we may assume that $D_1 = \frac{2}{5}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3$, $D_2 = \frac{1}{3}P_1 + \frac{3}{4}P_2 + \frac{3}{4}P_3$ or $D = \frac{1}{2}P_1 + \frac{3}{5}P_2 + \frac{5}{6}P_3$.

We may assume that $P_1 = 0$, $P_2 = \infty$ and $P_3 = 1$. By Fedder's criterion for pairs, it is enough to check that if $D = c_1 P_1 + c_2 P_2 + c_3 P_3$ for some $e > 0$ and $a_i = \lceil (p^e - 1)c_i \rceil$, then $x^{a_1} y^{a_2} (x+y)^{a_3}$ contains a monomial $x^i y^j$ where $i, j < p^e - 1$. (Note in fact that the pair $(\mathbb{P}^1, D')$ is strongly F-regular where $D' = \sum \frac{a_i}{p^e - 1} P_i$ and $D' \geq D$.)

Case 1: $D_1 = \frac{2}{5}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3$.

Let $e = 1$. Since

$$\begin{aligned}(p-1)D_1' = \lceil (p-1)D_1 \rceil \leq &\left(\frac{2}{5}(p-1) + \frac{4}{5}\right) P_1 \\ &+ \left(\frac{2}{3}(p-1) + \frac{2}{3}\right) P_2 + \left(\frac{5}{6}(p-1) + \frac{5}{6}\right) P_3,\end{aligned}$$

then $a_1 + a_2 + a_3 < 2p - 3$ for any $p > 34$. One also sees that the same inequality works for $p = 31$. In these cases the monomial $x^i y^j$ has non-zero coefficient for $j = \lfloor \frac{a_1+a_2+a_3}{2} \rfloor < p - 1$ and $i = a_1 + a_2 + a_3 - j < p - 1$. Therefore, we only need to check the cases $p \in \{7, 11, 13, 17, 19, 23, 29\}$.

When $p = 7$ and $e = 2$, we have $a_1 = 20$, $a_2 = 32$, $a_3 = 40$, and $x^{20}y^{32}(x+y)^{40}$ has the non-zero term $x^{46}y^{46}$.

When $p = 11$ and $e = 2$, we have $a_1 = 48$, $a_2 = 80$, and $a_3 = 100$, thus $x^{48}y^{80}(x+y)^{100}$ has the non-zero term $x^{115}y^{113}$.

When $p = 13$ and $e = 2$, we have $a_1 = 68$, $a_2 = 112$, $a_3 = 140$, and $x^{68}y^{112}(x+y)^{140}$ has the non-zero term $x^{166}y^{154}$.

When $p = 17$ and $e = 2$, we have $a_1 = 116$, $a_2 = 192$ and $a_3 = 240$, thus $x^{116}y^{192}(x+y)^{240}$ has the non-zero term $x^{287}y^{261}$.

When $p = 19$ and $e = 2$, we have $a_1 = 144$, $a_2 = 240$ and $a_3 = 300$, thus $x^{144}y^{240}(x+y)^{300}$ has the non-zero term $x^{357}y^{327}$.

When $p = 23$ and $e = 2$, we have $a_1 = 212$, $a_2 = 352$ and $a_3 = 440$, thus $x^{212}y^{352}(x+y)^{440}$ has the non-zero term $x^{491}y^{513}$.

When $p = 29$ and $e = 2$, we have $a_1 = 336$, $a_2 = 560$ and $a_3 = 700$, thus $x^{336}y^{560}(x+y)^{700}$ has the non-zero term $x^{775}y^{821}$.

Case 2: $D_2 = \frac{1}{3}P_1 + \frac{3}{4}P_2 + \frac{3}{4}P_3$.

Similarly, if $e = 1$,

$$\begin{aligned}(p-1)D_2' = \lceil (p-1)D_2 \rceil \leq &\left(\frac{1}{3}(p-1) + \frac{2}{3}\right) P_1 \\ &+ \left(\frac{3}{4}(p-1) + \frac{3}{4}\right) P_2 + \left(\frac{3}{4}(p-1) + \frac{3}{4}\right) P_3,\end{aligned}$$

then $a_1 + a_2 + a_3 < 2p - 3$ for any $p > 20$. One also sees that the same inequality works for $p \in \{13, 17, 19\}$. In these cases the monomial $x^i y^j$ has non-zero coefficient for $j = \lfloor \frac{a_1+a_2+a_3}{2} \rfloor$ and $i = a_1 + a_2 + a_3 - j$. Therefore, we only need to check for $p \in \{7, 11\}$.

When $p = 7$ and $e = 2$, we have $a_1 = 16$, $a_2 = 36$ and $a_3 = 36$. Thus $x^{16}y^{36}(x+y)^{36}$ has the non-zero term $x^{44}y^{44}$.

When $p = 11$ and $e = 2$, we have $a_1 = 40$, $a_2 = 90$ and $a_3 = 90$. Thus $x^{40}y^{90}(x+y)^{90}$ has non-zero term $x^{108}y^{112}$.

Case 3: $D_3 = \frac{1}{2}P_1 + \frac{3}{5}P_2 + \frac{5}{6}P_3$.

$$\begin{aligned}(p-1)D_3' = \lceil (p-1)D_3 \rceil \leq &\left(\frac{1}{2}(p-1) + \frac{1}{2}\right) P_1 \\ &+ \left(\frac{3}{5}(p-1) + \frac{4}{5}\right) P_2 + \left(\frac{5}{6}(p-1) + \frac{5}{6}\right) P_3,\end{aligned}$$

then $a_1 + a_2 + a_3 < 2p - 3$ for any $p > 48$. One also sees that the same inequality works for $p \in \{31, 37, 41, 47\}$.

When $p = 7$ and $e = 2$, we have $a_1 = 24$, $a_2 = 29$, $a_3 = 40$, and $x^{24}y^{29}(x+y)^{40}$ has the non-zero term $x^{46}y^{47}$.

When $p = 11$ and $e = 2$, we have $a_1 = 60$, $a_2 = 72$, and $a_3 = 100$, thus $x^{60}y^{72}(x+y)^{100}$ has the non-zero term $x^{116}y^{116}$.

When $p = 13$ and $e = 2$, we have $a_1 = 84$, $a_2 = 101$, $a_3 = 140$, and $x^{84}y^{101}(x+y)^{140}$ has the non-zero term $x^{159}y^{166}$.

When $p = 17$ and $e = 2$, we have $a_1 = 116$, $a_2 = 192$ and $a_3 = 240$, thus $x^{144}y^{173}(x+y)^{240}$ has the non-zero term $x^{282}y^{275}$.

When $p = 19$ and $e = 2$, we have $a_1 = 144$, $a_2 = 240$ and $a_3 = 300$, thus $x^{180}y^{216}(x+y)^{300}$ has the non-zero term $x^{347}y^{349}$.

When $p = 23$ and $e = 2$, we have $a_1 = 212$, $a_2 = 352$ and $a_3 = 440$, thus $x^{246}y^{317}(x+y)^{440}$ has the non-zero term $x^{494}y^{527}$.

When $p = 29$ and $e = 2$, we have $a_1 = 336$, $a_2 = 560$ and $a_3 = 700$, thus $x^{420}y^{504}(x+y)^{700}$ has the non-zero term $x^{797}y^{827}$. $\square$

4. Existence of pl-flips

4.1. **Normality of plt centers.** In characteristic 0, by a result of Kawamata, we know that plt centers (or more generally minimal log canonical centers) are normal. The proof uses the Kawamata-Viehweg vanishing theorem. The analogous result in characteristic $p > 0$ is not known. We prove a related result. The argument illustrates some of the techniques that will be used in the rest of this section.

Proposition 4.1. Let $(X, S+B)$ be a plt pair with $S = \lfloor S+B \rfloor$, $S^n \to S$ the normalization and write $(K_X+S+B)|_{S^n} = K_{S^n} + B_{S^n}$. If (S^n, B_{S^n}) is strongly F-regular, and $(X, S+B)$ has a log resolution $g : Y \to X$ which supports an effective $\mathbb{Q}$-divisor F which is g-exceptional and $-F$ is g-ample, then S is normal.

Proof. We may assume that X is affine. We define

$$K_Y + S' = g^*(K_X + S + B) + \mathbf{A}_Y \qquad \text{and} \qquad \mathbf{A}_{S'} = \mathbf{A}_Y|_{S'},$$

where S' is the birational transform of S.

We choose an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A on X whose support contains $g(\mathrm{Ex}(g))$ and $\mathrm{Supp}(B)$ but not $\mathrm{Supp}(S)$, such that $(S^n, B^*_{S^n} = B_{S^n} + A|_{S^n})$ is strongly F-regular (cf. (2.8)). We then pick Ξ an effective $\mathbb{Q}$-divisor with simple normal crossings support such that

(1) $S' + \{-\mathbf{A}_Y\} \le \Xi \le S' + \{-\mathbf{A}_Y\} + g^*A$,
(2) the index of $K_Y + \Xi$ is not divisible by p, and
(3) $\lceil \mathbf{A}_Y \rceil - (K_Y + \Xi) \sim_{\mathbb{Q}} -g^*(K_X + S + B) - (\Xi - S' - \{-\mathbf{A}_Y\})$ is g-ample.

To construct such a $\mathbb{Q}$-divisor Ξ, we proceed as follows: Let $\Xi' = S' + \{-\mathbf{A}_Y\} + \epsilon F$ for some $0 < \epsilon \ll 1$ so that (1) holds. Since $\lceil \mathbf{A}_Y \rceil - (K_Y + \Xi') \sim_{\mathbb{Q}} -g^*(K_X + S + B) - \epsilon F$ is g-ample for $0 < \epsilon \ll 1$, we may assume that (3) also holds. We may use (2.13) to slightly increase the coefficients of Ξ' to obtain Ξ so that (1-3) are satisfied.

By (2.18), there is a surjection

$$S^0 g_*(\sigma(Y, \Xi) \otimes \mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil)) \to S^0 h_*(\sigma(S', \Xi_{S'}) \otimes \mathcal{O}_{S'}(\lceil \mathbf{A}_{S'} \rceil))$$

where $\Xi_{S'} = (\Xi - S')|_{S'}$ and $h : S' \to S^n$ is the induced morphism. We claim that

$$S^0 h_*(\sigma(S', \Xi_{S'}) \otimes \mathcal{O}_{S'}(\lceil \mathbf{A}_{S'} \rceil)) = \mathcal{O}_{S^n}.$$

Grant this for the time being, then since

$$S^0 g_*(\sigma(Y,\Xi)\otimes \mathcal{O}_Y(\lceil \mathbf{A}_Y\rceil)) \subset g_*\mathcal{O}_Y(\lceil \mathbf{A}_Y\rceil) = \mathcal{O}_X$$

(as $\lceil \mathbf{A}_Y\rceil$ is effective and exceptional), it follows that the homomorphism $\mathcal{O}_X \to \mathcal{O}_{S^n}$ is surjective. This implies that $S = S^n$.

To see the claim, since $\{-\mathbf{A}_Y\} - \lceil \mathbf{A}_Y\rceil = \{-\mathbf{A}_Y\} + \lfloor -\mathbf{A}_Y\rfloor = -\mathbf{A}_Y$, note that the second inequality in (1) implies (after adding $-\lceil \mathbf{A}_Y\rceil$ and restricting to S') that

$$h^*(K_{S^n} + B_{S^n} + A|_{S^n}) \geq K_{S'} + \Xi_{S'} - \lceil \mathbf{A}_{S'}\rceil.$$

Since $B^*_{S^n} = B_{S^n} + A|_{S^n}$, it then follows that

$$(1-p^e)h^*(K_{S^n} + B^*_{S^n}) \leq (1-p^e)(K_{S'} + \Xi_{S'}) + p^e\lceil \mathbf{A}_{S'}\rceil$$

and so

$$F^e_*\mathcal{O}_{S^n}((1-p^e)(K_{S^n} + B^*_{S^n})) \subset h_*F^e_*\mathcal{O}_{S'}((1-p^e)(K_{S'} + \Xi_{S'}) + p^e\lceil \mathbf{A}_{S'}\rceil).$$

Consider the following commutative diagram.

$$\begin{tikzcd}
F^e_*\mathcal{O}_{S^n}((1-p^e)(K_{S^n} + B^*_{S^n})) \arrow[r] \arrow[d] & \mathcal{O}_{S^n} \arrow[d, "\cong"'] \\
h_*F^e_*\mathcal{O}_{S'}((1-p^e)(K_{S'} + \Xi_{S'}) + p^e\lceil \mathbf{A}_{S'}\rceil) \arrow[r] & h_*\mathcal{O}_{S'}(\lceil \mathbf{A}_{S'}\rceil)
\end{tikzcd}$$

Since $(S^n, B^*_{S^n} = B_{S^n} + A|_{S^n})$ is strongly F-regular, we have that $\sigma(S^n, B^*_{S^n}) = \mathcal{O}_{S^n}$ (cf. (2.19)) and hence the top horizontal arrow is surjective. It then follows that the bottom horizontal arrow is surjective so that the claim follows. $\square$

4.2. **b-divisors.** Assume that $f : (X, S+B) \to Z$ is a threefold pl-flipping contraction so that

(1) X is a normal threefold,
(2) $f : X \to Z$ is a small projective birational contraction with $\rho(X/Z) = 1$,
(3) $(X, S+B)$ is plt, and
(4) $-S$ and $-(K_X + S + B)$ are ample over Z.

We will assume that Z is affine. In particular any divisor which is ample over Z is in fact ample and if $\mathcal{F}$ is a coherent sheaf on X, then we may identify $f_*\mathcal{F}$ and $H^0(X, \mathcal{F})$.

Let $A \geq 0$ be an auxiliary $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X whose support contains $\mathrm{Supp}(B)$ and $\mathrm{Ex}(f)$ but not $\mathrm{Supp}(S)$, such that $(X, S+B+A)$ is plt. Let $S^n \to S$ be the normalization of S. Let $B_{S^n} = \mathrm{Diff}_{S^n}(X, B)$ and $B^*_{S^n} = \mathrm{Diff}_{S^n}(X, B+A)$, so that

$$(K_X + S + B)|_{S^n} = K_{S^n} + B_{S^n} \qquad \text{and} \qquad (K_X + S + B + A)|_{S^n} = K_{S^n} + B^*_{S^n}.$$

Let $g : Y \to X$ be a log resolution of $(X, S+A)$ (cf. (2.1)) and $h = f\circ g : Y \to Z$ the composition. For any divisor D on X, let $D' = g^{-1}_*D$ be the strict transform on Y. We assume that the restriction $g_{S'} : S' \to S^n$ factors through the terminal model $\bar{S}$ of (S^n, B_{S^n}) given by (2.22), i.e., $g_{S'} = \nu\circ\mu$ for morphisms $\mu : S' \to \bar{S}$ and $\nu : \bar{S} \to S^n$. We write

$$K_Y + S' = g^*(K_X + S + B) + \mathbf{A}_Y, \qquad K_{S'} = g^*_{S'}(K_{S^n} + B_{S^n}) + \mathbf{A}_{S'}$$

where $\mathbf{A}_{S'} = (\mathbf{A}_Y)|_{S'}$. Here, by abuse of notation, we use $\mathbf{A}$ to mean the discrepancy b-divisors of $(X, S+B)$ and of (S^n, B_{S^n}) (see [Corti07, 2.3.12(3)]). As the restriction of the discrepancy b-divisor of $(X, S+B)$ to the models of S^n gives the discrepancy b-divisor of (S^n, B_{S^n}), this should not cause any confusion. After possibly blowing up further, we may assume that g is given by a sequence of blow ups along smooth centers and thus there exists $F \geq 0$ a g-exceptional $\mathbb{Q}$-divisor on Y such that $-F$ is relatively g-ample. As in the proof of (4.1), we can pick Ξ such that

(1) $S' + \{-\mathbf{A}_Y\} \leq \Xi \leq S' + \{-\mathbf{A}_Y\} + g^*A$,
(2) the index of $K_Y + \Xi$ is not divisible by p, and
(3) $\lceil \mathbf{A}_Y \rceil - (K_Y + \Xi)$ is ample.

Lemma 4.2. *Let M be a relatively free $\mathbf{A}_Y$-saturated divisor on Y/Z (cf. Subsection (2.1)). Then there exists $M_Y \in |M|$ such that $M_{S'} := M_Y|_{S'}$ descends to $\bar{S}$ i.e. $M_{S'} = \mu^* M_{\bar{S}}$ where $M_{\bar{S}} = \mu_* M_{S'}$.*

Proof. Since Z is affine, $|M|$ is base point free. Since $f \circ g$ and $(f \circ g)|_{S'}$ are birational, $|M|$ induces a birational morphism $\psi : Y \to \mathbb{P}^N_Z$, whose restriction to S' is also birational. Note that there is a big open subset $U \subset \psi(S')$ (the complement of finitely many points) such that $\psi|_{S'}$ is finite over U. Let D be a general hyperplane divisor on $\psi(Y)$ and $M_Y \in |M|$ the corresponding divisor. We note that as we are in characteristic $p > 0$, M_Y is integral but not necessarily smooth. Since $D \cap \psi(S')$ is contained in U, we have that $M_{S'} = M_Y|_{S'} \to \psi(M_{S'})$ is finite and for any $x \in M_{S'}$, there exists $M'_{S'} \sim M_{S'}$ (obtained by considering another general hyperplane $\psi(x) \in D' \subset \psi(Y)$) such that $x \in M_{S'}$ and $M_{S'} \wedge M'_{S'} = 0$.

By (2.2), S' is an F-pure center for (Y, Ξ). If we let $L = M_Y + \lceil \mathbf{A}_Y \rceil$, then since M_Y is nef, $L - (K_Y + \Xi)$ is ample (cf. (3) above). Thus, by (2.3), we have that

$$S^0(Y, \sigma(Y, \Xi) \otimes \mathcal{O}_Y(L)) \to S^0(S', \sigma(S', \Xi_{S'}) \otimes \mathcal{O}_{S'}(L|_{S'}))$$

is surjective, where $\Xi_{S'} = \phi^{\Xi}_{S'} = (\Xi - S')|_{S'}$.

$\Xi_{S'}$ has simple normal crossings on S'. Since M is relatively $\mathbf{A}_Y$-saturated, for any $s \in H^0(Y, \mathcal{O}_Y(L))$, we have that s vanishes along $\lceil \mathbf{A}_Y \rceil$. It follows that any section $t \in S^0(S', \sigma(S', \Xi_{S'}) \otimes \mathcal{O}_{S'}(L|_{S'}))$ vanishes along $(\lceil \mathbf{A}_Y \rceil)|_{S'} = \lceil \mathbf{A}_{S'} \rceil$.

Claim 4.3. We may assume that $M_{S'}$ is smooth on a neighborhood of the support of $\lceil \mathbf{A}_{S'} \rceil \cap M_{S'}$.

Proof. Suppose that $M_{S'}$ is singular at some point x of the support of $\lceil \mathbf{A}_{S'} \rceil$. Let $m = \text{mult}_x M_{S'} \geq 2$. By choosing $M_{S'}$ sufficiently generally, we may assume that there exists $M'_{S'} \sim M_{S'}$ with $m = \text{mult}_x M'_{S'}$ and $M_{S'} \wedge M'_{S'} = 0$. Let $\pi : S'' \to S'$ be the blow up of S' at x with an exceptional divisor E. It follows easily that $\pi^* M_{S'} - mE$ is nef and hence the Seshadri constant of $M_{S'}$ at x satisfies $\epsilon(x, M_{S'}) \geq 2$. Since $L|_{S'} - (K_{S'} + \Xi_{S'}) - M_{S'}$ is ample, by [MS12], we have that the Frobenius-Seshadri constant satisfies

$$\epsilon_F(x, L|_{S'} - (K_{S'} + \Xi_{S'})) > \epsilon_F(x, M_{S'}) \geq \frac{1}{2}\epsilon(x, M_{S'}) \geq 1.$$

Since $(S', \Xi_{S'})$ is strongly F-regular, following the proof of [MS12, 3.1], we have that the image of

$$H^0(S', F^e_* \mathcal{O}_{S'}(p^e L|_{S'} + (1 - p^e)(K_{S'} + \Xi_{S'}))) \to H^0(S', \mathcal{O}_{S'}(L|_{S'}))$$

contains a section σ not vanishing at x. But any such section is in $S^0(S', \sigma(S', \Xi_{S'}) \otimes \mathcal{O}_{S'}(L|_{S'}))$. As we have observed above, any such σ must vanish along $\lceil \mathbf{A}_{S'} \rceil$. This is impossible and so $M_{S'}$ is smooth on a neighborhood of Supp$(\lceil \mathbf{A}_{S'} \rceil)$. ☐

After further blowing up along centers which do not intersect with the support of $\lceil \mathbf{A}_{S'} \rceil$, we may assume that $M_{S'} = M_1 + M_2$ has simple normal crossings where M_1 denotes the union of the Z-horizontal components of $M_{S'}$ and M_2 the Z-vertical components. Since $D \cap \psi(S')$ is irreducible, M_1 is irreducible and hence smooth.

By (2.2), M_1 is an F-pure center for $(Y, \Xi + M_Y)$. We let $\Gamma = (\Xi_{S'} + M_2)|_{M_1}$. Arguing as above, we have a surjective map

$$S^0(Y, \sigma(Y, \Xi + M_Y) \otimes \mathcal{O}_Y(L)) \to S^0(M_1, \sigma(M_1, \Gamma) \otimes \mathcal{O}_{M_1}(L|_{M_1})).$$

Notice that on a neighborhood of $\lceil \mathbf{A}_{S'} \rceil$, Γ is equal to $(\Xi - S')|_{M_1}$. We have that

$$S^0(M_1, \sigma(M_1, \Gamma) \otimes \mathcal{O}_{M_1}(L|_{M_1})) \supset S^0(M_1, \sigma(M_1, \{\Gamma\}) \otimes \mathcal{O}_{M_1}(L|_{M_1} - \lfloor \Gamma \rfloor))$$

(see for example the proof of (2.5)). As M_1 is affine, by (2.19), it then follows that $\mathcal{O}_{M_1}(L|_{M_1} - \lfloor \Gamma \rfloor)$ is globally generated by $S^0(M_1, \sigma(M_1, \{\Gamma\}) \otimes \mathcal{O}_{M_1}(L|_{M_1} - \lfloor \Gamma \rfloor))$ on a neighborhood of $\lceil \mathbf{A}_{S'} \rceil$. Since

$$S^0(Y, \sigma(Y, \Xi + M_Y) \otimes \mathcal{O}_Y(L)) \subset S^0(Y, \sigma(Y, \Xi) \otimes \mathcal{O}_Y(L)),$$

any section

$$s \in S^0(M_1, \sigma(M_1, \Gamma) \otimes \mathcal{O}_{M_1}(L|_{M_1}))$$

lifts to a section in $S^0(Y, \sigma(Y, \Xi) \otimes \mathcal{O}_Y(L))$ and hence s vanishes along $(\lceil \mathbf{A}_{S'} \rceil)|_{M_1}$. However, if P is a point contained in the support of $(\lceil \mathbf{A}_{S'} \rceil)|_{M_1}$, then since we may assume that the coefficients of $\Xi - S' - \{-\mathbf{A}_Y\}$ are sufficiently small, we have

$$\mathrm{mult}_P(\lfloor \Gamma \rfloor) = \mathrm{mult}_P(\lfloor (\Xi - S')|_{M_1} \rfloor) < \mathrm{mult}_P((\lceil \mathbf{A}_{S'} \rceil)|_{M_1}),$$

which is a contradiction. This implies that $(\lceil \mathbf{A}_{S'} \rceil)|_{M_{S'}} = (\lceil \mathbf{A}_{S'} \rceil)|_{M_1} = 0$. Since the support of $\lceil \mathbf{A}_{S'} \rceil$ is the $S' \to \bar{S}$ exceptional locus, it follows that $M_{S'}$ descends to $\bar{S}$. $\square$

We now define certain b-divisors following [Corti07, 2.4.1]. We fix an effective Cartier divisor $Q \sim k(K_X + S + B)$ on X where k is the Cartier index of $K_X + S + B$, such that the support of Q does not contain S. Let

$$\mathbf{N}_i = \mathbf{Mob}(iQ), \qquad \mathbf{M}_i = \mathbf{N}_i|_{S^n}, \text{ and } \qquad \mathbf{D}_i = \frac{1}{i}\mathbf{M}_i.$$

Note that by (2.1), for any $i > 0$ there exists $g : Y \to X$ (depending on i) such that $\mathbf{N}_{i,Y}$ is free and hence $\mathbf{N}_i$ descends to Y. We may assume that $Y \to X$ is a log resolution of Q and $|iQ|$ so that $\mathbf{N}_{i,Y}$ has simple normal crossings.

Lemma 4.4. *With the above notation $\mathbf{M}_i$ descends to $\bar{S}$.*

Proof. For any integer $i > 0$, we can choose a log resolution $g : Y \to X$ of the pair $(X, S + B)$ and of the linear system $|iQ|$. Thus, we can write $g^*(iQ) = N_i + F_i$, where $N_i \sim \mathbf{N}_{i,Y}$ is a free divisor. The divisor N_i is relatively $\lceil \mathbf{A}_Y \rceil$-saturated, since the inclusions

$$(f \circ g)_* \mathcal{O}_Y(N_i) \to (f \circ g)_* \mathcal{O}_Y(N_i + \lceil \mathbf{A}_Y \rceil) \to (f \circ g)_* \mathcal{O}_Y(g^*(iQ) + \lceil \mathbf{A}_Y \rceil)$$

are isomorphisms. The result now follows from (4.2). $\square$

In what follows, we will fix a model $g_0 : Y_0 \to X$ and the birational transform $S_0 \subset Y_0$ such that S_0 admits a morphism $\mu_0 : S_0 \to \bar{S}$. We also assume that the models $g : Y \to X$, factor through Y_0 and that $\rho : Y \to Y_0$ is an isomorphism over $X\backslash \mathrm{Supp}(A)$ (cf. (2.1)).

$$\begin{tikzcd}
 & S' \arrow[r] \arrow[d, "\rho_{S'}"] & Y \arrow[d, "\rho"] \arrow[ddr, "g"] & & \\
 & S_0 \arrow[r] \arrow[dl, "\mu_0"'] \arrow[d] & Y_0 \arrow[dr, "g_0"] & & \\
\bar{S} \arrow[r, "\nu"] & S^n \arrow[rr] & & X \arrow[r, "f"] & Z
\end{tikzcd}$$

Lemma 4.5. *For any effective $\mathbb{Q}$-divisor G on $\bar{S}$ we may fix an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A^* on X and rational numbers $0 < \epsilon' \ll \epsilon$, and for any Y as above we may choose $F \geq 0$ on Y such that*

(1) *the support of A^* contains the supports of B, Q, $\mathrm{Ex}(f)$ and the divisor E defined in* (2.13) *but not S,*
(2) *$-g^*(K_X + S + B) - \epsilon F + \rho^* A'$ is ample for any $-\epsilon' g_0^* A^* \leq A' \leq \epsilon' g_0^* A^*$,*
(3) *$-(g^*(K_X + S + B) + \epsilon F)|_{S'} + \mu^* A'$ is ample for any $-\epsilon' \nu^*(A^*|_{S^n}) \leq A' \leq \epsilon' \nu^*(A^*|_{S^n})$, and*
(4) *if the support of G contains $\nu^{-1}(\mathrm{Ex}(f) \cup g_0(F_0))$, then $\epsilon F|_{S'} \leq \mu^* G$.*

Proof. (1) is immediate. To see (2), notice that we may assume that there is an effective g_0-exceptional divisor F_0 on Y_0 such that $-F_0$ is ample over X. It then follows that $-g_0^*(K_X + S + B) - \epsilon F_0$ is ample for $0 < \epsilon \ll 1$ and it is easy to see that $-g_0^*(K_X + S + B) - \epsilon F_0 + A'$ is ample for any $-2\epsilon' g_0^* A^* \leq A' \leq 2\epsilon' g_0^* A^*$ and $0 < \epsilon' \ll \epsilon$. Let F_1 be a ρ-exceptional divisor on Y such that $-F_1$ is ample over Y_0. (2) now follows (by an easy compactness argument), letting $F = \rho^* F_0 + \lambda F_1$ where $0 < \lambda \ll 1$. The proof of (3) is similar. We also easily see that if ϵ and λ are sufficiently small, then

$$\epsilon F|_{S'} = \epsilon \rho_{S'}^*(F_0|_{S_0}) + \epsilon \lambda F_1|_{S'} \leq \frac{1}{2}\mu^* G + \frac{1}{2}\mu^* G = \mu^* G.$$

Note that the choice of λ depends on Y, but the choice of ϵ and ϵ' does not. $\square$

Note that the support of $g^* A^*$ contains the g-exceptional locus. For any $i, j > 0$, we define

$$L_{ij} = \left\lceil \frac{j}{i} \mathbf{N}_{i,Y} + \mathbf{A}_Y \right\rceil.$$

Lemma 4.6. *For any $0 < \epsilon' \ll \epsilon \ll 1$, we can pick a $\mathbb{Q}$-divisor Ψ on Y (depending on i, j and Y), with the following properties:*

(1) *$\Psi'_\epsilon \leq \Psi \leq \Psi'_\epsilon + \epsilon' g^* A^*$, where $\Psi'_\epsilon = \{-\frac{j}{i}\mathbf{N}_{i,Y} - \mathbf{A}_Y\} + S' + \epsilon F$,*
(2) *the index of $K_Y + \Psi$ is not divisible by p,*
(3) *$L_{ij} - (K_Y + \Psi) + \rho^* A'$ is ample for any $-\epsilon' g_0^* A^* \leq A' \leq \epsilon' g_0^* A^*$, and*
(4) *$(L_{ij} - (K_Y + \Psi))|_{S'} - \mu^* M$ is ample for any $\mathbb{Q}$-divisor M on $\bar{S}$ such that $-\epsilon' \nu^*(A^*|_{S^n}) \leq M - \frac{j}{i}\mathbf{M}_{i,\bar{S}} \leq \epsilon' \nu^*(A^*|_{S^n})$.*

Proof. To construct such a Ψ, we proceed as follows: By (2.13), for any $0 < \epsilon' \ll 1$, we may pick a $\mathbb{Q}$-divisor $\Psi'_\epsilon \leq \Psi \leq \Psi'_\epsilon + \epsilon' g^* A^*$ such that (1) and (2) hold. Since

$$\begin{aligned} L_{ij} - (K_Y + \Psi) &= \left\lceil \frac{j}{i}\mathbf{N}_{i,Y} + \mathbf{A}_Y \right\rceil - (K_Y + \Psi) \\ &= \frac{j}{i}\mathbf{N}_{i,Y} + \mathbf{A}_Y - (\Psi - \Psi'_\epsilon) - K_Y - S' - \epsilon F \\ &= \frac{j}{i}\mathbf{N}_{i,Y} - g^*(K_X + S + B) - \epsilon F - (\Psi - \Psi'_\epsilon), \end{aligned}$$

$\mathbf{N}_{i,Y}$ is nef and $0 < \epsilon' \ll 1$, (3) follows from (2) of (4.5).

Since $\mathbf{M}_{i,S'} = \mu^* \mathbf{M}_{i,\bar{S}}$, (4) easily follows from (3) of (4.5) and the equality

$$(L_{ij} - (K_Y + \Psi))|_{S'} - \mu^* M = -(g^*(K_X + S + B) + \epsilon F + \Psi - \Psi'_\epsilon)|_{S'} + \mu^* \left(\frac{j}{i}\mathbf{M}_{i,\bar{S}} - M \right).$$

□

Lemma 4.7. *The homomorphism*

$$S^0(Y, \sigma(Y, \Psi) \otimes \mathcal{O}_Y(L_{ij})) \to S^0(S', \sigma(S', \Psi_{S'}) \otimes \mathcal{O}_{S'}(L_{ij}|_{S'}))$$

is surjective where $\Psi_{S'} = (\Psi - S')|_{S'}$.

Proof. Since $L_{ij} - (K_Y + \Psi)$ is ample (cf. (3) of (4.6)), the surjectivity follows from (2.18) and (2.2). □

4.3. **Rationality of D.** Let $\mathbf{D} = \lim_i \mathbf{D}_i$. Our arguments follow the ideas explained in [Corti07]. However, there are added technical difficulties and the arguments are a little more delicate. In particular, it becomes necessary to extend sections from a curve to the ambient threefold (instead of just extending sections from a surface).

Lemma 4.8. *The* $\mathbb{R}$*-divisor* $\mathbf{D}_{\bar{S}}$ *is semi-ample over* $f(S)$.

Proof. See (2.21) and the argument in [Corti07, 2.4.11]. □

Let $h : \bar{S} \to f(S)$ be the induced morphism. Let $V \subset \mathrm{Div}(\bar{S}) \otimes \mathbb{R}$ be the smallest linear subspace defined over $\mathbb{Q}$ containing $\mathbf{D}_{\bar{S}}$. Since the nef cone of $\bar{S}$ over $f(S)$ is finitely generated, we may pick nef divisors $M_i \in V$ such that $\mathbf{D}_{\bar{S}}$ is contained in the convex cone generated by the M_i. Note that if Σ is any h-exceptional curve, then $M_i \cdot \Sigma = 0$ iff $\mathbf{D}_{\bar{S}} \cdot \Sigma = 0$. By (2.21), after replacing each M_i by a positive multiple, we may assume that each $|M_i|$ defines a birational morphism to a normal surface over $f(S)$, say $\alpha_i : \bar{S} \to S^i$. Notice that the exceptional set of α_i corresponds to the set of h-exceptional curves intersecting M_i trivially, and so this set is independent of i. Therefore the morphism α_i is independent of i and we denote it by $a : \bar{S} \to S^+$.

By Diophantine approximation, we may pick $j > 0$ and $M \in V$ such that (cf. [Corti07, 2.4.12]) $M = \sum a_i M_i$ where $a_i \in \mathbb{N}$, and $\|M - j\mathbf{D}_{\bar{S}}\| \leq \frac{\epsilon'}{2}$ (here $\|\cdot\|$ denotes the sup norm). It follows that M is relatively base point free and the map defined by $|M|$ is also given by $a : \bar{S} \to S^+$.

Note that if Σ is any proper curve contained in $\mathrm{Ex}(h)$, then $\Sigma \cdot \mathbf{D}_{\bar{S}} = 0$ if and only if $\Sigma \cdot M = 0$. Thus $\mathbf{D}$ descends to S^+. We can assume that C is a smooth general curve such that $C \sim M$. To see this, note that there is a big open subset U of S^+ (the complement of finitely many points) such that U is smooth and a is an isomorphism over U. C is then isomorphic to a general hyperplane C^+ of S^+, which is contained in U. We may also assume that $\Psi_{S'} + C'$ has simple normal crossings, where C' is the strict transform of C on S'.

Lemma 4.9. *Let $\Theta = \Psi_{S'} + C'$. If j is as above and $i \gg 0$, then*

$$S^0(S', \sigma(S', \Theta) \otimes \mathcal{O}_{S'}(L_{ij})) \to S^0(C', \sigma(C', \Theta_{C'}) \otimes \mathcal{O}_{C'}(L_{ij}))$$

is surjective, where $\Theta_{C'} = (\Theta - C')|_{C'}$.

Proof. Recall that $\Psi_{S'} + C'$ has simple normal crossings and since $-\epsilon'\nu^* A^*|_{S^n} \leq \frac{j}{i}\mathbf{M}_{i,\bar{S}} - M \leq \epsilon'\nu^* A^*|_{S^n}$ (for $i \gg 0$) it follows that $L_{ij}|_{S'} - (K_{S'} + \Theta)$ is ample (cf. (4) of (4.6)). The lemma now follows from (2.18). □

Lemma 4.10. *If j is as above and $i \gg 0$ is divisible by j, then $\lceil \frac{j}{i}\mathbf{M}_{i,\bar{S}} + \mathbf{A}_{\bar{S}} \rceil - \mathbf{M}_{j,\bar{S}}$ is a-exceptional.*

Proof. Combining (4.7) and (4.9) (and the fact that $\Theta \geq \Psi_{S'}$) it follows that the image of

$$H^0(Y, \mathcal{O}_Y(L_{ij})) \to H^0(C', \mathcal{O}_{C'}(L_{ij}))$$

contains the subspace $S^0(C', \sigma(C', \Theta_{C'}) \otimes \mathcal{O}_{C'}(L_{ij}))$ which generates $\mathcal{O}_{C'}(L_{ij})$ (cf. (2.19)).

On the other hand, since $\lceil \frac{j}{i}\mathbf{N}_{i,Y} + \mathbf{A}_Y \rceil \leq jg^*Q + \lceil \mathbf{A}_Y \rceil$ and $\lceil \mathbf{A}_Y \rceil$ is effective and g-exceptional, we have an isomorphism

$$(f \circ g)_* \mathcal{O}_Y(\mathbf{N}_{j,Y}) \to (f \circ g)_* \mathcal{O}_Y \left(\left\lceil \frac{j}{i}\mathbf{N}_{i,Y} + \mathbf{A}_Y \right\rceil \right),$$

which is induced by adding the effective divisor $\lceil \frac{j}{i}\mathbf{N}_{i,Y} + \mathbf{A}_Y \rceil - \mathbf{N}_{j,Y}$. Therefore, we conclude that the sections in the image of

$$H^0(Y, \mathcal{O}_Y(L_{ij})) \to H^0(C', \mathcal{O}_{C'}(L_{ij}))$$

vanish along $(\lceil \frac{j}{i}\mathbf{N}_{i,Y} + \mathbf{A}_Y \rceil - \mathbf{N}_{j,Y})|_{C'}$. Since, as we have seen above, they also generate $\mathcal{O}_{C'}(L_{ij})$, we must have $(\lceil \frac{j}{i}\mathbf{N}_{i,Y} + \mathbf{A}_Y \rceil - \mathbf{N}_{j,Y})|_{C'} = 0$. Therefore $(\lceil \frac{j}{i}\mathbf{M}_{i,S'} + \mathbf{A}_{S'} \rceil - \mathbf{M}_{j,S'})|_{C'} = 0$ and the lemma follows. □

Corollary 4.11. $\mathbf{D}_{\bar{S}}$ is rational and $a_*\mathbf{D}_{\bar{S}} = a_*\mathbf{D}_{j,\bar{S}}$ for some $j > 0$.

Proof. Suppose that $a_*\mathbf{D}_{\bar{S}}$ is not rational, then arguing as in [Corti07, 2.4.12], we may assume that the divisor $a_*(j\mathbf{D}_{\bar{S}}) - a_*M$ is not effective. Since $\lceil \mathbf{A}_{\bar{S}} \rceil = 0$, we may pick $\delta > 0$ such that the coefficients of $\mathbf{A}_{\bar{S}}$ are greater than $\delta - 1$. We may assume that $\delta > \epsilon'/2$ and hence for $i \gg 0$, we have $\|M - \frac{j}{i}\mathbf{M}_{i,\bar{S}}\| < \delta$. Since M is integral, by an easy computation, one sees that $M \leq \lceil \frac{j}{i}\mathbf{M}_{i,\bar{S}} + \mathbf{A}_{\bar{S}} \rceil$. It follows that

$$a_*M \leq a_* \left(\left\lceil \frac{j}{i}\mathbf{M}_{i,\bar{S}} + \mathbf{A}_{\bar{S}} \right\rceil \right) = a_*\mathbf{M}_{j,\bar{S}} \leq a_*(j\mathbf{D}_{\bar{S}}),$$

where the second equality follows from (4.10) and the last inequality follows easily from the definition of $\mathbf{D}_{\bar{S}}$ (cf. [Corti07, 2.3.47]). Thus $a_*\mathbf{D}_{\bar{S}}$ is rational and hence we have that $a_*(j\mathbf{D}_{\bar{S}}) = a_*\mathbf{M}_{j,\bar{S}}$. Since $\mathbf{D}_{\bar{S}}$ descend to S^+, we have that $\mathbf{D}_{\bar{S}}$ is rational. □

4.4. **Existence of pl-flips.** Up to this point, our arguments apply to any three dimensional pl-flip. However, in this subsection, we will require that the characteristic of the ground field is larger than 5 and the coefficients are in the standard set $\{\frac{n-1}{n}|n \in \mathbb{N}\}$.

Theorem 4.12. *Let $f : (X, S+B) \to Z$ be a pl-flipping contraction of a projective threefold defined over an algebraically closed field of characteristic $p > 5$ such that the coefficients of B belong to the standard set $\{1 - \frac{1}{n}|n \in \mathbb{N}\}$. Then the flip exists.*

Proof. Since, by adjunction,

$$K_{S^n} + B_{S^n} = (K_X + S + B)|_{S^n},$$

we have that the coefficients of (S^n, B_{S^n}) also lie in $\{1 - \frac{1}{n}|n \in \mathbb{N}\}$ (cf. [Kollár13, 3.36]) and so, by (3.1), (S^n, B_{S^n}) is strongly F-regular. By (4.1), we have that S is normal.

By [Corti07, 2.3.6], it suffices to show that the restricted algebra $R_{S/Z}(k(K_X + S + B))$ is finitely generated for some $k > 0$. Recall that the restricted algebra is a graded $\mathcal{O}_Z$-algebra whose degree m piece corresponds to the image of the restriction homomorphism

$$f_*\mathcal{O}_X(mk(K_X + S + B)) \to f_*\mathcal{O}_S(mk(K_S + B_S)),$$

or equivalently to the image of the restriction homomorphism

$$(f \circ g)_*\mathcal{O}_Y(mk(K_Y + S' + B_Y)) \to (f \circ g)_*\mathcal{O}_{S'}(mk(K_{S'} + B_{S'})),$$

where $B_Y = (-\mathbf{A}_Y)_{\geq 0}$ and $B_{S'} = (-\mathbf{A}_{S'})_{\geq 0}$. Recall that $Q \sim k(K_X + S + B)$ is Cartier. Replacing k by a multiple, we may assume that $a_*\mathbf{D}_{\bar{S}} = a_*\mathbf{D}_{j,\bar{S}}$ for all $j > 0$ by (4.11).

We have the following commutative diagram.

$$\begin{tikzcd}
S' \arrow[r, "\mu"] \arrow[dr] & \bar{S} \arrow[r, "a"] \arrow[d, "\nu"] & S^+ \arrow[d] \\
 & S \arrow[r] & f(S)
\end{tikzcd}$$

Pick a rational number $\delta > 0$ such that

(1) $(X, S + B + \delta A^*)$ is plt,
(2) $\nu : \bar{S} \to S$ is also a terminalization of $(S, B_S + \delta A^*|_S)$.

Lemma 4.13. *Denote by $\Psi_{S'} = (\Psi - S')|_{S'}$. Replacing j by a multiple, for any $i \gg 0$ divisible by j, we have*

$$S^0(S', \sigma(S', \Psi_{S'}) \otimes \mathcal{O}_{S'}(L_{ij}|_{S'})) = H^0(S^+, \mathcal{O}_{S^+}(j\mathbf{D}_{S^+})).$$

Proof. Possibly replacing j by a multiple, we can assume $|j\mathbf{D}_{\bar{S}}|$ induces the morphism to the normal surface S^+.

Let $G_\epsilon = (\Psi - \Psi'_\epsilon + \epsilon F)|_{S'}$ (see (4.6) for the definitions of these divisors) and

$$\Psi_{\bar{S}} := \mu_*\Psi_{S'} = B_{\bar{S}} + \mu_*G_\epsilon + j(\mathbf{D}_{\bar{S}} - \mathbf{D}_{i,\bar{S}})$$

for sufficiently large i and $\Psi_{S^+} = a_*\Psi_{\bar{S}}$. (We have used the fact that $B_{\bar{S}} = \{B_{\bar{S}}\} = \{-\mathbf{A}_{\bar{S}}\}$, that $j\mathbf{D}_{S'} = \mu^*(j\mathbf{D}_{\bar{S}})$ is integral, and that $\{-\frac{j}{i}\mathbf{M}_{i,S'}\} = j\mathbf{D}_{S'} - \frac{j}{i}\mathbf{M}_{i,S'}$ for $i \gg 0$.)

Since (S, B_S) is globally F-regular over Z (cf. (3.1)), so is $(\bar{S}, B_{\bar{S}})$ (cf. (2.11)). We have that $\|\Psi - \Psi'_\epsilon\| \ll 1$ (as $0 < \epsilon' \ll \epsilon \ll 1$) and $\|\{-\frac{j}{i}\mathbf{M}_{i,\bar{S}}\}\| \ll 1$ (as $i \gg 0$ and $j\mathbf{D}_{\bar{S}} = \lim \frac{j}{i} M_{i,\bar{S}}$ is integral). By (2.8), it follows that $(\bar{S}, \Psi_{\bar{S}})$ is globally F-regular, and so (S^+, Ψ_{S^+}) has strongly F-regular singularities (see (2.12)). Let E_{S^+} be an effective $\mathbb{Q}$-divisor on S^+ whose support contains the image of the locus where $S' \to S^+$ is not an isomorphism (this is possible as we have assumed that $Y \to Y_0$ is an isomorphism over $X \backslash \mathrm{Supp}(A)$), and the birational transform of $\mathrm{Supp}(B_S + A^*|_S)$. We may assume that

(1) $(\bar{S}, B^\sharp_{\bar{S}} = B_{\bar{S}} + 2a^* E_{S^+})$ is globally F-regular over S^+ and
(2) $G_\epsilon + j(\mathbf{D}_{S'} - \mathbf{D}_{i,S'}) \le \mu^* a^* E_{S^+}$ (for fixed j and $i \gg 0$).

To see (2), note that the support of $G_\epsilon + j(\mathbf{D}_{S'} - \mathbf{D}_{i,S'})$ is contained in the support of $\mu^* a^* E_{S^+}$ and we have that $\|j(\mathbf{D}_{S'} - \mathbf{D}_{i,S'})\| \ll 1$ for $i \gg 0$. From our choice of ϵ and F, A^* we can also assume that $\epsilon F|_{S'} \le \frac{1}{3}\mu^* a^* E_{S^+}$ (see (4.5.4)) and $(\Psi - \Psi'_\epsilon)|_{S'} \le g^* A^*|_{S'} \le \frac{1}{3}\mu^* a^* E_{S^+}$.

Claim 4.14. We have the following inclusion

$$S^0(S', \sigma(S', \Psi_{S'}) \otimes \mathcal{O}_{S'}(L_{ij}|_{S'})) \supset S^0(\bar{S}, \sigma(\bar{S}, B^\sharp_{\bar{S}}) \otimes \mathcal{O}_{\bar{S}}(j\mathbf{D}_{\bar{S}})).$$

Proof. Note that $\mu_* \Psi_{S'} = \Psi_{\bar{S}}$ and

$$\mu_*(L_{ij}|_{S'}) = \mu_* \left(\left\lceil \frac{j}{i}\mathbf{M}_{i,S'} + \mathbf{A}_{S'} \right\rceil \right) = j\mathbf{D}_{\bar{S}}$$

for any $i \gg 0$ sufficiently divisible (since $\mathbf{A}_{S'}$ is μ-exceptional, $\lceil \mathbf{A}_{\bar{S}} \rceil = 0$ and $\lim \frac{j}{i}\mathbf{M}_{i,\bar{S}} = j\mathbf{D}_{\bar{S}}$ which is integral). Thus there is a commutative diagram.

$$\begin{tikzcd}
\mu_* F^e_* \mathcal{O}_{S'}((1-p^e)(K_{S'} + \Psi_{S'}) + p^e L_{ij}|_{S'}) \arrow[r] \arrow[d, "\cong"'] & \mu_* \mathcal{O}_{S'}(L_{ij}|_{S'}) \arrow[d, "\cong"'] \\
F^e_* \mathcal{O}_{\bar{S}}((1-p^e)(K_{\bar{S}} + \Psi_{\bar{S}}) + p^e j\mathbf{D}_{\bar{S}}) \arrow[r] & \mathcal{O}_{\bar{S}}(j\mathbf{D}_{\bar{S}})
\end{tikzcd}$$

As in the argument of (2.15) for $e \gg 0$, the image of the map on global sections induced by the top arrow is $S^0(S', \sigma(S', \Psi_{S'}) \otimes \mathcal{O}_{S'}(L_{ij}|_{S'}))$, thus it suffices to show that for $i \gg 0$, the bottom left hand corner contains

$$F^e_* \mathcal{O}_{\bar{S}}((1-p^e)(K_{\bar{S}} + B^\sharp_{\bar{S}}) + p^e j\mathbf{D}_{\bar{S}}).$$

Let

$$\Psi^*_{\bar{S}} := B_{\bar{S}} + \mu_* G_\epsilon + j(\mathbf{D}_{\bar{S}} - \mathbf{D}_{i,\bar{S}}) + a^* E_{S^+} \le B^\sharp_{\bar{S}}$$

(cf. (2) above). It suffices to show that

$$\begin{aligned}
&(1-p^e)(K_{S'}+\Psi_{S'})+p^eL_{ij}|_{S'}-\mu^*((1-p^e)(K_{\bar S}+B^\sharp_{\bar S})+p^e(j\mathbf{D}_{\bar S}))\\
&\quad\geq (1-p^e)(K_{S'}+\Psi_{S'})+p^eL_{ij}|_{S'}-\mu^*((1-p^e)(K_{\bar S}+\Psi^*_{\bar S})+p^e(j\mathbf{D}_{\bar S}))\\
&\quad= (p^e-1)\left(\left\lceil \frac{j}{i}\mathbf{M}_{i,S'}+\mathbf{A}_{S'}\right\rceil -K_{S'}-\Psi_{S'}-\mu^*(j\mathbf{D}_{\bar S}-K_{\bar S}-\Psi^*_{\bar S})\right)\\
&\qquad+L_{ij}|_{S'}-\mu^*(j\mathbf{D}_{\bar S})\\
&\quad= (p^e-1)\left(\frac{j}{i}\mathbf{M}_{i,S'}+\mathbf{A}_{S'}-G_\epsilon-K_{S'}-\mu^*(j\mathbf{D}_{\bar S}-K_{\bar S}-\Psi^*_{\bar S})\right)\\
&\qquad+L_{ij}|_{S'}-\mu^*(j\mathbf{D}_{\bar S})\\
&\quad\geq (p^e-1)\left(\mu^*a^*E_{S^+}-G_\epsilon+\frac{j}{i}\mathbf{M}_{i,S'}-j\mu^*\mathbf{D}_{\bar S}\right)+L_{ij}|_{S'}-\mu^*(j\mathbf{D}_{\bar S})\\
&\quad\geq L_{ij}|_{S'}-\mu^*(j\mathbf{D}_{\bar S})
\end{aligned}$$

is effective. Here the first inequality follows as $\Psi^*_{\bar S}\leq B^\sharp_{\bar S}$, the second equality by definition of L_{ij}, the third equality since $\lceil \frac{i}{j}\mathbf{N}_{i,Y}+\mathbf{A}_Y\rceil-(\frac{i}{j}\mathbf{N}_{i,Y}+\mathbf{A}_Y)-\Psi'_\epsilon+S+\epsilon F=0$ and so restricting to S' we have that $\lceil \frac{i}{j}\mathbf{M}_{i,S'}+\mathbf{A}_{S'}\rceil-(\frac{i}{j}\mathbf{M}_{i,S'}+\mathbf{A}_{S'})+G_\epsilon-\Psi_{S'}=0$, the fourth inequality follows since as $\Psi^*_{\bar S}=B_{\bar S}+\mu_*G_\epsilon+j(\mathbf{D}_{\bar S}-\mathbf{D}_{i,\bar S})+a^*E_{S^+}$, then

$$\mu^*(K_{\bar S}+\Psi^*_{\bar S}-a^*E_{S^+})-K_{S'}+\mathbf{A}_{S'}=\mu^*\mu_*G_\epsilon+j(\mathbf{D}_{S'}-\mathbf{D}_{i,S'})\geq 0,$$

and the final inequality follows since by (2) we have that

$$\mu^*a^*E_{S^+}-G_\epsilon+\frac{j}{i}\mathbf{M}_{i,S'}-j\mathbf{D}_{S'}\geq 0.$$

Thus it suffices to show that $L_{ij}|_{S'}-\mu^*(j\mathbf{D}_{\bar S})\geq 0$. Note that for any fixed $\delta>0$ and $i\gg 0$ we have

$$\frac{j}{i}\mathbf{M}_{i,\bar S}\geq j\mathbf{D}_{\bar S}-\delta\nu^*(A^*|_S)$$

and so

$$\frac{j}{i}\mathbf{M}_{i,S'}+\mathbf{A}_{S'}=\frac{j}{i}\mu^*\mathbf{M}_{i,\bar S}+\mathbf{A}_{S'}\geq \mu^*(j\mathbf{D}_{\bar S}-\delta\nu^*(A^*|_S))+\mathbf{A}_{S'}.$$

Thus, as $j\mathbf{D}_{\bar S}$ is Cartier by (4.11), we have that

$$\left\lceil\frac{j}{i}\mathbf{M}_{i,S'}+\mathbf{A}_{S'}\right\rceil\geq \mu^*(j\mathbf{D}_{\bar S})+\lceil -\delta g^*(A^*|_{S'})+\mathbf{A}_{S'}\rceil\geq \mu^*(j\mathbf{D}_{\bar S}),$$

where for the last inequality we have used the fact that since $(S,B_S+\delta A^*|_S)$ is klt, we have that $\lceil -\delta g^*(A^*|_S)+\mathbf{A}_{S'}\rceil\geq 0$. Therefore, we have that $L_{ij}|_{S'}-\mu^*(j\mathbf{D}_{\bar S})\geq 0$ is an effective divisor. This concludes the proof. $\square$

By (2.20), we have that for sufficiently divisible j,

$$S^0(\bar S,\sigma(\bar S,B^\sharp_{\bar S})\otimes\mathcal{O}_{\bar S}(j\mathbf{D}_{\bar S}))=H^0(S^+,\mathcal{O}_{S^+}(j\mathbf{D}_{S^+})).$$

Thus we have shown that

$$S^0(S',\sigma(S',\Psi_{S'})\otimes\mathcal{O}_{S'}(L_{ij}|_{S'}))\supset H^0(S^+,\mathcal{O}_{S^+}(j\mathbf{D}_{S^+})).$$

The reverse inclusion is clear as $a_*\mu_*(L_{ij}|_{S'})=j\mathbf{D}_{S^+}$. Thus the lemma follows. $\square$

By construction, we have that

$$R_{S/Z}(k(K_X + S + B)) \subset R(S^+/Z; j\mathbf{D}_{S^+}).$$

Since

$$|L_{ij}| \subset |jk(K_Y + S' + B_Y) + \lceil \mathbf{A}_Y \rceil| = |jk(K_Y + S' + B_Y)| + \lceil \mathbf{A}_Y \rceil$$

we have that

$$S^0(Y, \sigma(Y, \Xi) \otimes \mathcal{O}_Y(L_{ij})) \subset H^0(Y, \mathcal{O}_Y(jk(K_Y + S' + B_Y))).$$

The above lemma together with (4.7) imply that

$$R_{S/Z}(jk(K_X + S + B)) \supset R(S^+/Z, j\mathbf{D}_{S^+})$$

for $j > 0$ sufficiently divisible and thus equality holds. Since

$$R(S^+/Z, j\mathbf{D}_{S^+}) \cong R(\bar{S}/Z, j\mathbf{D}_{\bar{S}})$$

is a finitely generated $\mathcal{O}_Z$-algebra (cf. (2.21)), so is $R_{S/Z}(k(K_X + S + B))$ and the theorem holds. $\square$

5. On the minimal model program for threefolds

5.1. The results of Keel.

Theorem 5.1 ([Keel99, 0.2])**.** *Let L be a nef line bundle on a scheme X, projective over an algebraically closed field of characteristic $p > 0$. L is semi-ample if and only if $L|_{\mathbb{E}(L)}$ is semi-ample. In particular, if the basefield is the algebraic closure of a finite field and $L|_{\mathbb{E}(L)}$ is numerically trivial, then L is semi-ample.*

Recall that for a nef line bundle L, $\mathbb{E}(L)$ is the closure of the union of all of those irreducible subvarieties with $L^{\dim Z} \cdot Z = 0$.

Theorem 5.2 ([Keel99, 0.5])**.** *Let X be a normal $\mathbb{Q}$-factorial threefold, projective over an algebraically closed field of positive characteristic. Let L be a nef and big line bundle on X. If $L - (K_X + \Delta)$ is nef and big for some boundary Δ with $\lfloor \Delta \rfloor = 0$, then L is EWM. If the basefield is the algebraic closure of a finite field, then L is semi-ample.*

Recall that a nef line bundle L on a scheme X proper over a field is *Endowed With a Map (EWM)* if there is a proper map $f : X \to Y$ to a proper algebraic space which contracts exactly $\mathbb{E}(L)$.

Theorem 5.3 ([Keel99, 0.6])**.** *Let X be a normal $\mathbb{Q}$-factorial threefold, projective over an algebraically closed field. Let Δ be a boundary on X. If $K_X + \Delta$ has non-negative Kodaira dimension, then there is a countable collection of curves $\{C_i\}$ such that*

(1) $\overline{NE}_1(X) = \overline{NE}_1(X) \cap (K_X + \Delta)_{\geq 0} + \mathbb{R}_{\geq 0} \cdot [C_i]$.
(2) *All but a finite number of the C_i are rational and satisfy* $0 < (K_X + \Delta) \cdot C_i \leq 3$.
(3) *The collection of rays $\{R \cdot [C_i]\}$ does not accumulate in the half-space $(K_X + \Delta)_{<0}$.*

We have the following easy consequence of (5.2).

Theorem 5.4. *Let (X, Δ) be a normal $\mathbb{Q}$-factorial threefold dlt pair projective over a field of positive characteristic $p > 0$ such that $K_X + \Delta$ is pseudo-effective. Let R be a $(K_X + \Delta)$-negative extremal ray.*

(1) *Then the corresponding contraction $f : X \to Z$ exists in the category of algebraic spaces.*
(2) *If $k = \overline{\mathbb{F}}_p$ is the algebraic closure of a finite field, then $f : X \to Z$ is a morphism to a projective threefold.*
(3) *If $(X, \Delta = S + B)$ is plt, S is normal and $S \cdot R < 0$, then $f : X \to Z$ is a morphism to a projective threefold with $\rho(X/Z) = 1$.*

Proof. Suppose that $K_X + \Delta$ is not nef, then by (5.3), it follows easily that there is an ample $\mathbb{Q}$-divisor H such that $H + K_X + \Delta$ is nef and

$$\overline{NE}_1(X) \cap (H + K_X + \Delta)^{\perp} = \mathbb{R}_{\geq 0}[R].$$

Since $K_X + \Delta$ is pseudo-effective, we have that $L := H + K_X + \Delta$ is big. Then (5.2) immediately implies (1) and (2).

(3) follows from (5.1). Note that $\mathbb{E}(L) \subset S$ and $L|_S = K_S + \mathrm{Diff}_S(B + H)$ is semi-ample by the contraction theorem in the surface case (see e.g. [KK, 2.3.5]). Using (5.1), then a standard argument implies $\rho(X/Z) = 1$ (see [KM98, 3.17]). □

5.2. **Existence of flips and minimal models.**

Definition 5.5. Let (X, Δ) be a dlt pair such that X is $\mathbb{Q}$-factorial. We say that $f : X \to Z$ an *extremal flipping contraction* if it is a projective birational morphism between normal quasi-projective varieties such that

(1) f is small (i.e. an isomorphism in codimension 1);
(2) $-(K_X + \Delta)$ is ample over Z;
(3) $\rho(X/Z) = 1$.

Proof of (1.1). Replacing Δ by $\Delta - \frac{1}{n}\lfloor \Delta \rfloor$ for some $n \gg 0$, we may assume that (X, Δ) is klt. We use Shokurov's reduction to pl-flips and special termination as explained in [Fujino07]. We simply indicate the changes necessary to the part of the argument that is not characteristic independent.

Step 2 of [Fujino07, 4.3.7] requires resolution of singularities. We use (2.1).

Step 3 of [Fujino07, 4.3.7] requires that we run a relative minimal model program. Since all divisors are relatively big in this context, by (5.3) the relevant negative extremal ray R always exists. We may assume that all induced contractions are $K_Y + S + B$-negative for an appropriate plt pair $(Y, S + B)$ where $S \cdot R < 0$ (if the pair $(X, S + B)$ is dlt, then we may replace B by $B - \frac{1}{n}\lfloor B \rfloor$ for $n \gg 0$). Thus the corresponding contraction morphism exists and is projective and all required divisorial contractions exist. Since all flipping contractions are pl-flipping contractions the corresponding flips exist by (4.12).

By Step 5 of [Fujino07, 4.3.7], we obtain $f^+ : (X^+, \Delta^+) \to Z$ a small birational morphism such that X^+ is $\mathbb{Q}$-factorial, (X^+, Δ^+) is dlt and $K_{X^+} + \Delta^+$ is nef over Z. However, it is clear that Δ^+ is the strict transform of Δ and hence $\lfloor \Delta^+ \rfloor = 0$ so that (X^+, Δ^+) is klt. Denote by p the number of exceptional divisors of a common resolution $Y \to X$. Since X and X^+ are $\mathbb{Q}$-factorial, we have

$$\rho(X) = \rho(Y) - p = \rho(X^+),$$

which implies $\rho(X^+/Z) = \rho(X/Z) = 1$ and hence that $K_{X^+} + \Delta^+$ is ample over Z. Thus $X \dashrightarrow X^+$ is the required flip. □

Theorem 5.6. *Let k be an algebraically closed field of characteristic $p > 5$, (X, Δ) a projective $\mathbb{Q}$-factorial threefold klt pair over k such that $K_X + \Delta$ is pseudo-effective* *and all coefficients of* Δ *are in the standard set* $\{1-\frac{1}{n}|n\in\mathbb{N}\}$. *Let* R *be a* $(K_X+\Delta)$-*negative extremal ray and* $f:X\to Z$ *the corresponding proper birational contraction to a proper algebraic space (given by* (5.4)*) such that a curve* C *is contracted if and only if* $[C]\in R$.

(1) *There is a small birational morphism* $f^+:X^+\to Z$ *such that* X^+ *is* $\mathbb{Q}$-*factorial and projective, and* $K_{X^+}+\Delta^+$ *is nef over* Z *where* Δ^+ *denotes the strict transform of* Δ.

(2) *If moreover,* f *is divisorial, then* $X^+=Z$ *and in particular* Z *is projective.*

Proof. (1) The extremal ray R is cut out by a big and nef $\mathbb{Q}$-divisor of the form $L=K_X+\Delta+H$ for some ample $\mathbb{Q}$-divisor H. By (5.3) L is EWM. Let $f:X\to Z$ be the corresponding birational contraction to a proper algebraic space. Since $K_X+\Delta+(1-\epsilon)H$ is big for any rational number $0<\epsilon\ll 1$, there is a positive (sufficiently divisible) integer $m\in\mathbb{N}$ and a divisor $S\sim m(K_X+\Delta+(1-\epsilon)H)$. Thus $S\cdot R<0$. Replacing S by a prime component, we find a prime divisor T with $T\cdot R<0$. In the divisorial contraction case, T is the contracted divisor.

Consider a log resolution $\nu:Y\to X$ of the pair $(X,\Delta+T)$, and let E be the reduced ν-exceptional divisor. Write

$$\Delta=\sum_{\Delta_i\neq T}a_i\Delta_i+bT=\Gamma+bT.$$

We run the $(K_Y+\Gamma'+E+T')$-MMP over Z, where $\Gamma'=\nu_*^{-1}\Gamma$ and $T'=\nu_*^{-1}T$ are the birational transforms of Γ and T. Note that running the MMP over Z means that at each step we only consider extremal curves C^j such that $C^j\cdot L^j=0$, where L^j is the strict transform of ν^*L. We obtain models

$$Y=Y^1 \dashrightarrow Y^2\dashrightarrow\cdots$$

We note that since f only contracts curves whose classes are in R and $K_X+\Delta+H=L\equiv_Z 0$, it easily follows that $K_X+\Delta\equiv_Z aT$ for some $a>0$ and so

$$K_Y+\Gamma'+T'+E\equiv_{\mathbb{Q}} f^*(K_X+\Delta)+\sum c_iE_i+(1-b)T'\equiv_Z\sum d_iE_i+cT',$$

where $d_i\geq c_i>0$ and $c>1-b>0$.

For each step of this MMP, we let C^j be a curve spanning the corresponding extremal ray and $\bullet^j$ is the push forward of the divisor $\bullet$ from Y to Y^j. We have

$$C^j\cdot\left(\sum d_iE_i^j+cT^j\right)<0,$$

and so at each step of the MMP, the curve C_j intersects a component of T^j+E^j negatively. By special termination, this MMP terminates after finitely many steps and we obtain a minimal model over Z, say $W=Y^m$.

Next, we run the $(K_W+\Gamma_W+E_W+bT_W)$-MMP with scaling of T_W which yields birational maps

$$W=W^1\dashrightarrow W^2\dashrightarrow\cdots$$

It is easy to see that each step is $(\sum d_iE_i^k)$-negative, and hence also of special type. By special termination, this MMP terminates after finitely many steps and we obtain a minimal model over Z, say $X^+=W^l$.

Let $p:U\to X$ and $q:U\to X^+$ be a common resolution and consider the divisor $p^*(K_X+\Delta)-q^*(K_{X^+}+\Gamma_{X^+}+bT_{X^+}+E_{X^+})$ which is easily seen to be anti-nef and exceptional over X. By the negativity lemma, this divisor is effective and so

$X \dashrightarrow X^+$ is a birational contraction and $E_{X^+} = 0$. It follows that $X \dashrightarrow X^+$ is a minimal model for $K_X + \Delta$ over Z.
(2) It remains to show that if $f : X \to Z$ is divisorial contraction, then $Z \cong X^+$. Assume that $f^+ : X^+ \to Z$ is not an isomorphism. Let C be a f^+-contracted curve, and H^+ an ample divisor on X^+, so $C \cdot H^+ > 0$. Denote by H the birational transform of H^+ on X. Let R be the exceptional ray and E be an f-exceptional divisor. Since, $E \cdot R < 0$, there exists a number $a \in \mathbb{Q}$ such that $(H + aE) \cdot R = 0$.

Let $p : U \to X$ and $q : U \to X^+$ be a common resolution, then $p^*(H + aE) - q^*(H^+) \equiv_{X^+} 0$ is q-exceptional and so $p^*(H + aE) = q^*H^+$ (by the negativity lemma). But then $q^*H^+ \equiv p^*(H + aE) \equiv_Z 0$ contradicting the fact that $C \cdot H^+ > 0$. □

In (5.6), if f is small then we will say that $X^+ \to Z$ is the corresponding *generalized flip*. We say that $X = X_1 \dashrightarrow X_2 \dashrightarrow \cdots$ is *a generalized* $(K_X + \Delta)$-*MMP*, if each map $X_i \dashrightarrow X_{i+1}$ is a $(K_X + \Delta)$-divisorial contraction or a $(K_X + \Delta)$-generalized flip.

Proof of (1.2). When $k = \overline{\mathbb{F}}_p$, we can run the usual $(K_X + \Delta)$-MMP by (5.4) and (1.1); and in general we can run the generalized $(K_X + \Delta)$-MMP defined as above by (5.6). The condition $N_\sigma(K_X + \Delta) \wedge \Delta = 0$ guarantees that no component of Δ is contracted and hence that (X, Δ) remains canonical at each step.

So it suffices to show that a sequence of the generalized $(K_X + \Delta)$-minimal model program terminates. In fact this directly follows from the argument of [KM98, 6.17]. We define the non-negative integer valued function $d(X, \Delta)$ as in [KM98, 6.20]. We easily see $d(X, \Delta) < d(X^+, \Delta^+)$ as long as $\mathrm{Supp}(\Delta^+)$ contains an exceptional curve of $X^+ \to Z$. Thus for a sequence of flips, the birational transform of Δ eventually will not contain flipping curves. Then the rest of the argument is precisely the same as the one in [KM98, 6.17] (cf. [Kollár13, Section 3.3]).

To see the finite generation of $R(K_X + \Delta)$, note that by the construction of Iitaka fibration (see [Mori85, 1.11-12]), there is a positive integer d, a smooth projective variety Y of dimension $\kappa(K_X + \Delta)$ and a big divisor D on Y, such that

$$R(X, d(K_X + \Delta)) \cong R(Y, D).$$

When $\kappa(K_X + \Delta) = 3$, the finite generation follows from the existence of minimal model and Keel's theorem (see (5.2)). When $\kappa(K_X + \Delta) = 2$, by Zariski decomposition for surfaces, we can write $D = P + N$ where P is a nef and big $\mathbb{Q}$-divisor with the property $R(Y, D) = R(Y, P)$. Then it is well known that P is semi-ample since k is the algebraic closure of a finite field (see e.g. [Keel99, 2.13] and (5.1)). When $\kappa(K_X + \Delta)$ is 0 or 1, the finite generation follows trivially. □

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