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On the Time Operator in Quantum Mechanics

— Three Typical Examples —

Tetsuo GOTŌ, Katsuhito YAMAGUCHI and Naoshi SUDŌ

Atomic Energy Research Institute

College of Science and Technology, Nihon University, Tokyo 101

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The so-called time operator in quantum mechanics which may be regarded as a coordinate conjugate to Hamiltonian operator is studied. Three typical examples of the time operators in one-dimensional problem are explicitly obtained; that is, a free particle, a harmonic oscillator and a particle in a square well potential. Our procedure to construct the time operator seems to be useful in a more general case. Qualitative nature of the time operator may be understood from the three examples.

§ 1. Introduction

The problem of finding the so-called time operator has been investigated by many authors in relation to the time-energy uncertainty relation.^{*)} Although, as has been well known, the time operator \hat{T} satisfying the commutation relation

$$[\hat{H}, \hat{T}] = -i \quad \hat{H} : \text{Hamiltonian} \quad (1.1)$$

cannot exist in Hilbert space if eigenvalues of Hamiltonian operator \hat{H} have a lower bound,²⁾ there is a well-known time operator of a free particle; namely, the operator

$$\hat{T}_0 = \frac{m}{2} \left[\frac{1}{\hat{p}} \hat{x} + \hat{x} \frac{1}{\hat{p}} \right] \quad (1.2)$$

and Hamiltonian

$$\hat{H}_0 = \frac{1}{2m} \hat{p}^2 \quad (1.3)$$

satisfy formally the following commutation relation:

$$[\hat{H}_0, \hat{T}_0] = -i. \quad (1.4)$$

Physical meaning of the operator \hat{T}_0 may be understood easily ($T_0 \sim x/v$). Mathematical justification of the operator is attempted by Rosenbaum who proposed an idea of the super Hilbert space.³⁾ According to this theory, physical states are represented by continuous linear functionals on a space of good functions which are everywhere differentiable any number of times and decrease

^{*)} A review of the phase-number and angle-angular momentum problems, as well as references to the time-energy controversy is given by Carruthers et al.¹⁾

at infinity faster than the inverse of any polynomial. Although he succeeded in admitting inverse operators of \hat{p} and \hat{x} and also discussed the time operator of a harmonic oscillator, his procedure seems to be very complicated and difficult to be used in a more general case. There is also known an exceptional example in the case of a particle under a constant field of force; that is, Hamiltonian

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + G\hat{x} \quad (G: \text{constant}) \quad (1.5)$$

and the operator

$$\hat{T} = -\hat{p}/G \quad (1.6)$$

satisfy the commutation relation (1.1).⁴⁾ Since the eigenvalue of \hat{H} in (1.5) varies continuously from $-\infty$ to $+\infty$, the well-known Pauli objection²⁾ does not hold. In the case of a harmonic oscillator, the classical time variable

$$T = -\frac{1}{\omega} \tan^{-1}(p/m\omega x) \quad (1.7)$$

is well known where Hamiltonian is given by

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 x^2. \quad (1.8)$$

However, it is well known¹⁾ that the quantum mechanical analogue to (1.7) cannot be defined in Hilbert space and Rosenbaum has given the corresponding operator in his super Hilbert space.³⁾

On the other hand, we have proposed recently an alternative formulation of quantum mechanics⁵⁾ in which the time dependent Schrödinger equation

$$(\hat{p}_t + \hat{H})|\Psi\rangle = 0, \quad (1.9)$$

where

$$[\hat{p}_t, \hat{t}] = -i \quad (1.10)$$

is regarded as a constraint imposed on physical states. Since our formalism is based on homogeneous canonical theory, the time coordinate \hat{t} as well as the space coordinate \hat{x} is considered as a dynamical variable. In our opinion, the standard formulation where the time coordinate appears as a parameter is based on our choice of the special gauge fixing condition (i.e., $\hat{t} - s = 0$, s : parameter). If we take another gauge fixing condition, we may have an alternative form of quantum mechanics. In our previous paper,⁵⁾ we have put the gauge fixing condition as follows:

$$\hat{x} - s = 0, \quad s: \text{parameter} \quad (1.11)$$

and the corresponding formulation has been developed. In this case the space

coordinate appears as a parameter and the time coordinate remains to be a dynamical variable. The inner products and normalizations of wave functions obeying (1.9) are given by the integral of the time variable instead of the space coordinate. More precisely, the inner product of two wave functions ψ_1 and ψ_2 satisfying Eq. (1.9) is defined by the following:

$$\int_{-\infty}^{\infty} dt \frac{1}{2mi} \left[\psi_1^*(x, t) \frac{\partial \psi_2(x, t)}{\partial x} - \frac{\partial \psi_1^*(x, t)}{\partial x} \psi_2(x, t) \right]. \quad (1.12)$$

It should be noticed that we have no restriction whether $\psi(x, t)$'s have to be square integrable or not as functions of x .

Though the time coordinate \hat{t} is now regarded as a dynamical variable, it is not an observable. Since an observable \hat{O} should be commutable with $\hat{\phi} = \hat{p}_t + \hat{H}$, i.e.,

$$[\hat{\phi}, \hat{O}] = 0, \quad (1.13)$$

the observable time operator \hat{t}_{ob} is defined by

$$\hat{t}_{ob} = \hat{t} - \hat{T}, \quad (1.14)$$

where

$$[\hat{H}, \hat{T}] = -i. \quad (1.15)$$

Therefore, in our point of view, the existence of the operator \hat{T} is very important to define the observable time operator \hat{t}_{ob} . Hereafter, we call the operator \hat{T} satisfying Eq. (1.15) the time operator.

The purpose of the present paper is explicitly to show three typical examples of the time operator and establish how to construct it. In this paper, we confine ourselves to the one-dimensional motion of a particle for the sake of simplicity. In § 2, we shall discuss the simplest case of a free particle in order to explain our basic postulates for constructing the time operator. The spectrum of Hamiltonian in this case is continuous and has a lower bound. As an example of a discrete spectrum of Hamiltonian operator, we shall study a harmonic oscillator in § 3. The square well potential is discussed as an example of the case accompanied with both scattering and bound states in § 4. If the potential is repulsive, we have only scattering states and the mathematical behaviour of the time operator in this case is qualitatively similar to that of a free particle. Section 5 is devoted to the discussions of several problems related to the time operator.

§ 2. Free particle

As the simplest example, let us study the case of a free particle where Hamiltonian is given by

$$\hat{H} = \frac{1}{2m} \hat{p}^2. \quad (2.1)$$

As is well known, the time operator T in this case is usually written as follows:

$$\hat{T} = \frac{m}{2} \left[\hat{x} \frac{1}{\hat{p}} + \frac{1}{\hat{p}} \hat{x} \right], \quad (2.2)^*)$$

which satisfies formally the commutation relation

$$[\hat{H}, \hat{T}] = -i \quad (2.3)$$

provided that

$$[\hat{p}, \hat{x}] = -i. \quad (2.4)$$

However, as has been frequently emphasized, the time operator satisfying (2.3) cannot exist in Hilbert space. Therefore, if we wish to define the time operator \hat{T} in (2.2) unambiguously, we should specify the domain in which the time operator is defined. At present, however, we are not aware of the right way to develop the mathematical formalism. Therefore, though we do not wish to discuss the mathematical structure of the time operator, we shall make our basic assumption clear in a rather heuristic way. Since we suppose that wave packets corresponding to physically realizable states are represented by functions being differentiable any number of times and having finite support, operators are assumed to be given by a collection of matrix elements between such wave packets. As an ideal limit of such a wave packet, we assume the existence of a wave packet represented by an eigenstate $|x\rangle$ of the space coordinate \hat{x} and the completeness of the states; that is, we assume

$$\hat{x}|x\rangle = x|x\rangle, \quad -\infty < x < \infty \quad (2.5)$$

$$\left. \begin{aligned} \int_{-\infty}^{\infty} dx |x\rangle \langle x| &= 1, \\ \langle x|y\rangle &= \delta(x-y). \end{aligned} \right\} \quad (2.6)$$

We also assume that physically relevant operators such as Hamiltonian \hat{H} , momentum \hat{p} and so on are determined by matrix elements in these states; namely, we have the following correspondence:

$$\hat{H} \leftrightarrow \langle x|\hat{H}|y\rangle \equiv H(x, y) = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \delta(x-y), \quad (2.7a)$$

$$\hat{p} \leftrightarrow \langle x|\hat{p}|y\rangle \equiv p(x, y) = -i \frac{\partial}{\partial x} \delta(x-y), \quad (2.7b)$$

$$\hat{x} \leftrightarrow \langle x|\hat{x}|y\rangle \equiv x(x, y) = x \delta(x-y), \quad (2.7c)$$

$$\hat{1} \leftrightarrow \langle x|\hat{1}|y\rangle = \delta(x-y). \quad (\text{unit operator}) \quad (2.7d)$$

*) The time operator in this form has been used by many authors.⁶⁾

Now, if we have the time operator \hat{T} conjugate to Hamiltonian, its matrix elements denoted by $T(x, y)$ may determine the time operator \hat{T} . The commutation relation (2.3) should be understood as follows:

$$\int_{-\infty}^{\infty} dz [H(x, z)T(z, y) - T(x, z)H(z, y)] = -i\delta(x - y). \tag{2.8}$$

Throughout this paper, we shall assume that all operators relevant to our discussions are represented in such a way as shown in Eq. (2.7). Schrödinger equation is now regarded primarily as

$$\int_{-\infty}^{\infty} dy H(x, y)\psi(y) = E\psi(x),$$

$$\psi(x) = \langle x | \psi \rangle, \tag{2.9}$$

which becomes

$$-\frac{1}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} = E\psi(x). \tag{2.10}$$

It is well known that the set of normalized solutions of Eq. (2.10) forms a complete orthonormal set; namely, we have

$$\int_0^{\infty} dk [\psi^{(+)}(x, k)\psi^{(+)*}(y, k) + \psi^{(-)}(x, k)\psi^{(-)*}(y, k)] = \delta(x - y), \tag{2.11}$$

where

$$\psi^{(\pm)}(x, k) = \frac{1}{\sqrt{2\pi}} e^{\pm ikx},$$

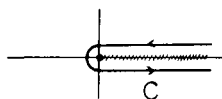
$$k = \sqrt{2mE}, \quad E > 0. \tag{2.12}$$

Now, we shall consider $\psi^{(\pm)}(x, k)$'s as functions of energy E instead of momentum k . Then, $\psi^{(\pm)'}(x, E)$'s have a branch point at $E=0$ and are regular except on the real positive axis (i.e., $E>0$). Therefore, it is easy to see that the completeness relation (2.11) can be written by a contour integral in the complex E -plane as follows:

$$-\frac{1}{2} \sqrt{\frac{m}{2}} \int_C \frac{dE}{\sqrt{E}} [\psi^{(+)}(x, E)\psi^{(-)}(y, E) + \psi^{(-)}(x, E)\psi^{(+)}(y, E)]$$

$$= \int_0^{\infty} dk [\psi^{(+)}(x, k)\psi^{(+)*}(y, k) + \psi^{(-)}(x, k)\psi^{(-)*}(y, k)] = \delta(x - y), \tag{2.13}$$

where the integral path C is shown in Fig. 1. It should be noticed that $\psi^{(\pm)}(x, E)$'s in Eq. (2.13) satisfy Schrödinger equation even if E is complex. Now, the time



E-plane

Fig. 1

operator T is given by

$$T(x, y) = \frac{1}{2} \sqrt{\frac{m}{2}} \int_C \frac{dE}{\sqrt{E}} \frac{i}{2} \times [\psi^{(+)}(x, E) \frac{\overleftarrow{\partial}}{\partial E} \psi^{(-)}(y, E) + \psi^{(-)}(x, E) \frac{\overleftarrow{\partial}}{\partial E} \psi^{(+)}(y, E)], \quad (2 \cdot 14)^*$$

where

$$g \frac{\overleftarrow{\partial}}{\partial E} f = \frac{\partial g}{\partial E} f - g \frac{\partial f}{\partial E}.$$

In virtue of Schrödinger equation

$$\begin{aligned} \int_{-\infty}^{\infty} dz H(x, z) \psi^{(\pm)}(z, E) &= E \psi^{(\pm)}(x, E), \\ \int_{-\infty}^{\infty} dz \psi^{(\pm)}(z, E) H(z, x) &= E \psi^{(\pm)}(x, E), \end{aligned} \quad (2 \cdot 15)$$

the following can be easily shown:

$$\int_{-\infty}^{\infty} dz [H(x, z) T(z, y) - T(x, z) H(z, y)] = -i \delta(x - y) \quad (2 \cdot 16a)$$

or, in short

$$[\hat{H}, \hat{T}] = -i. \quad (2 \cdot 16b)$$

Since $\psi^{(\pm)}$'s are so simple as given by (2·12), we can explicitly obtain the time operator T as follows:

$$T(x, y) = i \frac{m}{4} (x + y) \varepsilon(x - y), \quad (2 \cdot 17)$$

where

$$\varepsilon(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

It is also easy to see that if the singularity at $p=0$ in Eq. (2·2) is regularized by taking a principal value, T operator in Eq. (2·2) gives the same result as that of Eq. (2·17). It should be mentioned that even if we add an arbitrary function $\tau(x$

*) This expression seems to be similar to that given by Recami.⁷⁾

$-y)$ to $T(x, y)$ of Eq. (2·17), the commutation relation (2·16) does hold.

§ 3. Harmonic oscillator

Hamiltonian of harmonic oscillator

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m\omega^2 \hat{x}^2 \tag{3·1a}$$

is now understood as follows:

$$H(x, y) = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \delta(x-y) + \frac{1}{2} m\omega^2 x^2 \delta(x-y). \tag{3·1b}$$

For the sake of simplicity, changing variables x and E as

$$x \rightarrow \sqrt{2m\omega} x, \quad E \rightarrow E/\omega,$$

we obtain Hamiltonian (3·1b) and Schrödinger equation as follows:

$$H(x, y) = \left[-\frac{\partial^2}{\partial x^2} \delta(x-y) + \frac{1}{4} x^2 \delta(x-y) \right] \omega, \tag{3·2}$$

$$-\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{4} x^2 \psi = E\psi. \tag{3·3}$$

Eigenfunctions $h_n(x)$ and eigenvalues E_n of Eq. (3·3) under ordinary boundary condition ($|\psi| \rightarrow 0$ as $|x| \rightarrow \infty$) are well-known Hermite functions and half odd integers respectively. That is,

$$h_n(x) = e^{-(1/4)x^2} H_n(x), \quad (n=0, 1, 2\cdots)$$

$H_n(x)$: Hermite polynomials

$$E_n = n + \frac{1}{2}. \quad (n=0, 1, 2\cdots) \tag{3·4}$$

Completeness of h_n 's is expressed by the following:

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{1}{n!} h_n(x) h_n(y) = \delta(x-y). \tag{3·5}$$

In a similar way to Eq. (2·13), this is written by a contour integral in the complex E -plane; namely, we have

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_c d\lambda \frac{\pi}{\sin \pi\lambda} \frac{1}{\Gamma(\lambda+1)} \frac{1}{2} [D_\lambda(x) D_\lambda(-y) + D_\lambda(-x) D_\lambda(y)] \\ & = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{1}{n!} h_n(x) h_n(y) = \delta(x-y), \end{aligned} \tag{3·6}$$

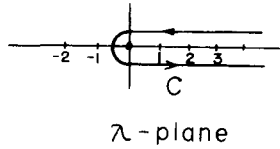


Fig. 2

where $D_\lambda(x)$ is Weber-Hermite function obeying Schrödinger equation

$$-\frac{d^2 D_\lambda}{dx^2} + \frac{1}{4} x^2 D_\lambda(x) = \left(\lambda + \frac{1}{2}\right) D_\lambda(x),$$

$$D_\lambda(x)|_{\lambda=n} = h_n(x), \quad n = 0, 1, 2, \dots \tag{3.7}$$

and the integral path C is shown in Fig. 2. Correspondingly, the time operator T is now given by

$$T(x, y) = \frac{-1}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_C d\lambda \frac{\pi}{\sin \pi\lambda} \cdot \frac{1}{\Gamma(\lambda+1)} \frac{i}{4}$$

$$\times \left[D_\lambda(x) \frac{\partial}{\partial \lambda} D_\lambda(-y) + D_\lambda(-x) \frac{\partial}{\partial \lambda} D_\lambda(y) \right]. \tag{3.8}$$

By making use of Eqs. (3.2), (3.6) and (3.7), it can be easily shown that

$$\int_{-\infty}^{\infty} dz [H(x, z) T(z, y) - T(x, z) H(z, y)] = -i\delta(x - y) \tag{3.9}$$

or

$$[H, T] = -i.$$

To obtain a more explicit expression of the time operator, we introduce the damping factor $e^{-\delta\lambda}$ into Eqs. (3.6) and (3.8) and we calculate the following expression:

$$T_\delta(x, y) = -\frac{1}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_C d\lambda \frac{\pi}{\sin \pi\lambda} \frac{e^{-\delta\lambda}}{\Gamma(\lambda+1)} \frac{i}{4}$$

$$\times \left[D_\lambda(x) \frac{\partial}{\partial \lambda} D_\lambda(-y) + D_\lambda(-x) \frac{\partial}{\partial \lambda} D_\lambda(y) \right]. \tag{3.10}$$

The damping factor assures the existence of the integral (3.10) in the ordinary sense. Employing the integral form of Weber-Hermite function

$$D_\lambda(x) = \frac{1}{\sqrt{2\pi}} e^{(1/4)x^2 - i(\pi/2)\lambda} \int_{-\infty}^{\infty} dt t^\lambda e^{-(t^2/2) + itx}, \quad (\text{Re } \lambda > -1) \tag{3.11}$$

(the integral path near the point $t=0$ is shown in Fig. 3), we obtain the following:



t-plane

Fig. 3

$$T_\delta(x, y) = -\frac{i}{\pi} \operatorname{sh}\left[\frac{1}{4}(x^2 - y^2)\right] \int_0^\infty dt e^{-\delta t^2} \ln t \cdot \cos[t(x - y)]. \quad (3 \cdot 12)$$

Even in the limit $\delta \rightarrow 0$, the integral (3.12) is meaningful as a distribution. Employing Eq. (3.12), we can examine explicitly that

$$\lim_{\delta \rightarrow \infty} \int_{-\infty}^\infty dz [H(x, z) T_\delta(z, y) - T_\delta(x, z) H(z, y)] = -i\delta(x - y). \quad (3 \cdot 13)$$

The expression (3.12) strongly suggests that, although the time operator does not exist in Hilbert space, it may be defined as an operator acting on a more restricted function space. As has been stated in the preceding section, we may expect that the time operator is a collection of matrix elements $T(x, y)$ which are meaningful as Schwarz's distribution. The singular nature of $T(x, y)$ may be easily understood from the factor $\operatorname{sh} \frac{1}{4}(x^2 - y^2)$ in (3.12) which increases very strongly as if $|x|$ or $|y|$ increases.

§ 4. Square well potential

Since most of interesting systems have both scattering and bound states, we wish here to study a particle motion in a square well attractive potential. Schrödinger equation is as follows:

$$-\frac{1}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi, \quad (4 \cdot 1)$$

$$V(x) = \begin{cases} -V_0, & (V_0 > 0) \text{ for } |x| < a, \\ 0 & \text{for } |x| > a. \end{cases}$$

Two independent scattering solutions of Eq. (4.1) are denoted by $\psi^{(+)}(x, k)$ and $\psi^{(-)}(x, k)$ respectively where $k = \sqrt{2mE}$ ($E > 0$). $\psi^{(+)}$ represents a scattering wave coming initially from left while $\psi^{(-)}$ is a wave coming from right. Explicitly, they are given as follows:

$$\psi^{(+)}(x, k) = \begin{cases} \frac{1}{\sqrt{2\pi}} [e^{ikx} + R(k)e^{-ikx}], & x < -a, \\ \frac{1}{\sqrt{2\pi}} e^{ika} T(k) [\cos \kappa(x - a) + i\frac{k}{\kappa} \sin \kappa(x - a)], & -a < x < a, \end{cases}$$

$$\left\{ \frac{1}{\sqrt{2\pi}} T(k) e^{ikx}, \quad x > a, \right. \quad (4.2)$$

where

$$k = \sqrt{2mE}, \quad x = \sqrt{2m(E + V_0)},$$

$$R(k) = \frac{imV_0}{kx} \sin 2\kappa a T(k), \quad (4.3)$$

$$T(k) = e^{-2i\kappa a} \frac{1}{\cos 2\kappa a - i \frac{k^2 + \kappa^2}{2k\kappa} \sin 2\kappa a} \quad (4.4)$$

and

$$\psi^{(-)}(x, k) = \psi^{(+)}(-x, k). \quad (4.5)$$

From the explicit forms (4.2)~(4.5) given above, we can see the analytic properties of $\psi^{(\pm)}(x, k)$ as functions of energy E . Namely, $\psi^{(\pm)}$'s have a branch point at $E=0$ and a finite number of poles on the negative real axis, the positions of which are determined by the equation

$$\cos 2\kappa a - i \frac{k^2 + \kappa^2}{2k\kappa} \sin 2\kappa a = 0. \quad (4.6)$$

As is well known, these poles correspond to bound state solutions of Eq. (4.1). It is also explicitly shown the following completeness relation:

$$\int_0^\infty dk [\psi^{(+)}(x, k) \psi^{(+)*}(y, k) + \psi^{(-)}(x, k) \psi^{(-)*}(y, k)]$$

$$+ \sum_{i=1}^n \phi_B(x, E_i) \phi_B^*(y, E_i) = \delta(x - y), \quad (4.7)$$

where ϕ_B 's stand for normalized bound state solutions. In the same way as that of § 2, we can write the completeness (4.7) by the contour integral in the complex E -plane. In fact, from Eqs. (4.2)~(4.7) we obtain

$$-\sqrt{\frac{m}{2}} \int_C \frac{dE}{\sqrt{E}} \frac{1}{T(E)} \frac{1}{2} [\psi^{(+)}(x, E) \psi^{(-)}(y, E) + \psi^{(-)}(x, E) \psi^{(+)}(y, E)]$$

$$= \int_0^\infty dk [\psi^{(+)}(x, k) \psi^{(+)*}(y, k) + \psi^{(-)}(x, k) \psi^{(-)*}(y, k)]$$

$$+ \sum_{i=1}^n \phi_B(x, E_i) \phi_B^*(y, E_i) = \delta(x - y), \quad (4.8)$$

where the contour C is shown in Fig. 4. Then, the time operator T is given by the following:

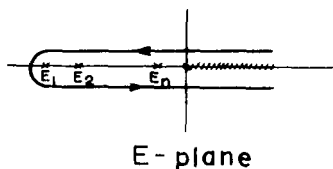


Fig. 4

$$T(x, y) = +\sqrt{\frac{m}{2}} \int_c \frac{dE}{\sqrt{E}} \frac{i}{4T(E)} \times \left[\psi^{(+)}(x, E) \frac{\partial}{\partial E} \psi^{(-)}(y, E) + \psi^{(-)}(x, E) \frac{\partial}{\partial E} \psi^{(+)}(y, E) \right]. \quad (4.9)$$

The following commutation relation is shown easily:

$$\int_{-\infty}^{\infty} dz [H(x, z)T(z, y) - T(x, z)H(z, y)] = -i\delta(x - y). \quad (4.10)$$

Here, it should be noticed that $\psi^{(\pm)}(x, E)$'s in Eqs. (4.8) and (4.9) satisfy Schrödinger equation (4.1) even if E is complex.

To obtain a more explicit expression of the time operator in Eq. (4.9), we shall study the limiting case where $a \rightarrow 0$, $V_0 \rightarrow +\infty$ and $2maV_0 = U_0$ (U_0 : finite). In this case, only one bound state survives. The solution (4.2) becomes the following simple form:

$$\psi^{(+)}(x, k) = \begin{cases} \frac{1}{\sqrt{2\pi}} [e^{ikx} + R(k)e^{-ikx}], & x < 0, \\ \frac{1}{\sqrt{2\pi}} T(k)e^{ikx}, & x > 0, \end{cases} \quad (4.11)$$

where $k = \sqrt{2mE}$ and

$$\left. \begin{aligned} R &= i \frac{U_0}{k - iU_0}, \\ T &= \frac{k}{k - iU_0}. \end{aligned} \right\} \quad (4.12)$$

The normalized bound state solution is

$$\psi_B(x, U_0) = \begin{cases} \sqrt{U_0} e^{+U_0x}, & x < 0, \\ \sqrt{U_0} e^{-U_0x}, & x > 0. \end{cases} \quad (4.13)$$

The completeness relations (4.7) and (4.8) are examined from (4.11) and (4.13). From (4.9), (4.11) and (4.12), we can obtain the time operator T as follows:

(a) $x, y < 0$

$$T(x, y) = \begin{cases} i \frac{m}{2} \left[x + \frac{1}{2U_0} (1 - e^{U_0(x-y)}) \right], & (x > y) \\ -i \frac{m}{2} \left[y + \frac{1}{2U_0} (1 - e^{-U_0(x-y)}) \right], & (x < y) \end{cases} \quad (4 \cdot 14a)$$

(b) $x < 0, y > 0$

$$T(x, y) = \begin{cases} \frac{im}{4} \frac{1}{U_0} [1 - e^{-U_0(x+y)}], & (x + y < 0) \\ -\frac{im}{4} \frac{1}{U_0} [1 - e^{U_0(x+y)}]. & (x + y > 0) \end{cases} \quad (4 \cdot 14b)$$

The expression for the cases $(x, y > 0)$ and $(x < 0, y > 0)$ are obtained by putting $x \rightarrow -y, y \rightarrow -x$ in the formula (4·14a) and (4·14b) respectively. In the limit $U_0 \rightarrow 0$, Eq. (4·14) reduces to Eq. (2·17) which gives the time operator of a free particle.

If we take $-U_0$ in place of U_0 appearing in Eqs. (4·11), (4·12) and (4·14), we obtain the time operator in the case of the repulsive potential where no bound-state exists. If $|x|$ or $|y|$ increases, $T(x, y)$ of the attractive potential increases exponentially while that of the repulsive potential increases linearly.

§ 5. Discussion

We have investigated three examples in which the completeness relation is written by the integral of energy E in the complex plane as follows:

$$\delta(x - y) = - \int_C dEI(E) \frac{1}{2} [\psi(x, E)\phi(y, E) + \phi(x, E)\psi(y, E)], \quad (5 \cdot 1)$$

where ψ and ϕ are two suitably chosen independent solutions of Schrödinger equation. $I(E)$ and the path C are also chosen suitably. Corresponding to Eq. (5·1), the time operator T is given by

$$T(x, y) = \int_C dEI(E) \frac{i}{4} \left[\psi(x, E) \frac{\overrightarrow{\partial}}{\partial E} \phi(y, E) + \phi(x, E) \frac{\overleftarrow{\partial}}{\partial E} \psi(y, E) \right]. \quad (5 \cdot 2)$$

If we notice that ψ and ϕ satisfy Schrödinger equation even in the complex energy E , we can easily see that

$$\int_{-\infty}^{\infty} dz [H(x, z)T(z, y) - T(x, z)H(z, y)] = -i\delta(x - y). \quad (5 \cdot 3)$$

This fact suggests that our procedure to construct the time operator is useful in general. We shall study this problem separately.

From our explicit representations, matrix element $T(x, y)$ has to be re-

garded as a distribution. Therefore, the product of T -operators such as T^2 , T^3 , etc. are not well defined in general. In fact, we cannot define T^2 as follows:

$$T^2(x, y) = \int_{-\infty}^{\infty} dz T(x, z) T(z, y), \tag{5.4}$$

which is not convergent in general. However, we can define $T^{(n)}(x, y)$'s obeying the following relations:

$$\int_{-\infty}^{\infty} dz [H(x, z) T^{(n)}(z, y) - T^{(n)}(x, z) H(z, y)] = -in T^{(n-1)}(x, y),$$

$$T^{(1)}(x, y) = T(x, y), \quad T^{(0)}(x, y) = \delta(x - y). \tag{5.5}$$

The operator $T^{(n)}$ is given by

$$T^{(n)}(x, y) = + \int_c dE I(E) \frac{1}{4} \left[\left(-i \frac{\partial}{\partial E} \right)^n \phi(x, E) \phi(y, E) \right. \\ \left. + \phi(x, E) \left(+i \frac{\partial}{\partial E} \right)^n \phi(y, E) \right. \\ \left. + \left(-i \frac{\partial}{\partial E} \right)^n \phi(x, E) \phi(y, E) + \phi(x, E) \left(+i \frac{\partial}{\partial E} \right)^n \phi(y, E) \right]. \tag{5.6}$$

Then, the operator $E(i\alpha T)$ corresponding to $\exp(i\alpha T)$ is also defined by the following matrix element:

$$E[i\alpha T](x, y) = \sum_{n=0}^{\infty} \frac{1}{n!} (i\alpha)^n T^{(n)}(x, y) \\ = + \int_c dE I(E) \frac{1}{4} [\phi(x, E + \alpha) \phi(y, E) + \phi(x, E) \phi(y, E - \alpha) \\ + \phi(x, E + \alpha) \phi(y, E) + \phi(x, E) \phi(y, E - \alpha)], \tag{5.7}$$

which may be regarded as a generating function of $T^{(n)}$'s, i.e.,

$$\left(-i \frac{\partial}{\partial \alpha} \right)^n E[i\alpha T] \Big|_{\alpha=0} = T^{(n)}. \tag{5.8}$$

The operator $E[i\alpha T]$ satisfies the equation

$$[H, E[i\alpha T]] = \alpha E[i\alpha T]. \tag{5.9}$$

Although T^2 , T^3 , etc. are not well-defined, the product of T with an operator having a finite support can be defined. For example, we can define $T \cdot H$ or $x \cdot T$ as follows:

$$\int dz H(x, z) T(z, y) = (HT)(x, y)$$

$$= \left[-\frac{1}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] T(x, y),$$

$$\int dz x \delta(x-z) T(z, y) = x T(x, y), \quad (5 \cdot 10)$$

where Hamiltonian $H(x, y)$ and position $x\delta(x-y)$ have finite supports.

Since, as has been seen from our examples, the time operator is meaningful only under very restricted conditions, it may be desirable to study the mathematical structure of the time operator in a more rigorous way. This is also necessary for developing our alternative formulation of quantum mechanics.

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