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On the Topological Orbit Equivalence in a Class of Substitution Minimal Systems

Hisatoshi YUASA

Keio University

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Abstract. We give numerical, complete invariants for topological orbit equivalence and Kakutani orbit equivalence in a class of substitution systems arising from primitive substitutions whose composition matrices have rational Perron-Frobenius eigenvalues.

1. Introduction.

In a series of the papers [4], [6], [8], it was shown that several topological orbit structures of Cantor minimal systems are completely classified in terms of associated dimension groups which are purely algebraic objects. On the other hand, dimension groups of Cantor minimal systems arinsing from primitive substitutions turned out in [1, 3] to be accesible in quite concrete forms. These results may leads to an attempt to investigate classifications up to topological orbit structures of substitution systems arising from primitive substitutions. In the present paper, we consider those classifications up to two kinds of topological orbit structures, which are restricted to the class of substitution systems arising from primitive substitutions whose composition matrices have rational Perron-Frobenius eigenvalues. Such restriction is not essential on the classification of substitution systems concerning topological orbit structures, i.e., a substitution system arising from a substitution whose composition matrix has a rational Perron-Frobenius eigenvalue never has the same topological orbit structures as one arising from a substitution whose composition matrix has an irrational Perron-Frobenius eigenvalue. In the class of substitution systems arising from substitutions whose composition matrices have rational Perron-Frobenius eigenvalues, we will find out numerical complete invariants for orbit equivalence and Kakutani orbit equivalence which are concretely obtained from information associated with given substitutions.

In Sections 2 through 5, several known facts and results are briefly summarized; we would like to refer the reader to [9] for details in Section 2; [1] for details in Sections 3 and 4; [4] for details in Section 5.

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2. Cantor minimal systems.

2.1. Stationary odometer systems. We say that (X, ϕ) is a *Cantor minimal system* if X is a *Cantor set*, i.e., a totally disconnected compact metric space without isolated points; equivalently, X has a countable basis of clopen (closed and open) sets and has no isolated points, and furthermore ϕ is a *minimal* homeomorphism on X, i.e., every orbit $\mathrm{Orb}_\phi(x) := \{\phi^n(x) : n \in \mathbf{Z}\}$ of ϕ is dense in X. A typical example of a Cantor minimal system is the so-called odometer system. Put $X = \prod_{i=1}^{\infty} X_i$ where $X_i = \{0, 1, \cdots, n_i - 1\}$, $n_i \geq 2$ and $i \geq 1$. The infinite product space X is a compact abelian group by addition with carries to the right. We define $\phi : X \to X$ by addition of $(1, 0, 0, \cdots) \in X_1 \times \prod_{i=2}^{\infty} X_i$ and call (X, ϕ) the *odometer system* with base $(n_1, n_2, n_3, \cdots)$. Then clearly (X, ϕ) is a uniquely ergodic Cantor minimal system with the unique Haar measure. In particular, when there exists $i_0 \geq 1$ such that $n_i = n_{i+1}$ for all $i \geq i_0$, we call (X, ϕ) a *stationary* odometer system.

2.2. Substitution systems. Let A be a finite set such that $|A| \geq 2$ and call it an *alphabet* and call elements of A *letters*. A *word* on A is a finite (possibly empty) sequence of letters. The *length*, as a sequence of letters, of a word u is denoted by $|u|$. We call a map $\sigma : A \to A^+$(the set of nonempty words on A) a *substitution* on A. We extend σ to A^+ or $A^{\mathbf{Z}}$ by concatenations in the following way. For $u = u_1 u_2 \cdots u_k \in A^+$,

$$\sigma(u) = \sigma(u_1)\sigma(u_2)\cdots\sigma(u_k)\,.$$

For $x = \cdots x_{-2}x_{-1}.x_0x_1x_2\cdots \in A^{\mathbf{Z}}$,

$$\sigma(x) = \cdots\sigma(x_{-2})\sigma(x_{-1}).\sigma(x_0)\sigma(x_1)\sigma(x_2)\cdots\,,$$

where the dot means the fixed separation between the -1-st coordinate and the 0-th coordinate. We say that a word u is a *factor* of a word $v = v_1 \cdots v_j$, $v_i \in A$ if $u = v_l \cdots v_k$ for some $1 \leq l \leq k \leq j$. We denote by $\mathcal{L}(\sigma)$ the *language* of σ, i.e., the set of words on A which are factors of $\sigma^n(a)$ for some integer $n \geq 1$ and letter $a \in A$.

DEFINITION 2.1. (1) The *composition matrix* $M(\sigma)$ of a substitution σ on an alphabet A is an $A \times A$ matrix whose (a, b)-entry is the number of occurrences of b in $\sigma(a)$ for $a, b \in A$.

(2) A substitution σ is said to be of *constant length* if $|\sigma(a)|$ is constant for each letter a.

(3) A substitution σ on A is said to be *primitive* if there exists some integer $n \geq 1$ such that a occurs in $\sigma^n(b)$ for every $a, b \in A$.

A substitution σ is primitive if and only if $M(\sigma)$ is *primitive*, i.e., there exists an integer $n \geq 1$ such that every element of $M(\sigma)^n$ is strictly positive.

Throughout the present paper, we always assume that any substitution σ on an alphabet A has the following properties:

(A) For every $a \in A$, it holds that $\lim_{n\to\infty} |\sigma^n(a)| = +\infty$;

(B) there never exists an integer $n_0 \geq 1$ such that for any integer $n \geq n_0$ $\mathcal{L}_n(\sigma) = \mathcal{L}_{n+1}(\sigma)$, where $\mathcal{L}_n(\sigma) = \{u \in \mathcal{L}(\sigma) : |u| = n\}$.

A substitution σ is said to be *aperiodic* if Condition (B) is satisfied. These properties ensure the well-definedness and the non-triviality of a topological dynamical system arising from a substitution, which is defined in a moment.

Let σ be a substitution on alphabet A. As $A \times A$ is a finite set, we can show that there exist two letters $a, b \in A$ and an integer $n \geq 1$ such that

(1) b is the last letter of $\sigma^n(b)$;

(2) a is the first letter of $\sigma^n(a)$;

(3) $ba \in \mathcal{L}(\sigma)$.

Then, there uniquely exists an $x \in A^{\mathbf{Z}}$ with the following properties:

(1) $x_{-1} = b, x_0 = a$;

(2) $\sigma^n(x) = x$.

We refer to such an $x \in A^{\mathbf{Z}}$ as an *admissible fixed point* of σ^n. We define

$$X_\sigma = \{x \in A^{\mathbf{Z}} : \text{every factor of } x \text{ belongs to } \mathcal{L}(\sigma)\},$$

which is well-defined thanks to Property (A). Since X_σ is a closed subset of $A^{\mathbf{Z}}$ and Condition (B) ensures that X_σ is an infinite set, X_σ is a Cantor set and invariant for the shift T where T is the full shift on $A^{\mathbf{Z}}$ defined by $(Tx)_i = x_{i+1}$, $i \in \mathbf{Z}$. We denote the restricted action of T on X_σ by T_σ. The dynamical system (X_σ, T_σ) is called the *substitution system* arising from a substitution σ. When σ is primitive, $X_\sigma = \overline{\mathrm{Orb}_T(x)}$ for any admissible fixed point x of σ^n because it holds that $\mathcal{L}(\sigma) = \mathcal{L}(\sigma^n)$ for every $n \geq 1$. It is well known that the substitution system arising from a primitive substitution is minimal and uniquely ergodic, see [9, Theorems V.2 and V.13].

3. Induced systems and return words.

DEFINITION 3.1. Let σ be a substitution on an alphabet A. We call a word w on A a *return word* to $b.a$ if the following properties are satisfied:

(1) $bwa \in \mathcal{L}(\sigma)$;

(2) a is the first letter of w and b is the last letter of w;

(3) the word ba never occurs in the word w.

REMARK 3.2. (1) The number of the return words to $b.a$ is finite because every $u \in \mathcal{L}(\sigma)$ occurs in any admissible fixed point of σ^n with a bounded gap, see the proof of [9, Theorem V. 2];

(2) the length of a return word w to $b.a$ represents the first return time to the cylinder set $[b.a] = \{x \in X_\sigma \mid x_{-1} = b,\ x_0 = a\}$ of the points in the cylinder set $[b.wa] = \{x \in X_\sigma \mid x_{[-1,|w|]} = bwa\}$;

(3) the set of return words to $b.a$ is a 'circular code', see [1, Lemma 17].

The following property of a substitution is a significant concept in order to construct a 'properly ordered Bratteli diagram' whose 'Bratteli-Vershik system' is topologically conjugate to a substitution system arising from a given primitive substitution, see [1].

DEFINITION 3.3. A substitution σ on an alphabet A is said to be *proper* if there exists an integer $n > 0$ and two letters $a, b \in A$ such that
(1) for every $c \in A$, b is the last letter of $\sigma^n(c)$;
(2) for every $c \in A$, a is the first letter of $\sigma^n(c)$.

REMARK 3.4. Every proper substitution has a unique admissible fixed point. Every substitution system arising from a primitive substitution is topologically conjugate to the substitution system arising from some proper substitution, see [1, Lemma 9] or [3, Lemma 15].

Let σ be a non-proper and primitive substitution on an alphabet A. Then, there exist letters $a, b \in A$ and an integer $k \geq 1$ such that
(1) a is the first letter of $\sigma^k(a)$;
(2) b is the last letter of $\sigma^k(b)$;
(3) $ba \in \mathcal{L}(\sigma)$.
Let $\{w_1, \cdots, w_r\}$ be the set of return words to $b.a$. We define a substitution τ on the alphabet $R = \{1, \cdots, r\}$ by

$$\tau(i) = l_1 \cdots l_j, \text{ if } \sigma^k(w_i) = w_{l_1} \cdots w_{l_j} .$$

PROPOSITION 3.5 ([1, Lemma 21]). *The substitution τ is a proper, primitive substitution whose substitution system (X_τ, T_τ) is topologically conjugate to the induced system on $[b.a]$ by T_σ.*

4. Properly ordered Bratteli diagrams associated with substitution systems.

In this section, we would like to describe how we should construct a properly ordered Bratteli diagram whose Bratteli-Vershik system is topologically conjugate to the substitution system arising from a given primitive substitution. Such realizations as transformations on diagrams of substitution systems will be useful in the proof of Proposition 6.4.

4.1. For proper substitutions. Let σ be a proper and primitive substitution on an alphabet A. Then, we construct an infinite graph (V, E) called a *Bratteli diagram*, where $V = \bigcup_{i=0}^{\infty} V_i$ is the vertex set and $E = \bigcup_{i=1}^{\infty} E_i$ is the edge set, so that
(1) V_0 is a singleton;
(2) $V_i = A$ for every $i \geq 1$;
(3) for each $a \in V_1$, a single edge connects the top vertex v_0 to a, where $V_0 = \{v_0\}$;
(4) each edge $e \in E_i$ connects a vertex in V_i to a vertex in V_{i-1} for every $i \geq 2$;
(5) there exists an edge in E_i which connects $a \in V_i$ to $b \in V_{i-1}$ if and only if b occurs in $\sigma(a)$ for every $i \geq 2$;

(6) the number of edges which connects $a \in V_i$ to $b \in V_{i-1}$ is equal to that of occurrences of b in $\sigma(a)$ for every $i \geq 2$.
We define maps $r, s : E \to V$ by

$$r(e) = v, s(e) = u \text{ for } e \in E_i \text{ if } e \text{ connects } v \in V_{i+1} \text{ to } u \in V_i .$$

Furthermore, we introduce a partial order $\leq$ on the edge set E so that

(7) $e_1, e_2 \in E$ are comparable if and only if $r(e_1) = r(e_2)$ in the same V_i;

(8) if $\sigma(a) = s(e_1)s(e_2)\cdots s(e_k)$, then $e_1 < e_2 < \cdots < e_k$, where $\{e_1, e_2, \cdots, e_k\} = r^{-1}\{a\}$, $a \in V_i$, $i \geq 2$.

We call the infinite graph $\mathcal{B}_\sigma = (E, V, \leq)$ with partial order a *properly ordered Bratteli diagram* associated with a substitution σ. We define

$$X_{\mathcal{B}_\sigma} = \left\{ (e_1, e_2, \cdots) \in \prod_{i=1}^{\infty} E_i : r(e_i) = s(e_{i+1}) \text{ for every } i \geq 1 \right\},$$

which is topologized so as to be a Cantor set by declaring that the collection of all the cylinder sets $[e_1, e_2, \cdots, e_k]$ defined as follows is an open basis:

$$[e_1, e_2, \cdots, e_k] = \{(f_1, f_2, \cdots) \in X_{\mathcal{B}_\sigma} : f_i = e_i \text{ for every } 1 \leq i \leq k\}$$

where $e_i \in E_i$ for every $1 \leq i \leq k$ and $r(e_i) = s(e_{i+1})$ for every $1 \leq i < k$. Now, we define the so-called *Vershik map* $V_{\mathcal{B}_\sigma}$ on $X_{\mathcal{B}_\sigma}$ as follows. If $x = (x_1, x_2, \cdots) \in X_{\mathcal{B}_\sigma}$ has an integer $i \geq 1$ such that x_i is not maximal in $r^{-1}r(x_i)$ we define $V_{\mathcal{B}_\sigma}(x) = (x'_1, x'_2, \cdots, x'_i, x_{i+1}, x_{i+2}, \cdots)$ where x'_i is the immediate successor of x_i in $r^{-1}r(x_i)$ and x'_j is the unique minimal edge in $r^{-1}(s(x'_{j+1}))$ for every $1 \leq j < i$. For $x = (x_1, x_2 \cdots) \in X_{\mathcal{B}}$ whose entry x_i is maximal in $r^{-1}r(x_i)$ for every $i \geq 1$, we define $V_{\mathcal{B}_\sigma}(x)$ to be the element in $X_{\mathcal{B}_\sigma}$ whose entries are all minimal. The map $V_{\mathcal{B}_\sigma} : X_{\mathcal{B}_\sigma} \to X_{\mathcal{B}_\sigma}$ turns out to be a homeomorphism which is topologically conjugate to (X_σ, T_σ), see [1, Proposition 20]. We call $(X_{\mathcal{B}_\sigma}, V_{\mathcal{B}_\sigma})$ the *Bratteli-Vershik system* associated with $\mathcal{B}_\sigma$.

4.2. For non-proper substitutions. Let σ be a non-proper and primitive substitution on an alphabet A, and let letters $a, b \in A$, an integer $k \geq 1$, $\{w_1, \cdots, w_r\}$ and τ be as in Section 3. From the previous subsection, we immediately obtain a properly ordered Bratteli diagram $\mathcal{B}_\tau$, whose associated Bratteli-Vershik system $(X_{\mathcal{B}_\tau}, V_{\mathcal{B}_\tau})$ is topologically conjugate to (X_τ, T_τ). As (X_τ, T_τ) is topologically conjugate to an induced system on $[b.a]$ by T_σ, we define a *properly ordered Bratteli diagram* $\mathcal{B}_\sigma$ associated with a substitution σ, so that associated Bratteli-Vershik system $(X_{\mathcal{B}_\sigma}, T_{\mathcal{B}_\sigma})$ is topologically conjugate to (X_σ, T_σ), by means of changing the single edge connecting a vertex $i \in R$ in the first level to the top vertex into $|w_i|$ edges. This operation corresponds to the so-called tower construction in ergodic theory. A partial order on the edge set of $\mathcal{B}_\sigma$, except the edge set of the first level, keeps that of $\mathcal{B}_\tau$ and it is trivial how a partial order on the first level edge set of $\mathcal{B}_\sigma$ should be given. In the same way as in the previous subsection, $V_{\mathcal{B}_\sigma}$ is defined on the infinite path space $X_{\mathcal{B}_\sigma}$. For either case when a given substitution σ is proper or non-proper, we can say that

THEOREM 4.1 ([1, Proposition 20]). *$(X_{B_\sigma}, V_{\mathcal{B}_\sigma})$ is topologically conjugate to (X_σ, T_σ).*

5. Dimension groups of Cantor minimal systems.

Let (X, ϕ) be a Cantor minimal system. We denote by $C(X, \mathbf{Z})$ the abelian group of integer valued continuous functions on X with pointwise addition and define $C(X, \mathbf{Z}_+) = \{f \in C(X, \mathbf{Z}) | f \geq 0\}$, $B_\phi = \{f \circ \phi - f : f \in C(X, \mathbf{Z})\}$ and $M_\phi = \{\phi$-invariant regular Borel probability measure$\}$.

DEFINITION 5.1. The *dimension group modulo the coboundary subgroup* of a Cantor minimal system (X, ϕ) is the pair of an abelian group $K^0(X, \phi)$ and its distinguished subset $K^0_+(X, \phi)$ which are defined by

$$K^0(X, \phi) = C(X, \mathbf{Z})/B_\phi;\ \ K^0_+(X, \phi) = C(X, \mathbf{Z}_+)/B_\phi\,.$$

DEFINITION 5.2. The *dimension group modulo the infinitesimal subgroup* of a Cantor minimal system (X, ϕ) is the pair of an abelian group $\tilde{K}^0(X, \phi)$ and its distinguished subset $\tilde{K}^0_+(X, \phi)$ which are defined by

$$\tilde{K}^0(X, \phi) = C(X, \mathbf{Z})/\operatorname{Inf}(X, \phi);\ \ \tilde{K}^0_+(X, \phi) = C(X, \mathbf{Z}_+)/\operatorname{Inf}(X, \phi)\,,$$

where $\operatorname{Inf}(X, \phi) = \{f \in C(X, \mathbf{Z}) : \int_X f d\mu = 0$ for every $\mu \in M_\phi\}$.

Let (G, G_+) be a dimension group arising from a Cantor minimal system (X, ϕ) in the sense of Definitions 5.1 or 5.2. We say that an element u of G_+ is an *order unit* if for every $a \in G_+$ there is an integer $n \geq 1$ such that $a \leq nu$, which means $nu - a \in G_+$. Let $[\chi_X]$ be the equivalence class of the characteristic function χ_X taking value 1 everywhere on X. We call $[\chi_X]$ the *distinguished order unit* of G. For dimension groups (G, G_+) and (H, H_+) arising from Cantor minimal systems in the sense of Definitions 5.1 or 5.2, a group isomorphism $\vartheta : G \to H$ is called an *order isomorphism* if $\vartheta(G_+) = H_+$.

A *state* p on a dimension group $\big(K^0(X, \phi), K^0_+(X, \phi)\big)$ is a group homomorphism from $K^0(X, \phi)$ to $\mathbf{R}$ such that $p(K^0_+(X, \phi)) \subset \mathbf{R}_+$ and $p([\chi_X]) = 1$. Denote by $\mathcal{S}$ the set of states on $(K^0(X, \phi), K^0_+(X, \phi))$. Then, there exists a bijection between M_ϕ and $\mathcal{S}$ which is defined so that each $\mu \in M_\phi$ is mapped to the state defined by $[f] \mapsto \int f d\mu$ for each $f \in C(X, \mathbf{Z})$, see [6, Theorem 5.5]. The inverse map of the bijection sends each $p \in \mathcal{S}$ to the unique $\mu \in M_\phi$ which satisfies that $\mu(A) = p([\chi_A])$ for any clopen set $A \subset X$.

Let (X, ϕ) be a Cantor minimal system with a unique invariant probability measure μ. As seen above, the unique state p on $K^0(X, \phi)$ is defined by $p([f]) = \int_X f d\mu$ for each $f \in C(X, \mathbf{Z})$. Then, we have that

$$\ker(p) = \left\{[f] \in K^0(X, \phi) : \int_X f d\mu = 0\right\} = \operatorname{Inf}(X, \phi)/B_\phi\,.$$

Since B_ϕ is a subgroup of $\operatorname{Inf}(X, \phi)$, it follows that $K^0(X, \phi)/\ker(p)$ is order isomorphic to $\tilde{K}^0(X, \phi)$ by an order isomorphism sending $[\chi_X] + \ker(p)$ to $[\chi_X] \in \tilde{K}^0(X, \phi)$. i.e., there exists a group isomorphism $\vartheta : K^0(X, \phi)/\ker(p) \to \tilde{K}^0(X, \phi)$ such that $\vartheta(K^0_+(X, \phi)/\ker(p)) = \tilde{K}^0_+(X, \phi)$ and $\vartheta([\chi_X] + \ker(p)) = [\chi_X]$. It also follows from the homomorphism theorem that $K^0(X, \phi)/\ker(p)$ is order isomorphic to $\mathrm{Im}(p) = \langle \mu(E) : E \subset X \text{ is clopen}\rangle$ by an order isomorphism sending $[\chi_X] + \ker(p)$ to 1. Finally, we have that

LEMMA 5.3. *If (X, ϕ) is a uniquely ergodic Cantor minimal system with a unique ϕ-invariant probability measure μ, then $(\tilde{K}^0(X, \phi), \tilde{K}^0_+(X, \phi), [\chi_X])$ is order isomorphic to $(\langle \mu(E) : E \subset X \text{ is clopen}\rangle, \langle \mu(E) : E \subset X \text{ is clopen}\rangle \cap \mathbf{R}_+, 1)$ by an order isomorphism preserving the order units.*

6. Results.

The following version of the Perron-Frobenius Theorem is necessary for the subsequent arguments.

THEOREM 6.1 (a portion of [7, Theorem 4.2.3]). *Let A be a nonnegative and primitive matrix. Then, A has a positive eigenvector v_A with corresponding eigenvalue $\lambda_A > 0$ which is simple. Any positive eigenvector of A is a positive multiple of v_A.*

We call λ_A the *Perron-Frobenius eigenvalue* of A and v_A a *Perron-Frobenius eigenvector* of A. The following is a known fact.

LEMMA 6.2. *Assume that a nonnegative, integral and primitive matrix M has a rational eigenvalue λ. Then, λ is integral and $\lambda > 1$ if λ is the Perron-Frobenius eigenvalue of M.*

PROOF. Put $\lambda = p/q$ where $p, q > 0$ are relatively prime integers. Let $f(x) = x^s + a_{s-1}x^{s-1} + \cdots + a_1 x + a_0$ be the characteristic polynomial of $M(\sigma)$, where s is the size of M. Since $f(\lambda) = 0$,

$$p^s + a_{s-1}p^{s-1}q + a_{s-2}p^{s-2}q^2 + \cdots + a_1 pq^{s-1} + a_0 q^s = 0\,.$$

Therefore, we have that

$$(a_{s-1}p^{s-1} + a_{s-2}p^{s-2}q + \cdots + a_1 pq^{s-2} + a_0 q^{s-1})q = -p^s\,.$$

Since $a_{s-1}p^{s-1} + a_{s-2}p^{s-2}q + \cdots + a_1 pq^{s-2} + a_0 q^{s-1}$ is integral, q divides p^s. This forces that $p = 1$ and hence λ is integral.

Furthermore, suppose that λ is the Perron-Frobenius eigenvalue of M. To the contrary, assume that $\lambda = 1$. Then, it follows from the Perron-Frobenius Theorem that there exists a vector $v > 0$ with $Mv = v$. Since M is primitive, there exists an integer $k \geq 1$ such that each entry of M^k is a strictly positive integer. Hence, $v_i = \sum_{j=1}^{s} M^k_{ij} v_j > v_i$ for $1 \leq i \leq r$, which is a contradiction. □

6.1 Invariant probability measure.

LEMMA 6.3. *Let σ be a non-proper, primitive substitution on an alphabet A and λ be the Perron-Frobenius eigenvalue of $M(\sigma)$. Let $k \geq 1$ and τ be as in Section 3. Then, the Perron-Frobenius eigenvalue of $M(\tau)$ is λ^k.*

PROOF. Let $\{w_1, \cdots, w_r\}$ and R as in Section 3 and put $s = |A|$. Define an $R \times A$ matrix S by

$$S_{i,c} = \text{ the number of occurrences of } c \text{ in } w_i\,, \text{ for every } i \in R \text{ and } c \in A\,.$$

It is verified that each (i, c)-entry of both $M(\tau)S$ and $SM(\sigma)^k$ are the number of occurrences of c in $\sigma^k(w_i)$ and we hence have that $M(\tau)S = SM(\sigma)^k$. Let $v \in \mathbf{R}^s_+$ be a right Perron-Frobenius eigenvector of $M(\sigma)$ whose entries are all strictly positive and put $u = Sv \in \mathbf{R}^r_+$. Then, we have that

$$M(\tau)u = M(\tau)Sv = SM(\sigma)^k v = \lambda^k Sv = \lambda^k u\,.$$

It follows from the Perron-Frobenius Theorem that λ^k is the Perron-Frobenius eigenvalue of $M(\tau)$. □

PROPOSITION 6.4. *Let σ be a primitive substitution and λ be the Perron-Frobenius eigenvalue of $M(\sigma)$. Then,*

(1) *when σ is proper, it follows that*

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen} \rangle = \langle \alpha_a/(d \cdot \lambda^n) : a \in A, n \geq 0 \rangle,$$

where μ is the unique T_σ-invariant probability measure and $\alpha = (\alpha_a)_{a \in A} \in \mathbf{N}^s$ is a left Perron-Frobenius eigenvector of $M(\sigma)$ and $d = \sum_{a \in A} \alpha_a$;

(2) *when σ is non-proper, it follows that*

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen} \rangle = \langle \beta_i/(d \cdot \lambda^{kn}) : i \in R, n \geq 0 \rangle\,,$$

where k, R, β_i and d are defined as follows: let k, $\{w_1, \cdots, w_r\}$, R and τ be as in Section 3 and let $\beta = (\beta_i)_{i \in R} \in \mathbf{N}^r$ be a left Perron-Frobenius eigenvector of $M(\tau)$. Then, we define $d = \sum_{i \in R} |w_i| \beta_i$.

PROOF. As the Bratteli-Vershik system associated with a properly ordered Bratteli diagram $\mathcal{B}_\sigma$ constructed in Section 4 is topologically conjugate to (X_σ, T_σ) and each clopen subset of $X_{\mathcal{B}_\sigma}$ is a finite union of cylinder sets, it is enough to know the measures of cylinder sets of $X_{\mathcal{B}_\sigma}$ evaluated by the unique $V_{\mathcal{B}_\sigma}$-invariant probability measure in order to compute $\langle \mu(E) : E \subset X \text{ is clopen} \rangle$.

First, we shall consider the case when a given primitive substitution σ is proper. Let $\mathcal{B}_\sigma$ be as in Subsection 4.1, and put $s = |A|$. Suppose that there is a sequence $p^{(1)}, p^{(2)}, p^{(3)}, \cdots$ of row vectors in $\mathbf{R}^s_+$ which satisfies that

$$\sum_{a \in A} p^{(1)}_a = 1 \quad \text{and} \quad p^{(k+1)} M(\sigma) = p^{(k)} \quad \text{for every } k \geq 1\,. \qquad (*)$$

It follows from these conditions that there exists a finitely additive probability measure ν on the algebra $\mathcal{C}$ generated by all the cylinder sets of $X_{\mathcal{B}_\sigma}$, which is exactly the algebra of clopen sets, such that

$$\nu([e_1, e_2, \cdots, e_k]) = p^{(k)}_{r(e_k)} \quad \text{for every cylinder set } [e_1, e_2, \cdots, e_k] \subset X_{\mathcal{B}_\sigma}.$$

Since it holds that $\nu([e_1, e_2, \cdots, e_k]) = \nu([e'_1, e'_2, \cdots, e'_k])$ for any cylinder sets $[e_1, e_2, \cdots, e_k]$ and $[e'_1, e'_2, \cdots, e'_k]$ with $r(e_k) = r(e'_k)$, the finitely additive measure ν is $V_{\mathcal{B}_\sigma}$-invariant. Then, by the Hopf extension theorem, we can uniquely extend ν to a $V_{\mathcal{B}_\sigma}$-invariant probability measure on the Borel σ-algebra of $X_{\mathcal{B}_\sigma}$. The observation in the first paragraph shows that

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen}\rangle = \langle \nu(E) : E \subset X_{\mathcal{B}_\sigma} \text{ a cylinder set}\rangle. \qquad (**)$$

Now, we construct the so-called 'Markov measure' ν on the Borel σ-algebra of $X_{\mathcal{B}_\sigma}$ which is invariant under $V_{\mathcal{B}_\sigma}$. Define an 'initial distribution' $\{p_a : a \in A\}$ by $p_a = \alpha_a/d$ and a 'transition probability' $\{p_{a,b} : a, b \in A\}$ by $p_{a,b} = 1/\lambda \cdot \alpha_b/\alpha_a$. Then, we define

$$\begin{aligned}\nu([e_1, e_2, \cdots, e_k]) &= p_{r(e_1)} p_{r(e_1),r(e_2)} \cdots p_{r(e_{k-1}),r(e_k)} \\ &= \frac{\alpha_{r(e_1)}}{d} \frac{\alpha_{r(e_2)}}{\lambda \cdot \alpha_{r(e_1)}} \frac{\alpha_{r(e_3)}}{\lambda \cdot \alpha_{r(e_2)}} \cdots \frac{\alpha_{r(e_k)}}{\lambda \cdot \alpha_{r(e_{k-1})}} \\ &= \frac{\alpha_{r(e_k)}}{d \cdot \lambda^{k-1}} \quad \text{for every cylinder set } [e_1, e_2, \cdots, e_k] \subset X_{\mathcal{B}_\sigma}.\end{aligned}$$

The sequence $\{p^{(k)}\}_{k \geq 1}$ of row vectors defined by $p^{(k)}_a = \alpha_a/(d \cdot \lambda^{k-1})$ for each $a \in A$ and $k \geq 1$ satisfies Property $(*)$ and, as we observed above, we consequently obtain a $V_{\mathcal{B}_\sigma}$-invariant probability measure ν such that $\nu([e_1, e_2, \cdots, e_k]) = \alpha_{r(e_k)}/(d \cdot \lambda^{k-1})$ for every cylinder set $[e_1, e_2, \cdots, e_k] \subset X_{\mathcal{B}_\sigma}$. It follows from $(**)$ that

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen}\rangle = \left\langle \frac{\alpha_a}{d \cdot \lambda^i} : i \geq 0, a \in A \right\rangle.$$

This completes the proof for the case when a given substitution σ is proper.

Next, we consider the case when σ is non-proper. Let $\mathcal{B}_\sigma$ be as in Subsection 4.2 and $k \geq 1$ be as in Section 3. Define an initial distribution $\{p_i : i \in R\}$ by $p_i = \beta_i/d$ and a transition probability $\{p_{i,j} : i, j \in R\}$ by $p_{i,j} = 1/\lambda^k \cdot \beta_j/\beta_i$. Then, we define

$$\begin{aligned}\nu([e_1, e_2, \cdots, e_l]) &= p_{r(e_1)} p_{r(e_1),r(e_2)} \cdots p_{r(e_{l-1}),r(e_l)} \\ &= \frac{\beta_{r(e_l)}}{d \cdot \lambda^{k(l-1)}} \quad \text{for every cylinder set } [e_1, e_2, \cdots, e_l] \subset X_{\mathcal{B}_\sigma}.\end{aligned}$$

The sequence $\{p^{(l)}\}_{l \geq 1}$ of row vectors defined by $p^{(l)}_i = \beta_i/(d \cdot \lambda^{k(l-1)})$ for each $i \in R$ and $l \geq 1$ satisfies that

$$\sum_{i \in R} |w_i| p^{(1)}_i = 1 \quad \text{and} \quad p^{(l+1)} M(\tau) = p^{(l)} \quad \text{for every } l \geq 1$$

because of Lemma 6.3 and, by the same argument as in the above case when σ is proper, we hence obtain a $V_{\mathcal{B}_\sigma}$-invariant probability measure ν on the Borel σ-algebra of $X_{\mathcal{B}_\sigma}$ such that

$$\nu([e_1, e_2, \cdots, e_l]) = \beta_{r(e_l)}/(d \cdot \lambda^{k(l-1)}) \quad \text{for every cylinder set } [e_1, e_2, \cdots, e_l] \subset X_{\mathcal{B}_\sigma} .$$

Finally, we have that

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen } \rangle = \left\langle \frac{\beta_i}{d \cdot \lambda^{kn}} : i \in R, n \geq 0 \right\rangle .$$

This completes the proof. □

DEFINITION 6.5. Let σ be a primitive substitution on an alphabet A whose composition matrix $M(\sigma)$ has a rational Perron-Frobenius eigenvalue λ. We define a subset $\Gamma(\sigma)$ of $\mathbf{N}$ by

$$\Gamma(\sigma) = \{\text{factor of } d \cdot \lambda^n \text{ for some } n \geq 1\} ,$$

where a natural number d is defined as in Proposition 6.4.

REMARK 6.6. Obviously, there is no ambiguity for the definition of d in the case when a given substitution σ is proper in the above definition. On the other hand, in the case when a given substitution σ is non-proper, it is verified from the following corollary that $\Gamma(\sigma)$ does not depend on choices of $a, b \in A$ and $k \geq 1$ although d depends on them.

COROLLARY 6.7. *Let σ be a primitive substitution whose composition matrix $M(\sigma)$ has a rational Perron-Frobenius eigenvalue λ. Then,*

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen} \rangle = \{n/m : n \in \mathbf{Z}, m \in \Gamma(\sigma)\} ,$$

where μ is the unique T_σ-invariant probability measure.

PROOF. First, suppose that σ is proper. Let α and d be as in (1) of Proposition 6.4. Since λ is an integer, it follows from Proposition 6.4 that

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen} \rangle = \langle \alpha_a/m : a \in A, m \in \Gamma(\sigma) \rangle .$$

Since the greatest common divisor of $\{\alpha_a : a \in A\}$ is one, there exists an integer n_a for each $a \in A$, such that $\sum_{a \in A} n_a \alpha_a = 1$. We have therefore that

$$\langle \alpha_a/m : a \in A, m \in \Gamma(\sigma) \rangle = \{n/m : n \in \mathbf{Z}, m \in \Gamma(\sigma)\} ,$$

and that

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen} \rangle = \{n/m : n \in \mathbf{Z}, m \in \Gamma(\sigma)\} .$$

In the same way, we can obtain a proof for the case when σ is non-proper. □

LEMMA 6.8 ([5, Lemma 2.4]). *Let (X, ϕ) be a Cantor minimal system. Suppose that $f \in C(X, \mathbf{Z})$ satisfy $0 < \int f d\mu < 1$ for every $\mu \in M_\phi$. Then, there exists a clopen subset A in X such that $\int f d\mu = \mu(A)$ for every $\mu \in M_\phi$. In other words $f - \chi_A \in \mathrm{Inf}(X, \phi)$.*

COROLLARY 6.9. *Let σ be a primitive substitution whose composition matrix $M(\sigma)$ has a rational Perron-Frobenius eigenvalue λ and μ be the unique invariant regular Borel probability measure for (X_σ, T_σ). Then,*

$$\{\mu(E) : E \subset X_\sigma \textit{ is clopen}\} = \{n/m : 0 \leq n \leq m, m \in \Gamma(\sigma)\}.$$

PROOF. To see that $\{\mu(E) : E \subset X_\sigma \text{ is clopen }\} = \{n/m : 0 \leq n \leq m, m \in \Gamma(\sigma)\}$, suppose that $n/m \in \{n/m : 0 \leq n \leq m, m \in \Gamma(\sigma)\}$. We may assume without loss of generality that $0 < n/m < 1$. As it is the map $[f] \mapsto \int f d\mu$ that implements the order isomorphism between $\tilde{K}^0(X_\sigma, T_\sigma)$ and $\langle \mu(E) : E \subset X_\sigma \text{ is clopen}\rangle$, there exists an $f \in C(X_\sigma, \mathbf{Z})$ such that $\int f d\mu = n/m$. From Lemma 6.8, we conclude therefore that there exists a clopen set $E \subset X_\sigma$ such that $\mu(E) = \int f d\mu = n/m$. The converse inclusion holds from Corollary 6.7. □

6.2. Invariant for orbit equivalence.

DEFINITION 6.10. We say that two Cantor minimal systems (X_1, ϕ_1) and (X_2, ϕ_2) are (*topologically*) *orbit equivalent* if there exists a homeomorphism $F : X_1 \to X_2$ such that $F(\mathrm{Orb}_{\phi_1}(x)) = \mathrm{Orb}_{\phi_2}(F(x))$ for each $x \in X_1$.

The next theorem is a portion of the theorem in [4].

THEOREM 6.11 ([4, Theorem 2.2]). *Let (X_i, ϕ_i) be Cantor minimal systems ($i = 1, 2$). The following are equivalent*:

(1) *(X_1, ϕ_1) and (X_2, ϕ_2) are orbit equivalent.*

(2) *The dimension groups $(\tilde{K}^0(X_i, \phi_i), \tilde{K}^0_+(X_i, \phi_i))$, $i = 1, 2$, are order isomorphic under an order isomorphism preserving the distinguished order units.*

THEOREM 6.12. *Let σ and σ' be primitive substitutions both of whose composition matrices have rational Perron-Frobenius eigenvalues. Then,*

(1) *(X_σ, T_σ) and $(X_{\sigma'}, T_{\sigma'})$ are orbit equivalent if and only if $\Gamma(\sigma) = \Gamma(\sigma')$;*

(2) *(X_σ, T_σ) is orbit equivalent to a stationary odometer system.*

PROOF. (1) If (X_σ, T_σ) and $(X_{\sigma'}, T_{\sigma'})$ are orbit equivalent, then it follows from Lemma 5.3 and Theorem 6.11 that $\langle \mu(E) : E \subset X_\sigma \text{ is clopen}\rangle$ and $\langle \mu'(E) : E \subset X_{\sigma'} \text{ is clopen }\rangle$ are order isomorphic by a group isomorphism preserving 1. It also follows from Corollary 6.7 that $\langle \mu(E) : E \subset X_\sigma \text{ is clopen}\rangle$ and $\langle \mu'(E) : E \subset X_{\sigma'} \text{ is clopen}\rangle$ are subgroups of $\mathbf{Q}$. A group isomorphism ϑ from a subgroup of $\mathbf{Q}$ to a subgroup of $\mathbf{Q}$ such that $\vartheta(1) = 1$ turns out to be only the identity map. We hence have the equality

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen}\rangle = \langle \mu'(E) : E \subset X_{\sigma'} \text{ is clopen}\rangle.$$

Therefore, Corollary 6.7 shows that

$$\{n/m : n \in \mathbf{Z}, m \in \Gamma(\sigma)\} = \{n/m : n \in \mathbf{Z}, m \in \Gamma(\sigma')\}.$$

Therefore, for an arbitrary $m \in \Gamma(\sigma)$, there exist integers $0 < n' \leq m$ and $m' \in \Gamma(\sigma')$ such that $1/m = n'/m'$. This implies that $m = m'/n' \in \Gamma(\sigma')$ because $m' \in \Gamma(\sigma')$. Finally, we conclude that $\Gamma(\sigma) = \Gamma(\sigma')$.

To show the converse implication, suppose that $\Gamma(\sigma) = \Gamma(\sigma')$. Then, it follows that $\{n/m : n \in \mathbf{Z}, m \in \Gamma(\sigma)\} = \{n/m : n \in \mathbf{Z}, m \in \Gamma(\sigma')\}$. This together with Corollary 6.7 shows that

$$\langle \mu(E) : E \subset X_\sigma \text{ is clopen}\rangle = \langle \mu'(E) : E \subset X_{\sigma'} \text{ is clopen}\rangle .$$

This together with Lemma 5.3 shows that $\tilde{K}^0(X_\sigma, T_\sigma)$ and $\tilde{K}^0(X_{\sigma'}, T_{\sigma'})$ are order isomorphic by an isomorphism preserving the distinguished order units. From Theorem 6.11, we conclude therefore that (X_σ, T_σ) and $(X_{\sigma'}, T_{\sigma'})$ are orbit equivalent.

(2) It follows from Lemma 5.3 and Corollary 6.7 that $\tilde{K}^0(X_\sigma, T_\sigma)$ is order isomorphic to $\{n/(d \cdot \lambda^k) : n \in \mathbf{Z}, k \geq 1\}$ for some integers $d, \lambda > 1$ by an isomorphism sending the distinguished order unit of $\tilde{K}^0(X_\sigma, T_\sigma)$ to 1. On the other hand, the dimension group modulo the infinitesimal subgroup of the stationary odometer system (X, ϕ) with base $(d, \lambda, \lambda, \cdots)$ is order isomorphic, by an isomorphism mapping the distinguished order unit of $\tilde{K}^0(X, \phi)$ to 1, to $\{n/(d \cdot \lambda^k) : n \in \mathbf{Z}, k \geq 1\}$. These together with Theorem 6.11 imply that (X_σ, T_σ) is orbit equivalent to the stationary odometer system (X, ϕ). □

COROLLARY 6.13. *If σ is a primitive substitution of constant length, then (X_σ, T_σ) is orbit equivalent to a stationary odometer system.*

PROOF. By the assumption, $\lambda := |\sigma(a)|$ is constant for any $a \in A$. Since the Perron-Frobenius eigenvalue of $M(\sigma)$ is not less than the smallest row sum and not bigger than the largest row sum, λ is the Perron-Frobenius eigenvalue of $M(\sigma)$. By (2) of Theorem 6.12, (X_σ, T_σ) turns out to be orbit equivalent to a stationary odometer system. □

EXAMPLE 6.14. There exist non-proper, primitive substitutions σ and σ' such that

(1) $M(\sigma) = M(\sigma')$;

(2) the Perron-Frobenius eigenvalue of $M(\sigma)$ is rational;

(3) (X_σ, T_σ) and (X'_σ, T'_σ) are not orbit equivalent.

We define two non-proper substitutions σ and σ' on the alphabet $\{a, b\}$ by $\sigma(a) = abb$, $\sigma(b) = aaa$; $\sigma'(a) = bab$, $\sigma'(b) = aaa$. Then it follows that T_σ is orbit equivalent to the stationary odometer system with base $(5, 3, 3, \cdots)$ and so is $T_{\sigma'}$ to the one with base $(2, 5, 3, 3, \cdots)$.

THEOREM 6.15. *For arbitrary integers $d > 1$ and $\lambda > 1$, there exists an aperiodic, proper and primitive substitution σ of constant length such that*

$$\Gamma(\sigma) = \{\text{factor of } d \cdot \lambda^n \text{for some } n \geq 1\} .$$

PROOF. Take an integer $m \geq 1$ such that $\lambda^m > d$. Take the d-dimensional integral vector $v = {}^t(\lambda^m, \lambda^m, \cdots, \lambda^m, (\lambda^m - d + 1)\lambda^m)$. Let M be the integral $d \times d$ matrix whose (i, j)-entry is the $\kappa^{j-1}(i)$-th entry of v for $1 \leq i, j \leq d$ where κ is the permutation on $\{1, 2, \cdots, d\}$ defined by $\kappa(d) = 1$ and $\kappa(i) = i + 1$ if $1 \leq i < d$. Every row sum of M is

λ^{2m}. Let σ be one of substitutions on the alphabet $\{1, 2, \cdots, d\}$ which satisfy the following conditions:

(i) $M(\sigma) = M$;
(ii) 1 is the first letter of $\sigma(i)$ for each $i \in \{1, 2, \cdots, d\}$;
(iii) d is the last letter of $\sigma(i)$ for each $i \in \{1, 2, \cdots, d\}$;
(iv) each of the words 11 and 12 occurs in some $\sigma(i)$.

It is easily seen that the substitution σ is proper, primitive and of constant length. It also follows from Theorem 6.12 that

$$\Gamma(\sigma) = \{\text{factor of } d \cdot \lambda^{2mn} \text{ for some } n \geq 1\} = \{\text{factor of } d \cdot \lambda^{n} \text{ for some } n \geq 1\}.$$

In the rest of the proof, we have to see the aperiodicity of σ. We have to show that $\lim_{n\to\infty} |\mathcal{L}_n(\sigma)| = +\infty$. Since $\lambda^m \neq (\lambda^m - d + 1)\lambda^m$, $i \neq j$ implies $\sigma(i) \neq \sigma(j)$ and also implies $\sigma^k(i) \neq \sigma^k(j)$ for every integer $k \geq 1$. Since $\sigma^k(2) \neq \sigma^k(1)$ and $|\sigma^k(1)\sigma^k(1)| = |\sigma^k(1)\sigma^k(2)|$ for every integer $k \geq 1$, there exists a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers such that $|\mathcal{L}_{n_i}(\sigma)| < |\mathcal{L}_{n_{i+1}}(\sigma)|$ for every integer $i \geq 1$. This completes the proof. □

The following is the converse of Corollary 6.13.

COROLLARY 6.16. *For arbitrary integers $d > 1$ and $\lambda > 1$, there exists a proper, primitive and aperiodic substitution of constant length whose associated substitution system is orbit equivalent to the odometer system with base $(d, \lambda, \lambda, \cdots)$.*

PROOF. It follows from Theorem 6.15 that there exists a proper, primitive and aperiodic substitution σ of constant length such that

$$\Gamma(\sigma) = \{\text{factor of } d \cdot \lambda^{n} \text{ for some } n \geq 1\}.$$

Then, the same argument in the proof of (2) of Theorem 6.12 shows that (X_σ, T_σ) is orbit equivalent to the odometer system with base $(d, \lambda, \lambda, \cdots)$. □

COROLLARY 6.17. *For any primitive substitution σ whose composition matrix $M(\sigma)$ has a rational Perron-Frobenius eigenvalue, there exists a proper, primitive substitution σ' of constant length whose associated substitution system $(X_{\sigma'}, T_{\sigma'})$ is orbit equivalent to (X_σ, T_σ).*

PROOF. Denote by λ the Perron-Frobenius eigenvalue of $M(\sigma)$. Then, $\Gamma(\sigma) = \{$factor of $d \cdot \lambda^n$ for some $n \geq 1\}$ for some integer $d > 0$. It follows from Theorem 6.15 that there exists a proper, primitive and aperiodic substitution σ' of constant length such that $\Gamma(\sigma) = \Gamma(\sigma')$. Therefore, Theorem 6.12 shows that (X_σ, T_σ) and $(X_{\sigma'}, T_{\sigma'})$ are orbit equivalent. □

Therefore, at least at the level of orbit equivalence, the class of substitution systems whose associated composition matrices have rational Perron-Frobenius eigenvalues coincides with the class of substitution systems arising from substitutions of constant length. In general, we cannot say that for a primitive substitution σ whose composition matrix has a rational Perron-Frobenius eigenvalue, there exists a primitive substitution of constant length whose associated substitution system is topologically conjugate to (X_σ, T_σ). This is verified from the following example: Let σ be a primitive substitution on the alphabet $\{a, b\}$ defined by $\sigma(a) = ababa$, $\sigma(b) = abb$. If there exists a primitive substitution ζ of constant length with length λ whose associated substitution system (X_ζ, T_ζ) is topologically conjugate to (X_σ, T_σ), then the set of eigenvalues in the measurable sense of T_σ is necessarily $\mathbf{Z}/h\mathbf{Z} \times \mathbf{Z}[\lambda^{-1}]$ for some integer $h \geq 1$, [9, Theorem VI. 15]. But the only eigenvalue of T_σ is 1, i.e., T_σ is weakly mixing, [9, Corollary VI. 23] and so this is a contradiction because weak mixing is invariant for measurable conjugacy, thus topological conjugacy. However, the author does not know whether at the level of strong orbit equivalence the counterpart to Corollary 6.17 holds, or not.

6.3. Invariant for Kakutani orbit equivalence.

DEFINITION 6.18. Let (X_i, ϕ_i) be Cantor minimal systems $(i = 1, 2)$. We say that (X_1, ϕ_1) and (X_2, ϕ_2) are *Kakutani orbit equivalent* if there exist two Kakutani equivalent, i.e., having conjugate induced systems, Cantor minimal systems (Y_i, ψ_i) $(i = 1, 2)$ such that (X_i, ϕ_i) is orbit equivalent to (Y_i, ψ_i), respectively for $i = 1, 2$.

THEOREM 6.19 ([4, Proposition 2.7]). *Let (X_i, ϕ_i) be Cantor minimal systems $(i = 1, 2)$. Then (X_1, ϕ_1) is Kakutani orbit equivalent to (X_2, ϕ_2) if and only if the dimension groups $\tilde{K}^0(X_i, \phi_i)$, $i = 1, 2$, are order isomorphic by an order isomorphism not necessarily preserving the distinguished order units.*

LEMMA 6.20. *If (X_1, ϕ_1) and (X_2, ϕ_2) are Kakutani orbit equivalent Cantor minimal systems, then there exist clopen subsets $A_1 \subset X_1$ and $A_2 \subset X_2$ such that the induced transformations $(\phi_1)_{A_1}$ and $(\phi_2)_{A_2}$ are orbit equivalent.*

PROOF. Let (Y_1, ψ_1) and (Y_2, ψ_2) be Kakutani equivalent Cantor minimal systems such that (X_i, ϕ_i) is orbit equivalent to (Y_i, ψ_i) for each $i = 1, 2$. There exist clopen subsets B_i of Y_i for each $i = 1, 2$ such that the induced transformations $(\psi_1)_{B_1}$ and $(\psi_2)_{B_2}$ are topologically conjugate. Since ϕ_1 and ψ_1 are orbit equivalent, there exists a homeomorphism $F_1 : X_1 \to Y_1$ such that $F_1(\mathrm{Orb}_{\phi_1}(x)) = \mathrm{Orb}_{\psi_1}(F_1(x))$ for all $x \in X_1$. Put $A_1 = F_1^{-1}(B_1)$ and define a homeomorphism $F_1' : A_1 \to B_1$ by $x \mapsto F_1(x)$. Then, for all $x \in A_1$,

$$
\begin{aligned}
F_1'(\mathrm{Orb}_{(\phi_1)_{A_1}}(x)) &= F_1(\mathrm{Orb}_{\phi_1}(x) \cap A_1) \\
&= F_1(\mathrm{Orb}_{\phi_1}(x)) \cap F_1(A_1) \\
&= \mathrm{Orb}_{\psi_1}(F_1(x)) \cap B_1 \\
&= \mathrm{Orb}_{(\psi_1)_{B_1}}(F_1(x)) \\
&= \mathrm{Orb}_{(\psi_1)_{B_1}}(F_1'(x)),
\end{aligned}
$$

which shows the orbit equivalence between $(\phi_1)_{A_1}$ and $(\psi_1)_{B_1}$. The same procedure as above shows the existence of a clopen set $A_2 \subset X_2$ such that $(\phi_2)_{A_2}$ and $(\psi_2)_{B_2}$ are orbit equivalent. We conclude therefore that ϕ_{A_1} and ϕ_{A_2} are orbit equivalent. □

LEMMA 6.21. *Let (X, ϕ) be a Cantor minimal system, μ be a ϕ-invariant probability measure and A be a nonempty clopen subset of X. Then,*

$$\langle \mu(E) : E \subset X \textit{ is clopen}\rangle = \langle \mu(E) : E \subset A \textit{ is clopen}\rangle .$$

PROOF. It is enough to show that $\mu(E) \in \langle \mu(E) : E \subset A \text{ is clopen}\rangle$ for any clopen set $E \subset X$. Let E be an arbitrary clopen subset of X. Since ϕ is minimal, there exists an $n_0 \geq 1$ such that $X = \bigcup_{i=0}^{n_0-1} \phi^i(A)$. Put $A_0 = A$, $A_1 = \phi(A) \setminus A_0$, $A_2 = \phi^2(A) \setminus (A_0 \cup A_1), \cdots , A_{n_0-1} = \phi^{n_0-1}(A) \setminus \bigcup_{i=0}^{n_0-2} A_i$, which compose a partition of X into clopen sets and it follows that $E = \bigcup_{i=0}^{n_0-1}(A_i \cap E)$ is a disjoint union. Put $E_i = \phi^{-i}(A_i \cap E)$ for each $0 \leq i < n_0$ which is a clopen subset of A. It follows from the above argument that

$$\mu(E) = \sum_{i=0}^{n_0-1} \mu(A_i \cap E) = \sum_{i=0}^{n_0-1} \mu(E_i) .$$

This completes the proof. □

COROLLARY 6.22. *Let (X, ϕ) be a uniquely ergodic Cantor minimal system with a unique invariant probability measure μ. Let A be a nonempty clopen subset of X and μ_A be the unique ϕ_A-invariant probability measure. Then,*

$$\langle \mu_A(E) : E \subset A \textit{ is clopen}\rangle = \mu(A)^{-1}\langle \mu(E) : E \subset X \textit{ is clopen}\rangle .$$

PROOF. The unique ϕ_A-invariant probability measure μ_A is defined by $\mu_A(E) = \mu(E)/\mu(A)$ for a measurable set $E \subset A$. Therefore, we have that

$$\begin{aligned}\langle \mu_A(E) : E \subset A \text{ is clopen}\rangle &= \langle \mu(E)/\mu(A) : E \subset A \text{ is clopen}\rangle \\ &= \mu(A)^{-1}\langle \mu(E) : E \subset X \text{ is clopen}\rangle .\end{aligned}$$

The last equality follows from Lemma 6.21. □

LEMMA 6.23. *Let M be a nonnegative, integral and primitive matrix and λ be the Perron-Frobenius eigenvalue of M. Then, there exists a right Perron-Frobenius eigenvector of M whose entries are all rational if and only if λ is rational.*

PROOF. Suppose that v is a right Perron-Frobenius eigenvector of M whose entries are all rational. Since $Mv = \lambda v$, $\sum_i v_i \in \mathbf{Q}$ and $(Mv)_i \in \mathbf{Q}$ for each i, we see that $\lambda = \sum_i (Mv)_i / \sum_i v_i \in \mathbf{Q}$.

Conversely, suppose that λ is rational. A right Perron-Frobenius eigenvector v of M is exactly a vector satisfying $(\lambda I - M)v = 0$. Since the eigenspace corresponding to the eigenvalue λ is one dimensional, we obtain that $\mathrm{rank}(\lambda I - M) = s - 1$ where s is the size of M. At last, there exist rational numbers r_i, $1 \leq i \leq s - 1$, such that $v_i = r_i v_s$ and it follows that $v = {}^t(r_1, \cdots , r_{s-1}, 1)$ is a right Perron-Frobenius eigenvector of M whose entries are all rational. □

THEOREM 6.24 ([7, Theorem 4.5.12]). *Let M be a nonnegative, primitive matrix with Perron-Frobenius eigenvalue λ. Let v, w be right, left Perron-Frobenius eigenvectors* *of M, i.e., vectors $v, w > 0$ such that $Mv = \lambda v$, $wM = \lambda w$, and normalized so that $wv = 1$. Then, for each i and j,*

$$(M^n)_{i,j} = [(v_i w_j) + \rho_{i,j}(n)]\lambda^n,$$

where $\rho_{i,j}(n) \to 0$ as $n \to \infty$.

From this theorem, it is verified that the vector $(\lim_{n\to\infty} \sum_j (M^n)_{i,j}/\lambda^n)_i$ is a right Perron-Frobenius eigenvector of M.

COROLLARY 6.25. *Let M be a nonnegative, primitive matrix with Perron-Frobenius eigenvalue λ. If strictly positive vector v satisfies that $M^k v = \lambda^k v$ for some $k \geq 1$, then $Mv = \lambda v$.*

PROOF. Suppose that $M^k v = \lambda^k v$ for some $k \geq 1$. It follows from Theorem 6.24 and the Perron-Frobenius Theorem that there exists a constant $c > 0$ such that

$$v_i = c \lim_{n\to\infty} \frac{\sum_j (M^{kn})_{i,j}}{\lambda^{kn}} \quad \text{for every } i\,.$$

It is easily seen that

$$\lim_{n\to\infty} \frac{\sum_j (M^{kn})_{i,j}}{\lambda^{kn}} = \lim_{n\to\infty} \frac{\sum_j (M^n)_{i,j}}{\lambda^n}\,.$$

We conclude therefore that v is a right Perron-Frobenius eigenvector of M. □

PROPOSITION 6.26. *Let σ, σ' be primitive substitutions, λ and λ' be the Perron-Frobenius eigenvalues of $M(\sigma)$ and $M(\sigma')$, respectively. If (X_σ, T_σ) and $(X_{\sigma'}, T_{\sigma'})$ are Kakutani orbit equivalent, then λ and λ' are both rational or both irrational simultaneously.*

PROOF. Suppose that (X_σ, T_σ) and $(X_{\sigma'}, T_{\sigma'})$ are Kakutani orbit equivalent and λ is rational. Let μ and μ' be the unique invariant probability measures of T_σ and $T_{\sigma'}$, respectively. Lemma 6.20 shows that there exist clopen sets $A \subset X_\sigma$ and $A' \subset X_{\sigma'}$ such that the induced transformations $(T_\sigma)_A$ and $(T_{\sigma'})_{A'}$ are orbit equivalent. It thus follows that

$$\langle \mu_A(E) : E \subset A \text{ is clopen}\rangle = \langle \mu'_{A'}(E) : E \subset A' \text{ is clopen}\rangle\,,$$

where μ_A and $\mu'_{A'}$ are the unique invariant probability measures of $(T_\sigma)_A$ and $(T_{\sigma'})_{A'}$, respectively, which are defined by $\mu_A(E) = \mu(E)/\mu(A)$ for a measurable $E \subset A$ and $\mu'_{A'}(E) = \mu(E)/\mu(A')$ for a measurable $E \subset A'$. It thus follows from Corollary 6.22 that

$$\mu(A)^{-1}\langle \mu(E) : E \subset X_\sigma \text{ is clopen}\rangle = \mu'(A')^{-1}\langle \mu'(E) : E \subset X_{\sigma'} \text{ is clopen}\rangle\,.$$

It follows from the assumption that $\mu'(A')^{-1}\langle \mu'(E) : E \subset X_{\sigma'} \text{ is clopen}\rangle \subset \mathbf{Q}$. Since Proposition 6.4 shows the existence of real numbers $r', d' > 0$, integers $k > 0$ ($k = 1$ if σ is proper; k is as in (2) of Proposition 6.4 if σ is non-proper.) and $n' \geq 0$ with $\mu'(A') = r'/(d' \cdot \lambda'^{kn'})$, it follows also from Proposition 6.4 that there exist real numbers α_i, $1 \leq i \leq i_0$, such that

$$\mu'(A')^{-1}\langle \mu'(E) : E \subset X_{\sigma'} \text{ is clopen}\rangle = \langle \alpha_i/\lambda'^{kn} : 1 \leq i \leq i_0, n \geq -n'\rangle\,.$$

Since $\langle \alpha_i/\lambda'^{kn} : 1 \leq i \leq i_0, n \geq -n' \rangle \subset \mathbf{Q}$, we have that $\alpha_i \in \mathbf{Q}$ for every $1 \leq i \leq i_0$ and therefore that $\lambda^k \in \mathbf{Q}$. It follows from Lemma 6.23 that there exists a right Perron-Frobenius eigenvector $v > 0$ of $M(\sigma')^k$ whose entries are all rational. Then, Corollary 6.25 shows that v is a right Perron-Frobenius eigenvector of $M(\sigma')$. Finally, it follows again from Lemma 6.23 that λ' is rational. □

LEMMA 6.27. *Let (X_1, ϕ_1) and (X_2, ϕ_2) be Cantor minimal systems with unique invariant probability measures μ_1 and μ_2, respectively. If there exist nonempty clopen sets $A_i \subset X_i$, $i = 1, 2$, such that the induced transformations $(\phi_1)_{A_1}$ and $(\phi_2)_{A_2}$ are orbit equivalent, then (X_1, ϕ_1) and (X_2, ϕ_2) are Kakutani orbit equivalent.*

PROOF. Assume that there exist nonempty clopen sets $A_i \subset X_i$, $i = 1, 2$, such that $(\phi_1)_{A_1}$ and $(\phi_2)_{A_2}$ are orbit equivalent. Let ν_i, $i = 1, 2$, be the unique invariant probability measures of $(\phi_1)_{A_1}$ and $(\phi_2)_{A_2}$, respectively, defined by $\nu_i(E) = \mu_i(E)/\mu_i(A_i)$ for a measurable $E \subset A_i$, for $i = 1, 2$. It follows from the assumption that

$$\langle \nu_1(E) : E \subset A_1 \text{ is clopen} \rangle = \langle \nu_2(E) : E \subset A_2 \text{ is clopen} \rangle . \qquad (***)$$

Here, we have from Lemma 6.21 that

$$\langle \nu_i(E) : E \subset A_i \text{ is clopen} \rangle = \mu_i(A_i)^{-1} \langle \mu_i(E) : E \subset X_i \text{ is clopen} \rangle ,$$

for each $i = 1, 2$. Therefore, it follows from $(***)$ that

$$\mu_1(A_1)^{-1} \langle \mu_1(E) : E \subset X_1 \text{ is clopen} \rangle = \mu_2(A_2)^{-1} \langle \mu_1(E) : E \subset X_2 \text{ is clopen} \rangle .$$

Hence, we conclude that there exists an $r > 0$ such that

$$\langle \mu_1(E) : E \subset X_1 \text{ is clopen} \rangle = r \langle \mu_2(E) : E \subset X_2 \text{ is clopen} \rangle$$

and that $\langle \mu_1(E) : E \subset X_1 \text{ is clopen} \rangle$ and $\langle \mu_2(E) : E \subset X_2 \text{ is clopen} \rangle$ are order isomorphic. At last, it follows from Lemma 5.3 and Theorem 6.19 that (X_1, ϕ_1) and (X_2, ϕ_2) are Kakutani orbit equivalent. □

THEOREM 6.28. *Let σ and σ' be primitive substitutions both of whose composition matrices have rational Perron-Frobenius eigenvalues λ and λ', respectively. Then, (X_σ, T_σ) and $(X_{\sigma'}, T_{\sigma'})$ are Kakutani orbit equivalent if and only if $\{$prime factor of $\lambda\} = \{$prime factor of $\lambda'\}$.*

PROOF. Suppose that (X_σ, T_σ) and $(X_{\sigma'}, T_{\sigma'})$ are Kakutani orbit equivalent. It follows from Lemma 6.20 and Corollary 6.22 that there exist nonempty clopen sets $A \subset X_\sigma$ and $A' \subset X_{\sigma'}$ such that

$$\mu(A)^{-1} \langle \mu(E) : E \subset X_\sigma \text{ is clopen} \rangle = \mu'(A')^{-1} \langle \mu'(E) : E \subset X_{\sigma'} \text{ is clopen} \rangle .$$

It follows from Corollary 6.7 that there exist integers m_0, n_0, m_1 and n_1 such that

$$\mu(A)^{-1}\langle\mu(E) : E \subset X_\sigma \text{ is clopen}\rangle = \left\{\frac{\lambda^{n_0}}{m_0}\frac{m}{\lambda^n} : m \in \mathbf{Z}, n \geq 0\right\},$$
$$\mu'(A')^{-1}\langle\mu'(E) : E \subset X_{\sigma'} \text{ is clopen}\rangle = \left\{\frac{\lambda'^{n_1}}{m_1}\frac{m}{\lambda'^n} : m \in \mathbf{Z}, n \geq 0\right\}.$$

Since $\mu(A)^{-1}\langle\mu(E) : E \subset X_\sigma \text{ is clopen}\rangle = \mu'(A')^{-1}\langle\mu'(E) : E \subset X_{\sigma'} \text{ is clopen}\rangle$, it follows that

$$\left\{\frac{\lambda^{n_0}}{m_0}\frac{m}{\lambda^n} : m \in \mathbf{Z}, n \geq 0\right\} = \left\{\frac{\lambda'^{n_1}}{m_1}\frac{m}{\lambda'^n} : m \in \mathbf{Z}, n \geq 0\right\}.$$

Therefore, for every integer $n \geq 0$ there exist integers $m(n)$ and $k(n)$ such that

$$\frac{1}{\lambda'^n} = \frac{\lambda^{n_0}}{m_0} \cdot \frac{m(n)}{\lambda^{k(n)}}.$$

We hence have that $m_0\lambda^{k(n)} = \lambda^{n_0} \cdot m(n) \cdot \lambda'^n$ for every $n \geq 0$ and that $\lambda'^n | m_0\lambda^{k(n)}$ for every $n \geq 0$. Let $p > 1$ be a prime factor of λ' and take $n \geq 0$ so that $m_0 < p^n$. Then, we necessarily have that p divides λ, which concludes that {prime factor of λ'} $\subset$ {prime factor of λ}. The converse inclusion can be shown in the same way.

To show the converse implication of the theorem, suppose that {prime factor of λ} $=$ {prime factor of λ'}. From Corollary 6.7, there exist integers $d, d' > 0$ such that

$$\langle\mu(E) : E \subset X_\sigma \text{ is clopen}\rangle = \{n/(d \cdot \lambda^m) : n \in \mathbf{Z}, m \geq 0\},$$
$$\langle\mu'(E) : E \subset X_{\sigma'} \text{ is clopen}\rangle = \{n/(d' \cdot \lambda'^m) : m \in \mathbf{Z}, n \geq 0\}.$$

It follows from Corollary 6.9 that there exist clopen sets $A \subset X_\sigma$ and $A' \subset X_{\sigma'}$ with $\mu(A) = 1/d$ and $\mu'(A') = 1/d'$. It follows from Corollary 6.22 that

$$\langle\mu_A(E) : E \subset A \text{ is clopen}\rangle = \{n/\lambda^m : n \in \mathbf{Z}, m \geq 0\},$$
$$\langle\mu'_{A'}(E) : E \subset A' \text{ is clopen}\rangle = \{n/\lambda'^m : n \in \mathbf{Z}, m \geq 0\},$$

where μ_A and $\mu_{A'}$ are the unique invariant probability measures of the induced transformations $(T_\sigma)_A$ and $(T_{\sigma'})_{A'}$, respectively. It follows from the assumption that $\langle\mu_A(E) : E \subset A \text{ is clopen}\rangle = \langle\mu'_{A'}(E) : E \subset A' \text{ is clopen}\rangle$. This shows the orbit equivalence between $(T_\sigma)_A$ and $(T_{\sigma'})_{A'}$ and it finally follows from Lemma 6.27 that T_σ and $T_{\sigma'}$ are Kakutani orbit equivalent. □

EXAMPLE 6.29. There are two proper and primitive substitutions whose composition matrices have a common irrational Perron-Frobenius eigenvalue but their associated substitution systems are not Kakutani orbit equivalent. We shall define substitutions σ and σ' by $\sigma(a) = abac, \sigma(b) = ca, \sigma(c) = aaccc; \sigma'(a) = aaabb, \sigma'(b) = aab$ respectively. Their composition matrices are respectively

$$M(\sigma) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{pmatrix} \quad \text{and} \quad M(\sigma') = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix},$$

whose common Perron-Frobenius eigenvalue is $2+\sqrt{5}$. The left Perron-Frobenius eigenvectors of $M(\sigma)$ and $M(\sigma')$ are given by $((\sqrt{5}-1)/2, (-\sqrt{5}+3)/2)$ and $(\sqrt{5}/5, (5-2\sqrt{5})/5, \sqrt{5}/5)$, respectively. A direct computation shows that

$$\tilde{K}^0(X_\sigma, T_\sigma) \cong \mathbf{Z} + \frac{\sqrt{5}}{5}\mathbf{Z} \quad \text{and} \quad \tilde{K}^0(X_{\sigma'}, T_{\sigma'}) \cong \mathbf{Z} + \frac{3-\sqrt{5}}{2}\mathbf{Z}$$

whose order structures and order units are both inherited from $(\mathbf{R}, \mathbf{R}_+, 1)$. We suppose that $G = \mathbf{Z} + \alpha\mathbf{Z}$ and $H = \mathbf{Z} + \beta\mathbf{Z}$ for irrational numbers α and β. The necessary and sufficient condition for G and H to be order isomorphic is that α is equivalent to β, i.e., for the continued fraction expansions $[a_0.a_1, a_2, \cdots]$ of α and $[b_0.b_1, b_2, \cdots]$ of β there exist integers $m, n \geq 1$ such that $a_{i+m} = b_{i+n}$ for every integer $i \geq 0$ where the dot means the separation between the integral part and the fractional part, see [2, Theorem 3.2]. The continued fraction expansions of $\sqrt{5}/5$ and $(3-\sqrt{5})/2$ are respectively $[0.2, 4, 4, \cdots]$ and $[0.2, 1, 1, \cdots]$. So the dimension groups $\mathbf{Z} + \sqrt{5}/5\mathbf{Z}$ and $\mathbf{Z} + (3-\sqrt{5})/2\mathbf{Z}$ are not order isomorphic and T_σ and $T_{\sigma'}$ are not Kakutani orbit equivalent from Theorem 6.19.

6.4. Examples.

(1) (Thue-Morse sequence) Put $A = \{a, b\}$ and define a primitive substitution σ on A by $\sigma(a) = ab$, $\sigma(b) = ba$. It follows that a is the first letter of $\sigma^2(a)$, b is the last letter of $\sigma^2(b)$ and $ba \in \mathcal{L}_2(\sigma)$. Then, the return words to $b.a$ are $w_1 = abb$, $w_2 = ab$, $w_3 = aabb$ and $w_4 = aab$. Put $R = \{1, 2, 3, 4\}$ and define a proper primitive substitution τ as in Section 3 by $\tau(1) = 1234$, $\tau(2) = 124$, $\tau(3) = 13234$ and $\tau(4) = 1324$. The composition matrix of τ is given by

$$M(\tau) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

whose left Perron-Frobenius eigenvector $\beta \in \mathbf{N}^4$ defined as in (2) of Proposition 6.5 is given by $(1, 1, 1, 1)$. We have that $d = 12$, where d is defined as in (2) of Proposition 6.5. Since the Perron-Frobenius eigenvalue of $M(\sigma)$ is 2, we have that

$$\Gamma(\sigma) = \{3^m \cdot 2^n : m \in \{0, 1\}, n \geq 0\}.$$

(2) (Rudin-Shapiro sequence) Put $A' = \{a, b, c, d\}$ and define a primitive substitution σ' by $\sigma'(a) = ab$, $\sigma'(b) = ac$, $\sigma'(c) = db$ and $\sigma'(d) = dc$. It follows that a is the first letter of ${\sigma'}^2(a)$ and b is the last letter of ${\sigma'}^2(b)$ and $ba \in \mathcal{L}_2(\sigma')$. Then, the return words to $b.a$ are $w_1 = ab$, $w_2 = acabdb$, $w_3 = acdcacab$, $w_4 = acabdbdcdb$, $w_5 = abdb$, $w_6 = acdcacdcdbdcacab$, $w_7 = abdbdcdbdcacdcdb$, $w_8 = acdcacdcdbdcacdcdb$ and $w_9 = abdbdcdbdcacab$. Put $R' = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define a proper and primitive substitution τ on R' as in Section 3 by $\tau(1) = 12$, $\tau(2) = 1345$, $\tau(3) = 1632$, $\tau(4) = 13475$, $\tau(5) = 145$, $\tau(6) = 168932$, $\tau(7) = 147985$, $\tau(8) = 168985$ and $\tau(9) = 147962$. The

composition matrix of τ is given by

$$M(\tau) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

whose left Perron-Frobenius eigenvector $\beta \in \mathbf{N}^9$ defined as in (2) of Proposition 6.5 is given by $(4, 2, 2, 2, 2, 1, 1, 1, 1)$. Therefore, we have that $d = 128 = 2^7$, where d is defined as in (2) of Proposition 6.5. Since the Perron-Frobenius eigenvalue of $M(\sigma')$ is 2, we have that

$$\Gamma(\sigma') = \{2^n : n \geq 0\}.$$

We hence conclude that (X_σ, T_σ) and $(X_{\sigma'}, T_{\sigma'})$ are not orbit equivalent but Kakutani orbit equivalent.

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Present Address:
DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY,
HIYOSHI, KOHOKU-KU, YOKOHAMA, 223–8522 JAPAN.
e-mail: hisatoc@math.keio.ac.jp