On the three types of relativistic equations for particles with nonzero mass

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In previous papers [1, 2] we have shown that there exist three types of the relativistic equations for the massless particles. Here we show that for the free particles and antiparticles with the mass $m > 0$ and the arbitrary spin $s \geq \frac{1}{2}$ there also exist three types of nonequivalent equations.

For the sake of brevity we shall only dwell upon the equations of motion for the particles with spin $s = \frac{1}{2}$. From the text it would be clear that all results of the paper can be formulated for arbitrary spin. Let us consider the eight-component equation of the Dirac type [3]

$$(\Gamma_\mu p^\mu - \Gamma_4 m)\Psi(t, \boldsymbol{x}) = 0, \qquad \mu = 0, 1, 2, 3,$$
$$p_0 = i\frac{\partial}{\partial t}, \qquad p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1, 2, 3, \tag{1}$$

where the 8×8 matrices Γ_μ, Γ_4, Γ_5, Γ_6 obey the Clifford algebra; Ψ is a eight-component wave function.

On the solutions of eq. (1) the generators of the Poincaré group $P_{1,3}$ have the form

$$P_0 \equiv \mathcal{H} = \Gamma_0\Gamma_a p_a + \Gamma_0\Gamma_4 m \equiv -2iS_{0k}p_k, \qquad p_4 \equiv m, \quad k = 1, 2, 3, 4,$$
$$P_a = p_a, \qquad J_{ab} = x_a p_b - x_b p_a + S_{ab}, \tag{2}$$
$$J_{0a} = x_0 p_a - \frac{1}{2}(x_0\mathcal{H} + \mathcal{H}x_a), \qquad S_{\mu\nu} = \frac{i}{4}(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu).$$

Using the generators (2) it can be shown that on the set of solutions eq. (1) a direct sum of four irreducible representations of the group $P_{1,3}$:

$$D^+_{s,0} \oplus D^-_{0,s} \oplus D^+_{0,s} \oplus D^-_{s,0}, \qquad s = \frac{1}{2}, \tag{3}$$

is realized. Here $D^\pm_{s,0}$ and $D^\pm_{0,s}$ denote the irreducible representations of the group $P_{1,3}$. The symbols $D_{s,0}$ and $D_{0,s}$ denote the irreducible representations of the group O_4. Elsewhere [3] we have shown that eq. (1) was invariant under the group O_6, and a usual Dirac equation was invariant under the group O_4.

From (3) it follows that we can obtain three types of nonequivalent four-component equations from eq. (1). It is evident that these three types of equations are equivalent to one eq. (1) with three subsidiary conditions. These relativistic invariant subsidiary condition have the form

$$P_1^-\Psi = 0 \quad \text{or} \quad P_1^+\Psi = 0, \qquad P_1^\pm = \frac{1}{2}(1 \pm 2S_{56}), \qquad S_{56} = \frac{i}{2}\Gamma_5\Gamma_6, \tag{4}$$

$$P_2^-\Psi = 0 \quad \text{or} \quad P_2^+\Psi = 0, \qquad P_2^\pm = \frac{1}{2}(1 \pm 2\hat{\varepsilon}S_{56}), \qquad \hat{\varepsilon} = \frac{\mathcal{H}}{E}, \tag{5}$$

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$$P_3^-\Psi = 0 \quad \text{or} \quad P_3^+\Psi = 0, \qquad P_3^\pm = \frac{1}{2}(1 \pm \hat{\varepsilon}), \qquad E = \sqrt{p_a^2 + m^2}. \tag{6}$$

As the projective operators $P_a^\pm$ commute with the generators (2), it means that subsidiary conditions (4), (6) are invariant under the Poincaré group. The conditions (5), (6) are nonlocal in configuration space since $\hat{\varepsilon}$ is the integrodifferential operator.

The eq. (1) together with the subsiduary condition (4) is equivalent to the usual Dirac equation. In this case the wave function is transformed under the representation

$$D_{s,0}^+ \oplus D_{0,s}^- \quad \text{if} \quad P_1^-\Psi = 0 \qquad \text{or} \qquad D_{s,0}^- \oplus D_{0,s}^+ \quad \text{if} \quad P_1^+\Psi = 0. \tag{7}$$

Equation (1) together with (4) is equivalent to the four-component equation which coincide on the form with the Dirac equation, however, the wave function in this equation is transformed under the representation

$$D_{s,0}^+ \oplus D_{s,0}^- \quad \text{if} \quad P_2^-\Psi = 0 \qquad \text{or} \qquad D_{0,s}^- \oplus D_{0,s}^+ \quad \text{if} \quad P_2^+\Psi = 0. \tag{8}$$

It is clear that the representations (8) are not equivalent to (7).

Equation (1) with subsidiary condition (6) is equivalent to the four-component equation of the form

$$i\frac{\partial\Psi^{(4)}(t,\boldsymbol{x})}{\partial t} = E\Psi^{(4)}(t,\boldsymbol{x}), \tag{9}$$

where the wave function $\Psi^{(4)}$ is transformed under the representation

$$D_{s,0}^+ \oplus D_{0,s}^+ \quad \text{if} \quad P_3^-\Psi = 0 \qquad \text{or} \qquad D_{s,0}^- \oplus D_{0,s}^- \quad \text{if} \quad P_3^+\Psi = 0. \tag{10}$$

It should be emphasized that only in the last equation of motion the Hamiltonian is the positive operator. If we compare the particle with the representation $D_{s,0}^+$ and the antiparticle with the representation $D_{0,s}^+$, then the eq. (9) describes free motion of a particle and antiparticle with positive energy. In this case the operator of a charge has the form $Q = \hat{\varepsilon}$. Equation (1) with subsidiary conditions (4)–(6) can be written in the form

$$(\Gamma_\mu p^\mu - \Gamma_4 m + \varkappa_a P_a^+)P_a^-\Psi(t,\boldsymbol{x}) = 0, \tag{11}$$

where $\varkappa_a$ are the arbitrary constant numbers. For eq. (11) the conditions (4)–(6) are automatically satisfied.

Equation (1) with the subsidiary conditions (4), (5), (6) (or three eqs. (11)) has different P-, T-, C-properties. These properties can be read easily from the following coupling scheme or irreducible representations of the Poincaré group

$$\begin{array}{ccc} D^+(s,0) & \overset{P}{\longleftrightarrow} & D^+(0,s) \\ T^p \updownarrow C & & C \updownarrow T^p \\ D^-(s,0) & \overset{P}{\longleftrightarrow} & D^-(0,s) \end{array}$$

T^p is the Pauli $\leftrightarrow$ time-reversal operator. These questions will be considered in more detail in another paper.

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