On the time to ruin for Erlang(2) risk processes

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Abstract

In this paper we consider a Sparre Andersen risk process for which the claim inter-arrival distribution is Erlang(2). Our purpose is to find expressions for moments of the time to ruin, given that ruin occurs. To do this, we define an auxiliary function ϕ along the lines of Gerber and Shiu (1998) and Gerber and Landry (1998). Our method of solution differs from that of Willmot and Lin (1999, 2000) who consider this problem for the classical risk model, in that we first solve for the auxiliary function ϕ.

1 Introduction

In this paper we consider the risk process studied by Dickson and Hipp (1998). The process is a Sparre Andersen process with claim inter-arrival times distributed as Erlang(2). We define a sequence of independent and identically distributed random variables $\{T_i\}_{i=1}^{\infty}$ representing the claim inter-arrival times, with T_1 being the time until the first claim. Each T_i has density function

$$k(t) = \beta^2 t e^{-\beta t} \quad \text{for } t > 0,$$

i.e. an Erlang(2) density with scale parameter β. This density belongs to the class of phase-type(2) or Coxian(2) densities. (See Dickson and Hipp (2000) or Willmot (1999) for details). In this paper we restrict our attention to a member of this class rather than the whole class purely for ease of presentation. The principles involved are unchanged for any other density in this class.

We next define a sequence of independent and identically distributed random variables $\{X_i\}_{i=1}^{\infty}$ where X_i denotes the amount of the ith claim. We denote by P the distribution function of X_i, and we assume throughout that X_i has a density function denoted p. We use the notation $m_k = E(X_i^k)$.

Let c denote the insurer's premium income per unit time. We assume that this premium income is received continuously. We further assume that $cE(T_i) > E(X_i)$ for all i.

We define the surplus process $\{U(t)\}_{t\geq 0}$ as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

where the counting process $\{N(t)\}_{t\geq 0}$ denotes the number of claims up to time t. Let T denote the time to ruin, so that

$$T = \begin{cases} \inf\{t|U(t) < 0\} \\ \infty \text{ if } U(t) > 0 \text{ for all } t > 0. \end{cases}$$

Then the probability of ultimate ruin from initial surplus u is defined as

$$\psi(u) = \Pr(T < \infty|U(0) = u).$$

We now define a function ϕ by

$$\phi(u) = E\left[e^{-\delta T} 1_{\{T<\infty\}}|U(0) = u\right], \tag{1.1}$$

where $1_{\{.\}}$ is the usual indicator function and δ is a non-negative parameter. For the most part, we consider the situation when $\delta > 0$. In this case, we can think of δ either as being a force of interest, or as a dummy variable in the context of Laplace transforms. (See Gerber and Shiu (1998) for details). Note that when $\delta = 0$, we have $\phi(u) = \psi(u)$. By noting that

$$(-1)^k \frac{d^k}{d\delta^k}\phi(u)\bigg|_{\delta=0} = E\left[T^k 1_{\{T<\infty\}}|U(0) = u\right]$$

we can find the moments of the time to ruin.

We remark that our function ϕ is a simple version of the more general function studied by Gerber and Shiu (1998) and Willmot and Lin (1999, 2000) for the classical risk model. They considered

$$E\left[w(U(T^-), |U(T)|)e^{-\delta T} 1_{\{T<\infty\}}|U(0) = u\right] \tag{1.2}$$

where w is a non-negative function and $U(T^-)$ denotes the surplus immediately prior to ruin. We have simply set $w(x, y) = 1$ for all x and y. Willmot and Lin (2000) use the fact that this function satisfies a defective renewal equation to find, amongst other things, a recursive scheme for deriving the moments of the time to ruin in the classical risk model.

In this paper, we take a different approach to both Gerber and Shiu (1998) and Willmot and Lin (2000), although there are similarities in results. We start by deriving an integro-differential equation satisfied by ϕ. We use this equation to find the Laplace transform of ϕ from which we derive a simple general expression for $\phi(0)$. In Section 4 we consider some solutions for the moments of the time to ruin for particular forms of P by first solving for ϕ. We also consider the case $u = 0$. We conclude by discussing in the final section some of the differences from the classical risk model and indicating further lines of inquiry.

2 An integro-differential equation for ϕ

In this section we show that ϕ satisfies an integro-differential equation. This equation will be the basis for our explicit solutions for ϕ in Section 4.

Theorem 2.1 *$\phi(u)$ satisfies the equation*

$$\begin{aligned} & c^2\frac{d^2}{du^2}\phi(u) - 2(\beta+\delta)c\frac{d}{du}\phi(u) + (\beta+\delta)^2\phi(u) \\ = \; & \beta^2\int_0^u \phi(u-x)p(x)dx + \beta^2\left(1-P(u)\right). \end{aligned} \tag{2.1}$$

Proof. By conditioning on the time and the amount of the first claim we have

$$\begin{aligned} \phi(u) &= \int_0^\infty k(t)e^{-\delta t}\int_0^{u+ct}\phi(u+ct-x)p(x)dxdt \\ &\quad + \int_0^\infty k(t)e^{-\delta t}\int_{u+ct}^\infty p(x)dxdt \end{aligned}$$

giving

$$\begin{aligned} c\phi(u) &= \int_u^\infty k\left(\frac{s-u}{c}\right)e^{-\delta(s-u)/c}\int_0^s \phi(s-x)p(x)dxds \\ &\quad + \int_u^\infty k\left(\frac{s-u}{c}\right)e^{-\delta(s-u)/c}\int_s^\infty p(x)dxds. \end{aligned}$$

Differentiation gives

$$
\begin{aligned}
c\frac{d}{du}\phi(u) &= \frac{-1}{c}\int_u^\infty k'\left(\frac{s-u}{c}\right)e^{-\delta(s-u)/c}\int_0^s \phi(s-x)p(x)dxds \\
&\quad \frac{-1}{c}\int_u^\infty k'\left(\frac{s-u}{c}\right)e^{-\delta(s-u)/c}\int_s^\infty p(x)dxds+\frac{\delta}{c}c\phi(u) \\
&= \frac{-1}{c}\int_u^\infty \beta^2 e^{-(\beta+\delta)(s-u)/c}\int_0^s \phi(s-x)p(x)dxds \\
&\quad -\frac{1}{c}\int_u^\infty \beta^2 e^{-(\beta+\delta)(s-u)/c}\int_s^\infty p(x)dxds \\
&\quad +\frac{\beta}{c}\int_u^\infty k\left(\frac{s-u}{c}\right)e^{-\delta(s-u)/c}\int_0^s \phi(s-x)p(x)dxds \\
&\quad +\frac{\beta}{c}\int_u^\infty k\left(\frac{s-u}{c}\right)e^{-\delta(s-u)/c}\int_s^\infty p(x)dxds+\delta\phi(u) \\
&= (\beta+\delta)\phi(u)-\frac{1}{c}\int_u^\infty \beta^2 e^{-(\beta+\delta)(s-u)/c}\int_0^s \phi(s-x)p(x)dxds \\
&\quad -\frac{1}{c}\int_u^\infty \beta^2 e^{-(\beta+\delta)(s-u)/c}\int_s^\infty p(x)dxds,
\end{aligned}
$$

and differentiating a second time we get

$$
\begin{aligned}
c\frac{d^2}{du^2}\phi(u) &= (\beta+\delta)\frac{d}{du}\phi(u)+\frac{\beta^2}{c}\int_0^u \phi(u-x)p(x)dx \\
&\quad +\frac{\beta^2}{c}\int_u^\infty p(x)dx+\left(\frac{\beta+\delta}{c}\right)\left[c\frac{d}{du}\phi(u)-(\beta+\delta)\phi(u)\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&c^2\frac{d^2}{du^2}\phi(u)-2(\beta+\delta)c\frac{d}{du}\phi(u)+(\beta+\delta)^2\phi(u) \\
&= \beta^2\int_0^u \phi(u-x)p(x)dx+\beta^2\left(1-P(u)\right).
\end{aligned}
$$

■

We note that when $\delta=0$, (2.1) becomes

$$
c^2\frac{d^2}{du^2}\psi(u)-2\beta c\frac{d}{du}\psi(u)+\beta^2\psi(u)=\beta^2\int_0^u \psi(u-x)p(x)dx+\beta^2\left(1-P(u)\right)
$$

in agreement with equation (2.1) of Dickson and Hipp (1998).

3 The Laplace transform of ϕ

Throughout this paper we denote the Laplace transform of a function γ by

$$\gamma^*(s) = \int_0^\infty e^{-sx}\gamma(x)dx.$$

Before deriving the Laplace transform of ϕ, we consider what, in the terminology of Gerber and Shiu (1998), is Lundberg's fundamental equation for our model.

Lemma 3.1 *Let δ be strictly positive and define*

$$l(s) = c^2s^2 - 2(\beta+\delta)cs + (\beta+\delta)^2.$$

Then there are two positive numbers $r_1 < (\beta+c)/\delta < r_2$ such that

$$l(r_i) = \beta^2 p^*(r_i),\ \ i = 1, 2. \tag{3.1}$$

Proof. We have

$$l(0) = (\beta+\delta)^2 > \beta^2 = \beta^2 p^*(0).$$

Also,

$$l'(s) = 2c^2s - 2(\beta+\delta)c$$

so that l has a turning point at $(\beta+\delta)/c$. Further, $l''(s) = 2c^2 > 0$, so that $l(s)$ has its minimum at $s = (\beta+\delta)/c$. We note that

$$l\left(\frac{\beta+\delta}{c}\right) = 0$$

and

$$\lim_{s\to\infty} l(s) = \infty.$$

Now

$$\frac{d}{ds}\beta^2 p^*(s) = -\beta^2 \int_0^\infty xe^{-sx}p(x)dx < 0$$

so that $\beta^2 p^*(s)$ is a decreasing function of s, and is always positive. Hence, for $s > 0$, $l(s)$ intersects $\beta^2 p^*(s)$ at two distinct points, one on each side of $(\beta+\delta)/c$. ■

We make two observations about Lundberg's fundamental equation. First, as $\delta \to 0^+$, it is easy to see from the above arguments that $r_1 \to 0^+$, and that as equation (3.1) becomes equation (3.2) of Dickson and Hipp (1998), r_2 goes to the parameter Dickson and Hipp denote s_0. (We will use this notation later.) Second, Lundberg's fundamental equation has in certain circumstances a negative root which we denote by $-R$, where $R > 0$. When $\delta = 0$, R is the adjustment coefficient.

We can now use Lundberg's fundamental equation to find the Laplace transform of ϕ. In addition, we use the derivation of the Laplace transform to find a general expression for $\phi(0)$.

Theorem 3.1 *The Laplace transform of ϕ is*

$$\phi^*(s) = \frac{\beta^2(s-r_1)(s-r_2)\eta^*(s)}{l(s) - \beta^2 p^*(s)} \tag{3.2}$$

where

$$\eta(y) = \int_y^\infty e^{-r_2(x-y)} g(x)dx,$$

and

$$g(y) = \int_y^\infty e^{-r_1(x-y)} \left(1 - P(x)\right) dx.$$

Proof. Taking the Laplace transform of (2.1) we get

$$\begin{aligned} & c^2\left(s^2\phi^*(s) - s\phi(0) - \phi'(0)\right) - 2(\beta+\delta)c\left(s\phi^*(s) - \phi(0)\right) + (\beta+\delta)^2\phi^*(s) \\ = & \ \beta^2\phi^*(s)p^*(s) + \beta^2 m_1 q^*(s) \end{aligned}$$

where

$$\phi'(0) = \frac{d}{du}\phi(u)\bigg|_{u=0}$$

and

$$q^*(s) = \frac{1}{m_1 s}(1 - p^*(s)).$$

(Thus, q^* is the Laplace transform of the ladder height density in the classical risk model. We later denote the kth moment of this distribution by μ_k.) Then

$$\begin{aligned} \phi^*(s) &= \frac{c^2 s\phi(0) + c^2\phi'(0) - 2(\beta+\delta)c\phi(0) + \beta^2 m_1 q^*(s)}{c^2s^2 - 2(\beta+\delta)cs + (\beta+\delta)^2 - \beta^2 p^*(s)} \\ &= \frac{c^2 s\phi(0) + c^2\phi'(0) - 2(\beta+\delta)c\phi(0) + \beta^2 m_1 q^*(s)}{l(s) - \beta^2 p^*(s)} \end{aligned} \tag{3.3}$$

Since r_1 is a zero of the denominator of (3.3), it must also be a zero of the numerator, giving

$$c^2 r_1 \phi(0) + \beta^2 m_1 q^*(r_1) = 2(\beta+\delta)c\phi(0) - c^2\phi'(0) \tag{3.4}$$

so that

$$\phi^*(s) = \frac{c^2\phi(0)(s-r_1) + \beta^2 m_1 (q^*(s) - q^*(r_1))}{l(s) - \beta^2 p^*(s)}. \tag{3.5}$$

Now define

$$g(y) = \int_y^\infty e^{-r_1(x-y)}(1-P(x))dx.$$

Then following arguments in Dickson and Hipp (1998),

$$m_1\left(q^*(s) - q^*(r_1)\right) = (r_1 - s)g^*(s)$$

so that

$$\phi^*(s) = \frac{(s-r_1)\left(c^2\phi(0) - \beta^2 g^*(s)\right)}{l(s) - \beta^2 p^*(s)}. \tag{3.6}$$

Now note that since $l(r_2) - \beta^2 p^*(r_2) = 0$, and $r_2 > r_1$,

$$c^2\phi(0) - \beta^2 g^*(r_2) = 0$$

giving

$$\begin{aligned}\phi^*(s) &= \frac{\beta^2(s-r_1)(g^*(r_2) - g^*(s))}{l(s) - \beta^2 p^*(s)} \\ &= \frac{\beta^2(s-r_1)(s-r_2)\eta^*(s)}{l(s) - \beta^2 p^*(s)}\end{aligned}$$

where

$$\eta(y) = \int_y^\infty e^{-r_2(x-y)} g(x)dx.$$

■

Corollary 3.1 *We can write $\phi(0)$ in terms of q^* as*

$$\phi(0) = \frac{\beta^2 m_1}{c^2}\frac{q^*(r_1) - q^*(r_2)}{r_2 - r_1} \tag{3.7}$$

Proof. This follows from (3.5). Since $\phi^*(s) > 0$ and r_2 is a zero of the denominator of (3.5), it is also a zero of the numerator. ∎

We will use this form of $\phi(0)$, rather that $\phi(0) = \beta^2 g^*(r_2)/c^2$, in Section 4.

Remark 3.2 *It is a straightforward task to show that*

$$\eta(y) = \frac{1}{r_2 - r_1} \int_y^\infty \left(e^{-r_2(z-y)} - e^{-r_1(z-y)}\right)(1 - P(z))dz$$

so that we could have derived (3.2) by interchanging r_2 and r_1 in the proof of Theorem 3.1.

4 Some explicit solutions

In this section we consider two individual claim amount distributions - exponential and a mixture of two exponentials. Willmot and Lin (2000) show how to the find the moments of the time to ruin in the classical model for each of these claim distributions. We do the same here for our model, but take a different approach. We find the functional form of ϕ and show how this can be used to find moments of the time to ruin, illustrating the method by finding the first two moments in each case.

We also consider the case $u = 0$ and show that we can find moments of the time to ruin even without an explicit solution for $\phi(u)$, $u > 0$.

4.1 Exponential individual claims

Result 4.1 *When $P(x) = 1 - \exp\{-\alpha x\}$, $x > 0$,*

$$\phi(u) = (1 - R/\alpha)\exp\{-Ru\}$$

where $-R$ is the negative root of Lundberg's fundamental equation.

Proof. This follows by a very standard argument (see, e.g., Gerber (1979)). For this form of P, (2.1) becomes

$$c^2 \frac{d^2}{du^2}\phi(u) - 2(\beta+\delta)c\frac{d}{du}\phi(u) + (\beta+\delta)^2\phi(u) = \beta^2 e^{-\alpha u}\left[\alpha \int_0^u \phi(x)e^{\alpha x}dx + 1\right]. \tag{4.1}$$

From this we find that

$$\begin{aligned} 0 \;=\; & c^2 \frac{d^3}{du^3}\phi(u) + \left[\alpha c^2 - 2(\beta+\delta)c\right]\frac{d^2}{du^2}\phi(u) \\ & + \left[(\beta+\delta)^2 - 2\alpha(\beta+\delta)c\right]\frac{d}{du}\phi(u) + \left[\alpha(\beta+\delta)^2 - \beta^2\right]\phi(u). \end{aligned} \tag{4.2}$$

Inserting for $p^*(s)$ in Lundberg's fundamental equation we get

$$c^2s^3 + \left[\alpha c^2 - 2(\beta+\delta)c\right] s^2 + \left[(\beta+\delta)^2 - 2\alpha(\beta+\delta)c\right] s + \alpha(\beta+\delta)^2 - \beta^2 = 0.$$

The roots of Lundberg's fundamental equation (r_1, r_2 and $-R$) are therefore also those of the characteristic equation of (4.2) which gives

$$\phi(u) = \kappa_1 e^{r_1 u} + \kappa_2 e^{r_2 u} + \kappa_3 e^{-Ru}.$$

The coefficients κ_1 and κ_2 must be zero, since $\phi(u) \to 0$ as $u \to \infty$, giving

$$\phi(u) = \phi(0)e^{-Ru}.$$

We find that

$$\phi(0) = 1 - R/\alpha$$

by inserting the functional form for ϕ in (4.1). ■

As noted in Section 1,

$$(-1)^k \frac{d^k}{d\delta^k}\phi(u)\bigg|_{\delta=0} = E\left[T^k 1_{\{T<\infty\}} | U(0) = u\right].$$

Thus, we can find the moments of the time to ruin by differentiating our functional form for ϕ an appropriate number of times. To emphasise dependence on δ, we now write $R = R_\delta$. Then

$$\frac{d}{d\delta}\phi(u) = -\frac{R'_\delta}{\alpha}e^{-R_\delta u} - (1 - R_\delta/\alpha)(R'_\delta u)e^{-R_\delta u}.$$

where, from now on, $'$ denotes differentiation with respect to δ.

R_0 is the adjustment coefficient, given by

$$R_0 = \frac{\alpha c - 2\beta + \sqrt{a^2c^2 + 4\alpha\beta c}}{2c}.$$

From Lundberg's fundamental equation,

$$c^2R_\delta^2 + 2(\beta+\delta)cR_\delta + (\beta+\delta)^2 = \beta^2(1 - R_\delta/\alpha)^{-1},$$

we find by differentiating that

$$2c^2R_\delta R'_\delta + 2(\beta+\delta)cR'_\delta + 2cR_\delta + 2(\beta+\delta) = \beta^2(1 - R_\delta/\alpha)^{-2}R'_\delta/\alpha. \quad (4.3)$$

c	$E(T\|T<\infty)$	$V(T\|T<\infty)$
1.1	$10.21+8.990u$	$1600+1500u$
1.3	$3.536+2.479u$	$66.70+55.53u$
1.5	$2.192+1.261u$	$16.03+11.98u$

Table 4.1: Mean and variance of $T|T<\infty$, exponential claims

Setting $\delta = 0$ we find that

$$R_0' = \frac{2(\beta + cR_0)}{\beta^2\alpha(\alpha - R_0)^{-2} - 2c^2R_0 - 2\beta c}.$$

Since $\psi(u) = (1 - R_0/\alpha)e^{-R_0 u}$ we have

$$\begin{aligned} E(T|T<\infty) &= \frac{R_0'}{\alpha}\frac{1}{1-R_0/\alpha} + R_0'u \\ &= R_0'\left(\frac{1}{\alpha - R_0} + u\right). \end{aligned}$$

Similarly, we get

$$\begin{aligned} \frac{1}{\psi(u)}\left.\frac{d^2}{d\delta^2}\phi(u)\right|_{\delta=0} &= \frac{2u(R_0')^2 - R_0''}{\alpha - R_0} - R_0''u + (R_0'u)^2 \\ &= E(T^2|T<\infty), \end{aligned}$$

where, by differentiating (4.3),

$$R_0'' = \frac{R_0'\left(2 + 4cR_0' + 2c^2(R_0')^2 - 2\beta^2\alpha(\alpha - R_0)^{-3}(R_0')^2\right)}{2(\beta + cR_0)}.$$

Table 4.1 shows the functions $E(T|T<\infty)$ and $V(T|T<\infty)$ for three values of c when $\alpha = 1$ and $\beta = 2$. We make the following comments about the results for this model:

(1) The solution for ϕ is of the same form under our model as it is under the classical risk model. Consequently, the form of our expressions for the first two moments of $T|T<\infty$ is the same as under the classical risk model. See Willmot and Lin (2000) for details.

(2) The formulae in Table 4.1 show that as c increases, both $E(T|T<\infty)$ and $V(T|T<\infty)$ decrease. As we increase c from 1.1 to 1.5 the probability or ruin decreases, but if ruin occurs, it is likely to occur sooner when $c = 1.5$ than when $c = 1.1$.

(3) In the previous section, we derived a general expression for $\phi(0)$ in terms of the positive roots of Lundberg's fundamental equation. For this choice of P, $\phi(0)$ can also be expressed in terms of the negative root. We have

$$\phi(0) = \frac{\beta^2}{c^2(\alpha + r_1)(\alpha + r_2)} = 1 - R/\alpha.$$

4.2 Mixed exponential individual claims

We now consider the case when P is a mixture of two exponential distributions. The method of solution for ϕ also applies for a mixture of more than two exponential distributions. We have chosen the simplest mixed exponential distribution simply to illustrate ideas. A full description of the techniques applicable in the general case is given by Dickson and Gray (1984). Their arguments for the classical risk model also apply to our model.

Result 4.2 *Let*

$$P(x) = 1 - \theta \exp\{-\alpha_1 x\} - (1-\theta)\exp\{-\alpha_2 x\},\ x \geq 0,$$

where $0 < \theta < 1$. *Then*

$$\phi(u) = \nu_{1,\delta} e^{-R_{1,\delta} u} + \nu_{2,\delta} e^{-R_{2,\delta} u}$$

where $-R_{1,\delta}(= -R)$ *and* $-R_{2,\delta}$ *are the negative roots of Lundberg's fundamental equation, and* $\nu_{1,\delta}$ *and* $\nu_{2,\delta}$ *are functions of* δ *satisfying*

$$\frac{\nu_{1,\delta}}{\alpha_k - R_{1,\delta}} + \frac{\nu_{2,\delta}}{\alpha_k - R_{2,\delta}} = \frac{1}{\alpha_k} \tag{4.4}$$

for $k = 1, 2$.

Proof. The proof follows by exactly the same arguments as in the case of exponential claims. Once again ϕ satisfies a differential equation whose characteristic equation has the same roots as Lundberg's fundamental equation. ■

We can solve for the moments of T following the method in the previous subsection. We have

$$\frac{d}{d\delta}\phi(u) = \sum_{i=1}^{2} \left(\nu'_{i,\delta} - \nu_{i,\delta} R'_{i,\delta} u\right) e^{-R_{i,\delta} u},$$

so that

$$E(T|T<\infty) = \frac{1}{\psi(u)} \sum_{i=1}^{2} \left(\nu_{i,0} R'_{i,0} u - \nu'_{i,0} \right) e^{-R_{i,0} u}.$$

Differentiating Lundberg's fundamental equation with respect to δ, then setting $\delta = 0$ gives

$$R'_{i,0} = \frac{2(\beta + cR_{i,0})}{\beta^2 \left(\theta\alpha_1(\alpha_1 - R_{i,0})^{-2} + (1-\theta)\alpha_2(\alpha_2 - R_{i,0})^{-2}\right) - 2c^2 R_{i,0} - 2\beta c}$$

for $i = 1, 2$. By differentiating (4.4) we find that

$$\sum_{i=1}^{2} \left(\frac{\nu'_{i,0}}{\alpha_k - R_{i,0}} + \frac{\nu_{i,0} R'_{i,0}}{(\alpha_k - R_{i,0})^2} \right) = 0$$

for $k = 1, 2$. Thus, we can solve for $\nu'_{1,0}$ and $\nu'_{2,0}$, and hence we can find $E(T|T<\infty)$.

Expressions for higher moments of $T|T<\infty$ can be found by further differentiation of ϕ, Lundberg's fundamental equation and (4.4).

Example 4.1 *Let $\theta = 0.25$, $\alpha_1 = 0.25$, $\alpha_2 = 0.75$, $\beta = 2$ and $c = 1.5$. Then we find the following:*

$R_{1,0} = 0.0824$	$R_{2,0} = 1.2983$
$R'_{1,0} = 3.0749$	$R'_{2,0} = 0.1915$
$\nu_{1,0} = 0.7520$	$\nu_{2,0} = 0.0391$
$\nu'_{1,0} = -9.3612$	$\nu'_{2,0} = 1.5333$

This gives

$$E(T|T<\infty) = \frac{9.3612 + 2.3124u - e^{-1.2159u}\left(1.5333 - 0.0075u\right)}{0.7520 + 0.0391e^{-1.2159u}}.$$

A graph of this function shows that $E(T|T<\infty)$ is approximately linear in u for $u \geq 5$.

4.3 Zero initial surplus

When the initial surplus is zero, we can find moments of the time to ruin for any individual claim amount distribution by differentiating (3.7) with respect to δ. Consider

$$\zeta(\delta) = \frac{q^*(r_1) - q^*(r_2)}{r_2 - r_1}. \tag{4.5}$$

Then

$$\begin{aligned}\frac{d}{d\delta}\zeta(\delta) &= \frac{-1}{r_2 - r_1}\int_0^\infty (r_1' e^{-r_1 x} - r_2' e^{-r_2 x}) x q(x) dx \\ &\quad - \frac{q^*(r_1) - q^*(r_2)}{(r_2 - r_1)^2}(r_2' - r_1').\end{aligned}$$

If we insert $r_1 = r_{1,\delta}$ in Lundberg's fundamental equation, then differentiate with respect to δ, we find that

$$r_{1,0}' = \left.\frac{d}{d\delta} r_{1,\delta}\right|_{\delta=0} = \frac{1}{c - 0.5\beta m_1}.$$

Similarly, we find that (in an obvious notation)

$$r_{2,0}' = \frac{2(\beta - c s_0)}{2\beta c - 2c^2 s_0 - \beta^2 \int_0^\infty x e^{-s_0 x} p(x) dx}.$$

It therefore follows that

$$\begin{aligned}E(T|T<\infty) &= \frac{\beta^2 m_1}{c^2 \psi(0)}\left[\frac{1}{s_0}\left(r_{1,0}'\mu_1 - r_{2,0}'\int_0^\infty x e^{-s_0 x} q(x) dx\right)\right. \\ &\quad \left. + \frac{1}{s_0^2}\left(1 - q^*(s_0)\right)\left(r_{2,0}' - r_{1,0}'\right)\right].\end{aligned}$$

We can find $\psi(0)$ from formula (3.3) of Dickson and Hipp (1998). Alternatively,

$$\psi(0) = \lim_{\delta \to 0^+} \phi(0) = \frac{\beta^2 m_1}{c^2}\frac{1 - q^*(s_0)}{s_0}.$$

Similarly, we find that

$$r_{1,0}'' = \frac{2r_{1,0}' - 4c(r_{1,0}')^2 + (r_{1,0}')^3\left(2c^2 - \beta^2 m_2\right)}{2\beta}$$

and

$$r_{2,0}'' = \frac{2r_{2,0}' - 4c(r_{2,0}')^2 + (r_{2,0}')^3\left[2c^2 - \beta^2\int_0^\infty x e^{-s_0 x} p(x) dx\right]}{2(\beta - c s_0)}.$$

c	$E(T\|T<\infty)$	$V(T\|T<\infty)$	s_0
1.1	16.58	4901	2.822
1.3	5.721	245.5	2.430
1.5	3.540	67.45	2.137

Table 4.2: Table Mean and variance of $T|T<\infty$, Pareto claims, $u=0$

Differentiating (4.5) a second time leads to

$$\begin{aligned} E(T^2|T<\infty) &= \frac{\beta^2 m_1}{c^2 s_0 \psi(0)} \Bigg[\frac{2(r'_{2,0}-r'_{1,0})}{s_0} \left(r'_{1,0}\mu_1 - r'_{2,0}\int_0^\infty x e^{-s_0 x} q(x)dx \right) \\ & \quad + (r'_{1,0})^2 \mu_2 - r''_{1,0}\mu_1 + r''_{2,0}\int_0^\infty x e^{-s_0 x} q(x)dx \\ & \quad - (r'_{2,0})^2 \int_0^\infty x^2 e^{-s_0 x} q(x) dx \\ & \quad + \left(\frac{2(r'_{2,0}-r'_{1,0})^2}{s_0^2} - \frac{r''_{2,0}-r''_{1,0}}{s_0} \right)(1-q^*(s_0)) \Bigg]. \end{aligned}$$

Example 4.2 *Let p be the Pareto density with parameters 4 and 3, so that q is the Pareto density with parameters 3 and 3. Given a value for c, we can solve numerically for s_0. Table 4.2 shows the functions $E(T|T<\infty)$ and $V(T|T<\infty)$ and the value of s_0 for the same three values of c as in Table 4.1, again with $\beta=2$.*

5 Concluding remarks

In this paper we have shown that some well-known techniques can be used to solve for the function ϕ, of which the function ψ is a special case. In cases where we can solve explicitly for ϕ we can then also solve for the moments of the time to ruin. This differs from the method of Willmot and Lin (2000), who consider the classical risk model. Their method requires an explicit solution for the ultimate ruin probability in order to obtain explicit solutions for the moments of the time to ruin. The approach presented in this paper also applies to the classical risk model.

In the case of the classical risk model, the function corresponding to ϕ satisfies a defective renewal equation - see Gerber and Shiu (1998). The same is true for a classical risk model perturbed by a diffusion process - see Gerber and Landry (1998). However, the techniques presented in these papers do not seem to lead to a defective renewal equation for ϕ. Curiously, ϕ appears to satisfy an excessive renewal equation. This results from inversion of ϕ^*.

However, at least in the special case $\delta = 0$, we know that ϕ does satisfy a defective renewal equation.

In the classical risk model, the defective renewal equation for the function corresponding to ϕ is the starting point for the analysis by Willmot and Lin (1999, 2000). This approach allows them to derive results such as the covariance of the time to ruin and the deficit at ruin in a unified manner. It is not apparent that their approach readily extends to our model. However, if we define ϕ by (1.2) rather than (1.1), we may be able to derive explicit solutions to the sort of problems they consider.

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