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ON THE TOPOLOGY OF MINIMAL ORBITS IN COMPLEX FLAG MANIFOLDS

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Abstract. We compute the Euler-Poincaré characteristic of the homogeneous compact manifolds that can be described as minimal orbits for the action of a real form in a complex flag manifold.

1. Introduction. A *complex flag manifold* is a simply connected homogeneous compact complex manifold that is also a projective variety. It is the quotient $\hat{M} = \hat{\mathbf{G}}/\mathbf{Q}$ of a connected complex semisimple Lie group $\hat{\mathbf{G}}$ by a parabolic subgroup $\mathbf{Q}$. Let a connected real form $\mathbf{G}$ of $\hat{\mathbf{G}}$ act on $\hat{M}$ by left translations. This action decomposes $\hat{M}$ into a finite number of $\mathbf{G}$-orbits. Among these, there is a unique orbit of minimal dimension, which is also the only one that is compact (cf. [Wol69]).

In this paper we compute the Euler-Poincaré characteristic of the minimal orbit M. This was already well known in the two cases where either $M = \hat{M}$, i.e., when $\mathbf{G}$ is transitive on $\hat{M}$, or M is totally real, i.e., when $\mathbf{Q}\cap\mathbf{G}$ is a real form of $\mathbf{Q}$, and, in particular, a real parabolic subgroup of $\mathbf{G}$. In these cases, indeed, explicit cell decompositions of M were obtained by several authors (see, e.g., [CS99, DKV83, Koc95]). The Euler characteristic of M was also computed in [MN01] for the case where M is a *standard* CR manifold. These are indeed special cases of minimal orbits, in which, although $\mathbf{Q} \cap \mathbf{G}$ is not a real form of $\mathbf{Q}$, M is diffeomorphic to a *real* flag manifold.

Our treatment of the general case, here, utilizes several notions developed in [AMN06a] for the study of the CR geometry of the minimal orbits. As in that paper, we shall use their representation in terms of the cross-marked Satake diagrams associated to their *parabolic* CR *algebras*. This makes easier to deal effectively with their $\mathbf{G}$-equivariant fibrations, by reducing the computation of the structure of the fibers to combinatorics on the Satake diagrams.

After observing that we may reduce to the case where $\mathbf{G}$ is simple, we show that in this case the Euler characteristic is different from zero, and hence positive, when $\mathbf{G}$ is compact, or of the complex type (in these cases M is diffeomorphic to a complex flag manifold), or of the real types A II, D II and E IV and for some special real flag manifolds of the real types A I, D I and E I. We explicitly compute $\chi(M)$ when $\mathbf{Q}$ is maximal parabolic and explain how, to compute $\chi(M)$ for general M, we may always reduce to that special case.

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The paper is organized as follows. In Sections 2 and 3 we rehearse the basic notions on complex flag manifolds and minimal orbits, and prove some results about **G**-equivariant fibrations. In Section 4 we establish some general criteria and tools that will be used to compute the Euler characteristic of the minimal orbits, and then in Section 5 we prove our main results. In Section 6 we further illustrate our method through the discussion of some examples. The final section is an appendix, containing a table that collects all the basic information on real semisimple Lie algebras that is required for computing $\chi(M)$.

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NOTATION. Throughout this paper, a *hat* means that we are considering some *complexification* of the corresponding bare object: For instance we use $\hat{\mathfrak{g}}$ for the complexification $\boldsymbol{C} \otimes_{\boldsymbol{R}} \mathfrak{g}$ of the real Lie algebra $\mathfrak{g}$, or $\hat{M}$ for the complex flag manifold that contains the minimal orbit M. For the labels of real simple Lie algebras and Lie groups we follow [Hel78, Table VI, Chapter X]. For the labels of the roots and the description of the root systems we refer to [Bou68].

2. Complex flag manifolds. A *complex flag manifold* is the quotient $\hat{M} = \hat{\mathbf{G}}/\mathbf{Q}$ of a complex semisimple Lie group $\hat{\mathbf{G}}$ by a parabolic subgroup $\mathbf{Q}$. We recall that $\mathbf{Q}$ is parabolic in $\hat{\mathbf{G}}$ if and only if its Lie algebra $\mathfrak{q}$ contains a Borel subalgebra, i.e., a maximal solvable subalgebra, of the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{\mathbf{G}}$. We also note that $\hat{\mathbf{G}}$ is necessarily a linear group, and that $\mathbf{Q}$ is connected, contains the center of $\hat{\mathbf{G}}$ and equals the normalizer of $\mathfrak{q}$ in $\hat{\mathbf{G}}$:

$$\mathbf{Q} = \{g \in \hat{\mathbf{G}} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{q}) = \mathfrak{q}\}\,. \tag{2.1}$$

In particular, a different choice of a connected $\hat{\mathbf{G}}'$ and of a parabolic $\mathbf{Q}'$, with Lie algebras $\hat{\mathfrak{g}}'$ and $\mathfrak{q}'$ isomorphic to $\hat{\mathfrak{g}}$ and $\mathfrak{q}$, yields a complex flag manifold $\hat{M}'$ that is complex-projectively isomorphic to $\hat{M}$. Thus a flag manifold $\hat{M}$ is better described in terms of the pair of Lie algebras $\hat{\mathfrak{g}}$ and $\mathfrak{q}$.

Fix a Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ that is contained in $\mathfrak{q}$. Let $\mathcal{R}$ be the root system with respect to $\hat{\mathfrak{h}}$ and denote by $\hat{\mathfrak{g}}^{\alpha} = \{Z \in \hat{\mathfrak{g}} \mid [H, Z] = \alpha(H)Z \text{ for any } H \in \hat{\mathfrak{h}}\}$ the root subspace of $\alpha \in \mathcal{R}$. Then we can choose a lexicographic order "$\prec$" of $\mathcal{R}$ such that $\hat{\mathfrak{g}}^{\alpha} \subset \mathfrak{q}$ for all positive α. Let $\mathcal{B}$ be the corresponding system of positive simple roots. All $\alpha \in \mathcal{R}$ are linear combinations of elements of the basis $\mathcal{B}$:

$$\alpha = \sum_{\beta \in \mathcal{B}} k_{\alpha}^{\beta} \beta\,, \quad k_{\alpha}^{\beta} \in \mathbf{Z} \tag{2.2}$$

and we define the support $\mathrm{supp}_{\mathcal{B}}(\alpha)$ of α with respect to $\mathcal{B}$ as the set of $\beta \in \mathcal{B}$ for which $k_{\alpha}^{\beta} \neq 0$. The set $\mathcal{Q} = \{\alpha \in \mathcal{R} \mid \hat{\mathfrak{g}}^{\alpha} \subset \mathfrak{q}\}$ is a *parabolic set*, i.e., is closed under root addition and $\mathcal{Q} \cup (-\mathcal{Q}) = \mathcal{R}$. Let $\Phi \subset \mathcal{B}$ be the subset of simple roots α for which $\hat{\mathfrak{g}}^{-\alpha} \not\subset \mathfrak{q}$. Then $\mathcal{Q}$ and $\mathfrak{q}$ are completely determined by Φ. Indeed,

$$\mathcal{Q} = \mathcal{Q}_{\Phi} := \{\alpha \succ 0\} \cup \{\alpha \prec 0 \mid \mathrm{supp}_{\mathcal{B}}(\alpha) \cap \Phi = \emptyset\} = \mathcal{Q}_{\Phi}^{r} \cup \mathcal{Q}_{\Phi}^{n}\,, \tag{2.3}$$

where

$$\mathcal{Q}^r_\Phi = \{\alpha \in \mathcal{R} \mid \operatorname{supp}_\mathcal{B}(\alpha) \cap \Phi = \emptyset\} \tag{2.4}$$

$$\mathcal{Q}^n_\Phi = \{\alpha \in \mathcal{R} \mid \alpha \succ 0 \text{ and } \operatorname{supp}_\mathcal{B}(\alpha) \cap \Phi \neq \emptyset\}\,, \tag{2.5}$$

and for the parabolic subalgebra $\mathfrak{q}$ we have the decomposition:

$$\mathfrak{q} = \mathfrak{q}_\Phi = \hat{\mathfrak{h}} + \sum_{\alpha \in \mathcal{Q}_\Phi} \hat{\mathfrak{g}}^\alpha = \mathfrak{q}^r_\Phi \oplus \mathfrak{q}^n_\Phi\,, \tag{2.6}$$

where

$$\mathfrak{q}^n_\Phi = \sum_{\alpha \in \mathcal{Q}^n_\Phi} \hat{\mathfrak{g}}^\alpha \quad \text{is the nilradical of } \mathfrak{q}_\Phi\text{, and} \tag{2.7}$$

$$\mathfrak{q}^r_\Phi = \hat{\mathfrak{h}} + \sum_{\alpha \in \mathcal{Q}^r_\Phi} \hat{\mathfrak{g}}^\alpha \quad \text{is a reductive complement of } \mathfrak{q}^n_\Phi \text{ in } \mathfrak{q}_\Phi\,. \tag{2.8}$$

We also set

$$\hat{\mathfrak{h}}'_\Phi = \hat{\mathfrak{h}} \cap [\mathfrak{q}^r_\Phi, \mathfrak{q}^r_\Phi]\,, \tag{2.9}$$

$$\hat{\mathfrak{h}}''_\Phi = \{H \in \hat{\mathfrak{h}} \mid [H, \mathfrak{q}^r_\Phi] = 0\}\,. \tag{2.10}$$

Then

$$\hat{\mathfrak{h}} = \hat{\mathfrak{h}}'_\Phi \oplus \hat{\mathfrak{h}}''_\Phi \tag{2.11}$$

and $\hat{\mathfrak{h}}''_\Phi$ is the center of the reductive Lie subalgebra $\mathfrak{q}^r_\Phi$.

All Cartan subalgebras of $\hat{\mathfrak{g}}$ are equivalent, modulo inner automorphisms, and all simple basis of a fixed root system $\mathcal{R}$ are equivalent for the transpose of inner automorphisms of $\hat{\mathfrak{g}}$ normalizing $\hat{\mathfrak{h}}$. Thus the correspondence $\Phi \leftrightarrow \mathfrak{q}_\Phi$ is one-to-one between the subsets Φ of an assigned system $\mathcal{B}$ of simple roots of $\mathcal{R}$ and the complex parabolic Lie subalgebras of $\hat{\mathfrak{g}}$, modulo inner automorphisms. In other words, the flag manifolds associated to a connected semisimple complex Lie group with Lie algebra $\hat{\mathfrak{g}}$ are parametrized by the subsets Φ of a basis $\mathcal{B}$ of simple roots of its root system $\mathcal{R}$, relative to any Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$.

The choice of a Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ contained in $\mathfrak{q}$ yields a canonical Chevalley decomposition of the parabolic subgroup $\mathbf{Q}$:

PROPOSITION 2.1. *With the notation above, we have a Chevalley decomposition*

$$\mathbf{Q} = \mathbf{Q}^n_\Phi \ltimes \mathbf{Q}^r_\Phi\,, \tag{2.12}$$

where the unipotent radical $\mathbf{Q}^n_\Phi$ is the connected and simply connected Lie subgroup of $\hat{\mathbf{G}}$ with Lie algebra $\mathfrak{q}^n_\Phi$, and $\mathbf{Q}^r_\Phi$ is the reductive[1] *complement with Lie algebra $\mathfrak{q}^r_\Phi$. The reductive $\mathbf{Q}^r_\Phi$ is characterized by*

$$\mathbf{Q}^r_\Phi = \mathbf{Z}_{\hat{\mathbf{G}}}(\hat{\mathfrak{h}}''_\Phi) = \{g \in \hat{\mathbf{G}} \mid \operatorname{Ad}_{\hat{\mathfrak{g}}}(g)(H) = H \textit{ for all } H \in \hat{\mathfrak{h}}''_\Phi\}\,. \tag{2.13}$$

[1] According to [Kna02] we call reductive a linear Lie group $\mathbf{G}$, having finitely many connected components, with a reductive Lie algebra $\mathfrak{g}$, and such that $\operatorname{Ad}_{\hat{\mathfrak{g}}}(\mathbf{G}) \subset \operatorname{Int}(\hat{\mathfrak{g}})$.

Moreover, $\mathbf{Q}_{\Phi}^{r}$ *is a subgroup of finite index in* $\mathbf{N}_{\hat{\mathbf{G}}}(\mathfrak{q}_{\Phi}^{r}) = \{g \in \hat{\mathbf{G}} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(\mathfrak{q}_{\Phi}^{r}) = \mathfrak{q}_{\Phi}^{r}\}$ *and* $\mathbf{Q} \cap \mathbf{N}_{\hat{\mathbf{G}}}(\mathfrak{q}_{\Phi}^{r}) = \mathbf{Q}_{\Phi}^{r}$.

PROOF. A complex parabolic subgroup can also be considered as a *real* parabolic subgroup. The Chevalley decomposition (2.12) reduces then to the Langlands decomposition $\mathbf{Q} = \mathbf{MAN}$, with $\mathbf{N} = \mathbf{Q}_{\Phi}^{n}$ and $\mathbf{MA} = \mathbf{Q}_{\Phi}^{r}$. Thus our statement reduces to [Kna02, Proposition 7.82(a)].

Next we note that $\mathfrak{q}_{\Phi}^{r}$ is the centralizer of $\hat{\mathfrak{h}}_{\Phi}''$ in $\hat{\mathfrak{g}}$ and is its own normalizer. This yields the inclusion $\mathbf{Q}_{\Phi}^{r} \subset \mathbf{N}_{\hat{\mathbf{G}}}(\mathfrak{q}_{\Phi}^{r})$. Since $\mathbf{N}_{\hat{\mathbf{G}}}(\mathfrak{q}_{\Phi}^{r})$ is semi-algebraic, it has finitely many connected components. Thus its intersection with $\mathbf{Q}_{\Phi}^{n}$ is discrete and finite, and thus trivial because $\mathbf{Q}_{\Phi}^{n}$ is connected, simply connected and unipotent. □

3. The structure of minimal G-orbits. Let $\hat{M} = \hat{\mathbf{G}}/\mathbf{Q}$ be a flag manifold for the transitive action of the connected semisimple complex linear Lie group $\hat{\mathbf{G}}$, and $\mathbf{G}$ a connected real form of $\hat{\mathbf{G}}$. Note that $\mathbf{G}$ is semi-algebraic, being a topological connected component of an algebraic group. We know from [Wol69] that there are finitely many $\mathbf{G}$-orbits. Fix any orbit M and a point $x \in M$. We can assume that $\mathbf{Q} \subset \hat{\mathbf{G}}$ is the stabilizer of x for the action of $\hat{\mathbf{G}}$ in $\hat{M}$. We keep the notation in §2, and we also set $\mathbf{G}_{+} = \mathbf{Q} \cap \mathbf{G}$ for the stabilizer of x in $\mathbf{G}$, so that $M \simeq \mathbf{G}/\mathbf{G}_{+}$. Let $\mathfrak{g} \subset \hat{\mathfrak{g}}$ be the Lie algebra of $\mathbf{G}$ and $\mathfrak{g}_{+} = \mathfrak{q} \cap \mathfrak{g}$ the Lie algebra of $\mathbf{G}_{+}$.

We summarize the results of [AMN06a, p. 491] by stating the following

PROPOSITION 3.1. *With the notation above*, $\mathfrak{g}_{+}$ *contains a Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$. *If* $\mathfrak{h}$ *is any Cartan subalgebra of* $\mathfrak{g}$ *contained in* $\mathfrak{g}_{+}$, *there are a Cartan involution* $\vartheta : \mathfrak{g} \to \mathfrak{g}$ *and a decomposition*

$$\mathfrak{g}_{+} = \mathfrak{n} \oplus \mathfrak{w} = \mathfrak{n} \oplus \mathfrak{l} \oplus \mathfrak{z} \tag{3.1}$$

such that

(i) $\mathfrak{n}$ *is the nilpotent ideal of* $\mathfrak{g}_{+}$, *consisting of the elements* $X \in \mathfrak{g}_{+}$ *for which* $\mathrm{ad}_{\mathfrak{g}}(X) : \mathfrak{g} \to \mathfrak{g}$ *is nilpotent*,

(ii) $\mathfrak{w} = \mathfrak{l} \oplus \mathfrak{z}$ *is reductive*,

(iii) $\mathfrak{z} \subset \mathfrak{h}$ *is the center of* $\mathfrak{w}$ *and* $\mathfrak{l} = [\mathfrak{w}, \mathfrak{w}]$ *its semisimple ideal*,

(iv) $\mathfrak{h}$, $\mathfrak{n}$, $\mathfrak{z}$ *and* $\mathfrak{l}$ *are invariant under the Cartan involution* ϑ *of* $\mathfrak{g}$.

We have the following

PROPOSITION 3.2. *Keep the notation introduced above. The isotropy subgroup* $\mathbf{G}_{+}$ *is the closed real semi-algebraic subgroup of* $\mathbf{G}$:

$$\mathbf{G}_{+} = \mathbf{N}_{\mathbf{G}}(\mathfrak{q}_{\Phi}) = \{g \in \mathbf{G} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{q}_{\Phi}) = \mathfrak{q}_{\Phi}\}\,. \tag{3.2}$$

The isotropy subgroup $\mathbf{G}_{+}$ *admits a Chevalley decomposition*

$$\mathbf{G}_{+} = \mathbf{W} \rtimes \mathbf{N}\,, \tag{3.3}$$

where

(i) $\mathbf{N}$ *is a unipotent*, *closed*, *connected*, *and simply connected subgroup with Lie algebra* $\mathfrak{n}$,

(ii) $\mathbf{W}$ *is a reductive Lie subgroup, with Lie algebra* $\mathfrak{w}$, *and is the centralizer of* $\mathfrak{z}$ *in* $\mathbf{G}$:

$$\mathbf{W} = \mathbf{Z}_{\mathbf{G}}(\mathfrak{z}) = \{g \in \mathbf{G} \mid \mathrm{Ad}_{\mathfrak{g}}(g)(H) = H \textit{ for all } H \in \mathfrak{z}\} \,. \tag{3.4}$$

PROOF. Let $g \in \mathbf{G}_+$. Then $\mathrm{Ad}_{\mathfrak{g}}(g)(\mathfrak{w})$ is a reductive complement of $\mathfrak{n}$ in $\mathfrak{g}_+$. Since all reductive complements of $\mathfrak{n}$ are conjugated by an inner automorphism from $\mathrm{Ad}_{\mathfrak{g}_+}(\mathbf{N})$, we can find a $g_n \in \mathbf{N}$ such that $\mathrm{Ad}_{\mathfrak{g}_+}(g_n^{-1}g)(\mathfrak{w}) = \mathfrak{w}$. Consider the element $g_r = g_n^{-1}g$. We then have:

$$\begin{aligned} \mathrm{Ad}_{\mathfrak{g}}(g_r)(\mathfrak{w}) &= \mathfrak{w}\,, \quad \mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\mathfrak{q}_\Phi) = \mathfrak{q}_\Phi\,, \\ \mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\mathfrak{q}^n_\Phi) &= \mathfrak{q}^n_\Phi\,, \quad \mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\bar{\mathfrak{q}}_\Phi) = \bar{\mathfrak{q}}_\Phi\,, \end{aligned}$$

because $g_r \in \mathbf{Q} \cap \bar{\mathbf{Q}}$. We consider the parabolic subalgebra of $\hat{\mathfrak{g}}$ defined by

$$\mathfrak{q}_{\Phi'} = \mathfrak{q}^n_\Phi \oplus (\mathfrak{q}^r_\Phi \cap \bar{\mathfrak{q}}_\Phi) = \mathfrak{q}^n_\Phi + (\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi)\ .$$

It has the property that $\mathfrak{q}^r_{\Phi'} = \bar{\mathfrak{q}}^r_{\Phi'}$ is the complexification of $\mathfrak{w}$. Clearly, $\mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\mathfrak{q}_{\Phi'}) = \mathfrak{q}_{\Phi'}$ and $\mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\mathfrak{q}^r_{\Phi'}) = \mathfrak{q}^r_{\Phi'}$. By Proposition 2.1, $g_r \in \mathbf{Z}_{\hat{\mathbf{G}}}(\hat{\mathfrak{h}}''_{\Phi'})$. The statement follows because $g_r \in \mathbf{G}$ and $\hat{\mathfrak{h}}''_{\Phi'}$ is the complexification of $\mathfrak{z}$. □

Among the $\mathbf{G}$-orbits in $\hat{M}$ there is one, and only one, say M, that is closed, and that we shall call henceforth *the minimal orbit*. Fix a point $x \in M$. We can assume that $\mathbf{Q}$ is the stabilizer of x in $\hat{\mathbf{G}}$. Then the orbit M is completely determined by the datum of the real Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ and of the complex Lie subalgebra $\mathfrak{q}$ of $\hat{\mathfrak{g}}$ corresponding to $\mathbf{Q}$. In [AMN06a] we called the pair $(\mathfrak{g}, \mathfrak{q})$, consisting of the real Lie algebra $\mathfrak{g}$ and of the parabolic complex Lie subalgebra $\mathfrak{q}$ of its complexification $\hat{\mathfrak{g}}$, a *parabolic minimal* CR algebra. This is a special instance of the notion of CR algebra that was introduced in [MN05] (for the general orbits and their corresponding *parabolic* CR algebras, we refer the reader to [AMN06b]).

We recall that $(\mathfrak{g}, \mathfrak{q})$ is *effective* if $\mathfrak{g}_+$ does not contain any ideal of $\mathfrak{g}$. We remark that this means that the action of $\mathbf{G}$ on M is almost effective.

Moreover, we have (see [AMN06a, p. 490]) the following

PROPOSITION 3.3. *Let M be the minimal orbit associated to the pair $(\mathfrak{g}, \mathfrak{q})$. If $\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i$ is the decomposition of $\mathfrak{g}$ into the direct sum of its simple ideals, then*

(1) $\mathfrak{q}_i = \mathfrak{q} \cap \hat{\mathfrak{g}}_i$ *is parabolic in* $\hat{\mathfrak{g}}_i$,

(2) $\mathfrak{q} = \bigoplus_{i=1}^m \mathfrak{q}_i$,

(3) $M = M_1 \times \cdots \times M_m$, *where M_i is a minimal orbit associated to the pair* $(\mathfrak{g}_i, \mathfrak{q}_i)$,

(4) $(\mathfrak{g}, \mathfrak{q})$ *is effective if and only if all* $(\mathfrak{g}_i, \mathfrak{q}_i)$ *are effective, i.e., if* $\mathfrak{q}_i \neq \hat{\mathfrak{g}}_i$ *for all* $i = 1, \ldots, m$. □

We showed in [AMN06a, Proposition 5.5] that $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ contains a maximally noncompact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Fix such a maximally noncompact Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_+$ of $\mathfrak{g}$, and, accordingly, a Cartan involution ϑ and a decomposition (3.1) as in Proposition 3.1.

Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{3.5}$$

be the Cartan decomposition defined by ϑ. Then $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$, with $\mathfrak{h}^+ = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{h}^- = \mathfrak{h} \cap \mathfrak{p}$. Moreover, $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup $\mathbf{K}$ of $\mathbf{G}$. The group $\mathbf{K}$ is connected and semi-algebraic. Hence the isotropy subgroup $\mathbf{K}_+ = \mathbf{K} \cap \mathbf{Q}$ has finitely many connected components and thus, since, by [Mon50], $\mathbf{K}$ acts transitively on the minimal orbit M,

$$M = \mathbf{K}/\mathbf{K}_+ . \tag{3.6}$$

Let $\mathcal{R}$ be the root system of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{h}}$. By duality, the conjugation in $\hat{\mathfrak{g}}$ defined by the real form $\mathfrak{g}$ defines an involution $\alpha \to \bar{\alpha}$ in the root system $\mathcal{R}$. A root α is *real* when $\bar{\alpha} = \alpha$, *imaginary* when $\bar{\alpha} = -\alpha$, and *complex* when $\bar{\alpha} \neq \pm\alpha$. The condition that $\mathfrak{h}$ is maximally noncompact is equivalent to the fact that all imaginary roots α are compact, i.e., that $\hat{\mathfrak{g}}^\alpha \subset \hat{\mathfrak{k}} = \boldsymbol{C} \otimes \mathfrak{k}$. We indicate by $\mathcal{R}_\bullet$ the set of imaginary roots.

We also showed (see [AMN06a, Proposition 6.2]) that, by choosing a suitable lexicographic order in $\mathcal{R}$, we have, with the notation in §2:

(1) $\mathcal{R}^+ = \{\alpha \succ 0\} \subset \mathcal{Q}$,

(2) $\bar{\alpha} \succ 0$ for all complex α in $\mathcal{R}^+$.

Let $\mathcal{B}$ be the system of simple roots in $\mathcal{R}^+$. The involution $\alpha \to \bar{\alpha}$ defines an involution $\alpha \to \varepsilon(\alpha)$ on $\mathcal{B} \setminus \mathcal{R}_\bullet$, with the property that $\bar{\alpha} = \varepsilon(\alpha) + \sum_{\beta \in \mathcal{B} \cap \mathcal{R}_\bullet} t_\alpha^\beta \beta$. It is described on the corresponding Satake diagrams (cf. [Ara62]) by joining by a curved arrow all pairs of distinct simple roots $(\alpha, \varepsilon(\alpha))$.

Let $\Phi \subset \mathcal{B}$ and $\mathfrak{q} = \mathfrak{q}_\Phi$ be as in §2. Then the Satake diagram $\mathcal{S}$ of $\mathfrak{g}$, with cross-marks corresponding to the roots in Φ, yields a complete graphic description of the minimal orbit M (see [AMN06a, §6]). We call the pair $(\mathcal{S}, \Phi)$ the *cross-marked Satake diagram* associated to M, or equivalently, to the pair $(\mathfrak{g}, \mathfrak{q})$.

An inclusion $\mathbf{Q}_\Phi \subset \mathbf{Q}_{\Phi'}$ of parabolic subgroups of $\hat{\mathbf{G}}$ defines a natural $\hat{\mathbf{G}}$-equivariant fibration $(\hat{\mathbf{G}}/\mathbf{Q}_\Phi) \to (\hat{\mathbf{G}}/\mathbf{Q}_{\Phi'})$, yielding by restriction a $\mathbf{G}$-equivariant fibration $M \to M'$ of the corresponding minimal orbits. In the following proposition we describe these $\mathbf{G}$-equivariant fibrations in terms of the associated cross-marked Satake diagrams.

PROPOSITION 3.4. *Let M and M' be minimal orbits for the same* $\mathbf{G}$, *associated to the pairs* $(\mathfrak{g}, \mathfrak{q}_\Phi)$ *and* $(\mathfrak{g}, \mathfrak{q}_{\Phi'})$, *respectively, with* $\Phi' \subset \Phi$. *Let $\mathcal{E}$ be the set of all roots $\alpha \in \mathcal{B}$ with* $\big(\{\alpha\} \cup \operatorname{supp}_{\mathcal{B}}(\bar{\alpha})\big) \cap \Phi' \neq \emptyset$. *Consider the Satake diagram $\mathcal{S}''$ obtained from the Satake diagram $\mathcal{S}$ of $\mathfrak{g}$ by erasing all nodes corresponding to the set $\mathcal{E}$ and all lines and arrows issued from them.*

Then the $\mathbf{G}$*-equivariant fibration $M \to M'$ has connected fibers that are minimal orbits M'', corresponding to the cross-marked Satake diagram $(\mathcal{S}'', \Phi'')$, where $\Phi'' = \Phi \setminus \mathcal{E}$.*

PROOF. Let $\mathbf{H} = \mathbf{Z}_{\mathbf{G}}(\mathfrak{h}) = \{h \in \mathbf{G} \mid \operatorname{Ad}_{\mathfrak{g}}(h)(H) = H, \text{for all } H \in \mathfrak{h}\}$ be the Cartan subgroup of $\mathbf{G}$ corresponding to $\mathfrak{h}$. We have $\operatorname{Ad}_{\hat{\mathfrak{g}}}(h)(\mathfrak{q}_\Phi) = \mathfrak{q}_\Phi$ and $\operatorname{Ad}_{\hat{\mathfrak{g}}}(h)(\mathfrak{q}_{\Phi'}) = \mathfrak{q}_{\Phi'}$ for all $h \in \mathbf{H}$. Hence $\mathbf{H} \subset \mathbf{G}_+ \subset \mathbf{G}'_+$, where $\mathbf{G}_+ = \mathbf{G} \cap \mathbf{Q}_\Phi$ and $\mathbf{G}'_+ = \mathbf{G} \cap \mathbf{Q}_{\Phi'}$. We decompose $\mathbf{G}'_+ = \mathbf{W}' \rtimes \mathbf{N}'$ according to (3.3), with a $\mathbf{W}'$ that satisfies (3.4). Then $\mathbf{H} \subset \mathbf{W}'$ and, since $\mathfrak{h}$ is maximally noncompact in $\mathfrak{g}'_+$, it is also maximally noncompact in $\mathfrak{w}' = \mathrm{Lie}(\mathbf{W}')$. Thus, by [Kna02, Proposition 7.90], all connected components of $\mathbf{W}'$, and hence also of $\mathbf{G}'_+$, intersect $\mathbf{H}$ and, a fortiori, $\mathbf{G}_+$. Therefore the fiber $\mathbf{G}'_+/\mathbf{G}_+$ is connected.

The fact that M'' is the minimal orbit associated to $(\mathcal{S}'', \Phi'')$ is the contents of [AMN06a, Proposition 7.3]. $\square$

In the following two lemmata we give sufficient conditions, in terms of cross-marked Satake diagrams, in order that two minimal orbits be diffeomorphic.

LEMMA 3.5. *We keep the notation introduced above. Let*

$$\Pi = \Phi \cup \{\alpha \in \mathcal{B} \setminus \mathcal{R}_\bullet \mid \mathrm{supp}(\bar{\alpha}) \cap \Phi \neq \emptyset\}\,, \tag{3.7}$$

and M^* the minimal orbit corresponding to $(\mathfrak{g}, \mathfrak{q}_\Pi)$. Then the canonical $\mathbf{G}$-equivariant map $M^* \to M$ is a diffeomorphism.

PROOF. By Proposition 3.4, $M^* \to M$ is a $\mathbf{G}$-equivariant fibration whose fiber reduces to a point, and hence a diffeomorphism. $\square$

From Lemma 3.5 we obtain

LEMMA 3.6. *We keep the notation introduced above. Let M_1, M_2 be minimal orbits associated to pairs $(\mathfrak{g}, \mathfrak{q}_{\Phi_1})$, $(\mathfrak{g}, \mathfrak{q}_{\Phi_2})$, respectively, for the same semisimple real Lie algebra $\mathfrak{g}$, and with suitable $\Phi_1, \Phi_2 \subset \mathcal{B}$. Let*

$$\Pi_1 = \Phi_1 \cup \{\alpha \in \mathcal{B} \mid \mathrm{supp}(\bar{\alpha}) \cap \Phi_1 \neq \emptyset\}\,,$$
$$\Pi_2 = \Phi_2 \cup \{\alpha \in \mathcal{B} \mid \mathrm{supp}(\bar{\alpha}) \cap \Phi_2 \neq \emptyset\}\,.$$

If $\Pi_1 = \Pi_2$, then there is a $\mathbf{G}$-equivariant diffeomorphism $M_1 \to M_2$.

PROOF. Indeed, by Lemma 3.5, we have a chain of $\mathbf{G}$-equivariant diffeomorphisms $M_1 \xleftarrow{\sim} M_1^* = M_2^* \xrightarrow{\sim} M_2$. $\square$

We also have the following

PROPOSITION 3.7. *We keep the notation introduced above. By erasing all nodes corresponding to roots in Π and all lines and arrows issuing from them, we obtain a new Satake diagram $\mathcal{S}''_\Phi$, that is the Satake diagram of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$. Then $\mathcal{R}''_\Phi = \mathcal{Q}^r_\Phi \cap \bar{\mathcal{Q}}^r_\Phi$ is the root system of the complexification $\hat{\mathfrak{l}}$ of $\mathfrak{l}$ with respect to its Cartan subalgebra $\hat{\mathfrak{h}} \cap \hat{\mathfrak{l}}$, that is, the complexification of the maximally noncompact Cartan subalgebra $\mathfrak{h} \cap \mathfrak{l}$ of $\mathfrak{l}$.*

PROOF. Since $\mathfrak{q}_\Pi \cap \mathfrak{g} = \mathfrak{q}_\Phi \cap \mathfrak{g} = \mathfrak{g}_+$, we can as well assume that $\Phi = \Pi$. The intersection $\mathfrak{w} = \mathfrak{q}^r \cap \bar{\mathfrak{q}}^r \cap \mathfrak{g}$ is a reductive complement of the nilradical of $\mathfrak{g}_+$ and its semisimple ideal $\mathfrak{l} = [\mathfrak{w}, \mathfrak{w}]$ is a Levi subalgebra of $\mathfrak{g}_+$. The associated root system of $\mathfrak{q}^r \cap \bar{\mathfrak{q}}^r$, with respect to $\hat{\mathfrak{h}}$, and of $\hat{\mathfrak{l}}$ with respect to its Cartan subalgebra $\hat{\mathfrak{h}} \cap \hat{\mathfrak{l}}$, is $\mathcal{Q}^r_\Pi \cap \bar{\mathcal{Q}}^r_\Pi$. We observe that $\bar{\alpha} \in \mathcal{Q}^r_\Pi$ for all simple $\alpha \in \mathcal{B} \setminus \Pi$. Hence, for $\alpha \in \mathcal{Q}^r_\Pi$, also $\bar{\alpha} \in \mathcal{Q}^r_\Pi$, because $\operatorname{supp}_{\mathcal{B}}(\bar{\alpha}) \subset \bigcup_{\beta \in \operatorname{supp}_{\mathcal{B}}(\alpha)} \operatorname{supp}_{\mathcal{B}}(\bar{\beta})$ and hence, when $\operatorname{supp}_{\mathcal{B}}(\alpha) \cap \Pi = \emptyset$, also $\operatorname{supp}_{\mathcal{B}}(\bar{\alpha}) \cap \Pi = \emptyset$. This shows that $\mathcal{Q}^r_{\Pi} = \bar{\mathcal{Q}}^r_{\Pi}$ and that $\operatorname{supp}_{\mathcal{B}}(\alpha) \subset \mathcal{B} \setminus \Pi$ for all $\alpha \in \mathcal{Q}^r_{\Pi}$. Since $\mathcal{B} \setminus \Pi \subset \mathcal{Q}^r_{\Pi}$, we proved that $\mathcal{B} \setminus \Pi$ is a system of simple roots for $\mathcal{R}''_{\Phi} = \mathcal{Q}^r_{\Pi} = \mathcal{Q}^r_{\Pi} \cap \bar{\mathcal{Q}}^r_{\Pi}$. Since the nodes of $\mathcal{S}''_{\Phi}$ are exactly those corresponding to the simple roots in $\mathcal{B} \setminus \Pi$, this proves our contention. □

From Lemma 3.6, we obtain in particular the following

PROPOSITION 3.8. *If $\mathfrak{g}$ is a simple Lie algebra of the complex type, then every minimal orbit M of $\mathbf{G}$ is diffeomorphic to a complex flag manifold.*

PROOF. The Satake diagram of $\mathfrak{g}$ consists of two disjoint connected graphs, whose nodes correspond to two sets of simple roots, each root of one set being strongly orthogonal to all roots of the other, $\mathcal{B}' = \{\alpha'_1, \dots, \alpha'_l\}$ and $\mathcal{B}'' = \{\alpha''_1, \dots, \alpha''_l\}$, with curved arrows joining α'_j to α''_j. Let $J \subset \{1, \dots, l\}$ be the set of indices for which either α'_j or α''_j are cross-marked, i.e., belongs to $\Phi \subset \mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$. By Lemma 3.6, our M is diffeomorphic to the M' corresponding to the parabolic $\mathfrak{q}_{\Phi'}$ with $\Phi' = \{\alpha'_j \mid j \in J\}$. By [AMN06a, Theorem 10.2], M' is complex and, hence, a complex flag manifold. □

4. Euler characteristic of minimal orbits. Let $M = \mathbf{K}/\mathbf{K}_+$ be a homogeneous space for the transitive action of a compact connected Lie group $\mathbf{K}$. It is known (see, e.g., [GHV73, p. 182]) that its Euler characteristic $\chi(M)$ is nonnegative. Moreover, it is positive exactly when the rank of the isotropy subgroup $\mathbf{K}_+$ equals the rank of $\mathbf{K}$. In this case the identity component $\mathbf{K}^0_+$ of the isotropy $\mathbf{K}_+$ contains the center of $\mathbf{K}$ and hence $\tilde{M} = \mathbf{K}/\mathbf{K}^0_+$ is the universal covering of M. Indeed, we can reduce to the case of a semisimple $\mathbf{K}$ and thus assume that $\mathbf{K}$ is simply connected. The number of sheets of $\tilde{M} \to M$ equals then the order $|\pi_1(M)|$ of the fundamental group of M. By using [MT91, Ch. VII, Theorem 3.13] for instance, we obtain

$$\chi(\tilde{M}) = \frac{|\mathbf{W}(\mathbf{K})|}{|\mathbf{W}(\mathbf{K}^0_+)|}, \tag{4.1}$$

$$\chi(M) = \frac{|\mathbf{W}(\mathbf{K})|}{|\mathbf{W}(\mathbf{K}^0_+)| \cdot |\pi_1(M)|}. \tag{4.2}$$

We have the following

PROPOSITION 4.1. *Let $M = \mathbf{G}/\mathbf{G}_+ = \mathbf{K}/\mathbf{K}_+$, as in* (3.6), *be a minimal orbit. Then $\mathbf{K}_+ = \mathbf{K} \cap \mathbf{G}_+$ is a maximal compact subgroup of $\mathbf{G}_+$, contained in the maximal compact subgroup $\mathbf{K}$ of $\mathbf{G}$. Also, the following are equivalent*:

(1) $\chi(M) > 0$.
(2) $\operatorname{rk}(\mathbf{K}) = \operatorname{rk}(\mathbf{K}_+)$, *i.e.*, $\mathbf{K}_+$ *contains a maximal torus of* $\mathbf{K}$.
(3) $\mathfrak{g}_+$ *contains a maximally compact Cartan subalgebra of* $\mathfrak{g}$.

PROOF. The proof of the equivalence (1) $\Leftrightarrow$ (2) is contained in [Wan49]. Thus we need only to prove that (2) $\Leftrightarrow$ (3). We also observe that $\mathbf{K}_+$ is a maximal compact subgroup of $\mathbf{G}_+$ because of (3.6).

Let $\mathfrak{k}$ and $\mathfrak{k}_+$ be the Lie algebras of $\mathbf{K}$ and $\mathbf{K}_+$, respectively. Assume that $\mathfrak{k}_+$ contains a maximal torus $\mathfrak{t}$ of $\mathfrak{k}$. Take a maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}_+$, consisting of $\mathrm{ad}_{\mathfrak{g}}$-semisimple elements, and with $\mathfrak{a} \supset \mathfrak{t}$. We claim that $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}_+$ and therefore also of $\mathfrak{g}$, and clearly it will also be maximally compact in $\mathfrak{g}$. Indeed, since $\mathfrak{g}_+$ contains a Cartan subalgebra of $\mathfrak{g}$, all Cartan subalgebras of $\mathfrak{g}_+$ are also Cartan subalgebras of $\mathfrak{g}$. Since $\mathfrak{g}_+$ is $\mathrm{ad}_{\mathfrak{g}}$-splittable (see [AMN06a, Proposition 5.4]), its Cartan subalgebras are its maximal Abelian Lie subalgebras consisting of $\mathrm{ad}_{\mathfrak{g}}$-semisimple elements (cf., e.g., [Bou75, Chap. VII, §5, Prop. 6]).

Vice versa, if $\mathfrak{a}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_+$, then $\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a} \cap \mathfrak{k}_+$ is a maximal torus of $\mathfrak{k}$ and $\mathfrak{k}_+$. Thus $\mathbf{K}$ and $\mathbf{K}_+$ have the same rank. □

In the following we shall keep the notation in §3. In particular, we fix a maximally noncompact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_+$, standard with respect to the Cartan decomposition (3.5) associated to the Cartan involution ϑ of Proposition 3.1. To express the equivalent conditions (1), (2) and (3) of Proposition 4.1 in terms of the description in §3, we need to rehearse first the construction of the Cartan subalgebras of a real semisimple Lie algebra from [Kos55, Sug59, Kna02].

LEMMA 4.2. *With the notation above, every Cartan subalgebra of* $\mathfrak{g}$ *is equivalent, modulo an inner automorphism, to a Cartan subalgebra* $\mathfrak{a}$ *which is standard with respect to the triple* $(\mathfrak{k}, \mathfrak{p}, \mathfrak{h}^-)$. *This means that* $\mathfrak{a}$ *has noncompact part* $\mathfrak{a}^- \subset \mathfrak{h}^-$ *and compact part* $\mathfrak{a}^+ \subset \mathfrak{k}$.

All standard Cartan subalgebras $\mathfrak{a}$ *are obtained in the following way*:

(1) *fix a system* $\alpha_1, \dots, \alpha_r$ *of strongly orthogonal real roots in* $\mathcal{R}$;

(2) *fix* $X_{\pm\alpha_i} \in \hat{\mathfrak{g}}^{\pm\alpha_i} \cap \mathfrak{g}$ *with* $[X_{-\alpha_i}, X_{\alpha_i}] = H_{\alpha_i}$, $[H_{\alpha_i}, X_{\pm\alpha_i}] = \pm 2X_{\pm\alpha_i}$, *for* $i = 1, \dots, r$;

(3) *let* $\mathbf{d} = \mathbf{d}_{\alpha_1} \circ \cdots \circ \mathbf{d}_{\alpha_r}$, *where* $\mathbf{d}_{\alpha_i} = \mathrm{Ad}_{\hat{\mathfrak{g}}}\left(\exp(i\pi(X_{-\alpha_i} - X_{\alpha_i})/4)\right)$, *for* $i = 1, \dots, r$ ($\mathbf{d}$ *is the Cayley transform with respect to* $\alpha_1, \dots, \alpha_r$);

(4) *set* $\mathfrak{a} = \mathbf{d}(\hat{\mathfrak{h}}) \cap \mathfrak{g}$. □

NOTATION. For a real semisimple Lie algebra $\mathfrak{g}$, with associated Satake's diagram $\mathcal{S}$, we shall denote by $\nu = \nu(\mathfrak{g}) = \nu(\mathcal{S})$ the maximum number of strongly orthogonal real roots in $\mathcal{R}$.

From Lemma 4.2 we deduce the criterion:

PROPOSITION 4.3. *Let* M *be the minimal orbit corresponding to the pair* $(\mathfrak{g}, \mathfrak{q}_\Phi)$. *Let* $\mathfrak{l}$ *be a Levi subalgebra of* $\mathfrak{g}_+$. *Then* $\chi(M) > 0$ *if and only if one of the following equivalent conditions is satisfied*:

(4.3) $\mathcal{Q}^r_\Phi$ *contains a maximal system of strongly orthogonal real roots of* $\mathcal{R}$.

(4.4) $\nu(\mathfrak{l}) = \nu(\mathfrak{g})$.

PROOF. The Cartan subalgebras of $\mathfrak{g}$ contained in $\mathfrak{g}_+$ are conjugated, modulo inner automorphisms of $\mathfrak{g}_+$, to standard Cartan subalgebras that are contained in $\mathfrak{w} = \mathfrak{q}^r \cap \mathfrak{g}$. Decompose the reductive real Lie algebra $\mathfrak{w}$ as $\mathfrak{w} = \mathfrak{l} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{w}$ and $\mathfrak{l} = [\mathfrak{w}, \mathfrak{w}]$ its semisimple ideal, that is a Levi subalgebra of $\mathfrak{g}_+$. We have $\mathfrak{z} \subset \mathfrak{h}$ and $\mathfrak{h} = \mathfrak{z} \oplus (\mathfrak{h} \cap \mathfrak{l})$. Thus a maximally compact Cartan subalgebra of $\mathfrak{g}_+$ will be conjugate to one of the form $\mathfrak{z} \oplus \mathfrak{e}$, with $\mathfrak{e}$ a maximally compact Cartan subalgebra of $\mathfrak{l}$. By Lemma 4.2, these are obtained via a Cayley transform $\mathbf{d} = \mathbf{d}_{\alpha_1} \circ \cdots \circ \mathbf{d}_{\alpha_r}$ for a system of strongly orthogonal real roots $\alpha_1, \ldots, \alpha_r$ in $\mathcal{Q}^r_\Phi$. Hence the statement follows. □

5. Classification of the minimal orbits with $\chi(M) > 0$. Throughout this section, we shall consistently employ the notation of the previous sections. In particular, $\mathfrak{l}$ will always denote a Levi subalgebra of $\mathfrak{g}_+$, $\mathfrak{k}$ the compact Lie subalgebra in the decomposition (3.5). We set $\mathfrak{k}^s_+$ for the maximal compact subalgebra $\mathfrak{k} \cap \mathfrak{l}$ of $\mathfrak{l}$.

By using the result of Proposition 3.4, the computation of $\chi(M)$ for a minimal orbit M can be reduced to the the case where, for the associated CR algebra $(\mathfrak{g}, \mathfrak{q})$, the real Lie algebra $\mathfrak{g}$ is simple and the parabolic $\mathfrak{q}$ is maximal, i.e., $\Phi = \{\alpha\}$ for some $\alpha \in \mathcal{B}$. Thus we begin by considering this special case:

THEOREM 5.1. *Let M be the minimal orbit associated to the effective pair* $(\mathfrak{g}, \mathfrak{q}_{\{\alpha\}})$, *with $\mathfrak{g}$ simple and a maximal parabolic $\mathfrak{q}_{\{\alpha\}} \subset \hat{\mathfrak{g}}$ for $\alpha \in \mathcal{B}$.*

Then $\chi(M) > 0$ if and only if either one of the following conditions holds:

(i) *$\mathfrak{g}$ is either of the complex type, or compact, or of the real non split types* A II, D II, E IV *and α is any root in $\mathcal{B}$.*

(ii) *$\mathfrak{g}$ is of the real types* A I, D I, E I, *and $\alpha \in \mathcal{B}$ is chosen as in Table* 1.

Here we list all pairs of real noncompact $\mathfrak{g}$ and $\alpha \in \mathcal{B}$ for which $\chi(M) > 0$, also computing $\chi(M)$ in the different cases.

PROOF. When $\mathfrak{g}$ is either of the complex type, or compact, or of the real types A II, D II, E IV, we have $\nu(\mathfrak{g}) = \nu(\mathcal{S}) = 0$ and thus the necessary and sufficient condition of Proposition 4.3 to have $\chi(M) > 0$ is trivially satisfied. Moreover, we know from [AMN06a, Theorem 8.6] that M is simply connected, and therefore $\tilde{M} = M$ and $\chi(\tilde{M}) = \chi(M)$.

Before discussing the remaining cases, we note that a necessary condition for $\chi(M) > 0$ is that $\mathrm{rk}(\mathbf{K}) < \mathrm{rk}(\mathbf{G})$. Indeed, when $\mathrm{rk}(\mathbf{K}) = \mathrm{rk}(\mathbf{G})$, Condition (2) of Proposition 4.1 implies that $\mathfrak{g}_+$ contains a compact Cartan subalgebra. This in turn implies that the orbit M is complex and thus coincides with the complex flag manifold $\hat{M}$. But this may occur only if $\mathfrak{g}$ is either of the complex type, or compact, or of real types A II, D II. The first two cases have already been considered, while in the remaing two cases $\mathrm{rk}(\mathbf{K}) < \mathrm{rk}(\mathbf{G})$.

Thus, to complete the proof, we need only to consider the cases where we may have $\mathrm{rk}(\mathbf{K}) < \mathrm{rk}(\mathbf{G})$, namely, [A I], [A II], [D I], [D II], [E I], [E IV]. We shall do this by comparing $\nu(\mathcal{S})$ with $\nu(\mathcal{S}''_{\{\alpha\}})$ for the different types of $\mathfrak{g}$.

[A I] Here $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{R})$ and $\alpha = \alpha_i$ with $1 \leq i < n$. Then $\mathfrak{l} \simeq \mathfrak{sl}(i, \boldsymbol{R}) \oplus \mathfrak{sl}(n-i, \boldsymbol{R})$. Hence $\nu(\mathcal{S}''_{\{\alpha_i\}}) = [i/2] + [(n-i)/2]$ and thus $[n/2] = [i/2] + [(n-i)/2]$, i.e., $i(n-i) \in 2\boldsymbol{Z}$,

TABLE 1.

type	$\mathfrak{g}$	α	condition	$\chi(\tilde{M})$	$\chi(M)$
A I	$\mathfrak{sl}(n, \boldsymbol{R})$	α_i	$i \cdot (n-i) \in 2\boldsymbol{Z}$	$2\binom{[n/2]}{[i/2]}$	$\binom{[n/2]}{[i/2]}$
A II	$\mathfrak{sl}(n, \boldsymbol{H})$	α_{2i-1}	$1 \leq i \leq n$	$2i\binom{n}{i}$	$2i\binom{n}{i}$
		α_{2i}	$1 \leq i < n$	$\binom{n}{i}$	$\binom{n}{i}$
D I	$\mathfrak{so}(p, 2n-p)$ $n \geq 4$ $2 \leq p \leq n$	α_1	$p+1 \in 2\boldsymbol{Z}$	4	2
D II	$\mathfrak{so}(1, 2n-1)$ $n \geq 4$	α_1		2	2
		α_i	$2 \leq i \leq n-2$	$2^i\binom{n-1}{i-1}$	$2^i\binom{n-1}{i-1}$
		α_i	$n-1 \leq i \leq n$	2^{n-1}	2^{n-1}
E I	$\mathfrak{e}_{\mathrm{I}}$	α_i	$i \in \{1, 6\}$	6	3
E IV	$\mathfrak{e}_{\mathrm{IV}}$	α_i	$i \in \{1, 6\}$	3	3
		α_i	$i = 2, 3, 5$	192	192
		α_4		144	144

is the necessary and sufficient condition in order that $\nu(\mathcal{S}) = \nu(\mathcal{S}''_{\{\alpha\}})$. We have $\mathfrak{k} \simeq \mathfrak{so}(n)$ and $\mathfrak{k}^s_+ \simeq \mathfrak{so}(i) \oplus \mathfrak{so}(n-i)$. Thus, when $\chi(M) > 0$, we have

$$\chi(\tilde{M}) = \frac{2^{[(n+1)/2]-1}[n/2]!}{2^{[(i+1)/2]-1}[i/2]! \cdot 2^{[(n-i+1)/2]-1}[(n-i)/2]!} = 2\binom{[n/2]}{[i/2]}.$$

We also have $\pi_1(M) \simeq \boldsymbol{Z}_2$ (see, e.g., [Wig98]), and hence $\chi(M) = \binom{[n/2]}{[i/2]}$.

[A II] Here $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{H})$, and $\nu(\mathcal{S}) = 0$ yields $\chi(M) > 0$ for any choice of α. If $\alpha = \alpha_{2i-1}$, for $1 \leq i \leq n$, then $\mathfrak{l} \simeq \mathfrak{sl}(i-1, \boldsymbol{H}) \oplus \mathfrak{sl}(n-i, \boldsymbol{H})$. Hence $\mathfrak{k} \simeq \mathfrak{sp}(n)$ and $\mathfrak{k}^s_+ \simeq \mathfrak{sp}(i-1) \oplus \mathfrak{sp}(n-i)$. Thus

$$\chi(M) = \chi(\tilde{M}) = \frac{2^n n!}{2^{i-1}(i-1)! \cdot 2^{n-i}(n-i)!} = 2i\binom{n}{i}.$$

If $\alpha = \alpha_{2i}$ with $1 \leq i < n$, then $\mathfrak{l} \simeq \mathfrak{sl}(i, \boldsymbol{H}) \oplus \mathfrak{sl}(n-i, \boldsymbol{H})$. Thus $\mathfrak{k}^s_+ \simeq \mathfrak{sp}(i) \oplus \mathfrak{sp}(n-i)$ and

$$\chi(M) = \chi(\tilde{M}) = \frac{2^n n!}{(2^i(i)!)(2^{n-i}(n-i)!)} = \binom{n}{i}.$$

[D I] We have $\mathfrak{g} \simeq \mathfrak{so}(p, 2n-p)$ with $2 \le p \le n$ and $\nu(\mathcal{S}) = 2[p/2]$. Because of the symmetry of $\mathcal{S}$, the minimal orbits corresponding to $\Phi = \{\alpha_{n-1}\}$ and to $\Phi = \{\alpha_n\}$ are diffeomorphic. Thus we can assume in the following that $i \neq n-1$. We obtain:

$$\mathfrak{l} \simeq \begin{cases} \mathfrak{sl}(i, \boldsymbol{R}) \oplus \mathfrak{so}(p-i, 2n-p-i) \quad \text{if } 1 \le i \le p \\ \qquad\qquad \Longrightarrow \nu(\mathcal{S}''_\alpha) = \left[\dfrac{i}{2}\right] + 2\left[\dfrac{p-i}{2}\right] \\ \mathfrak{sl}(p, \boldsymbol{R}) \oplus \mathfrak{su}(i-p) \oplus \mathfrak{so}(2n-2i) \quad \text{if } p < i \le n,\ i \neq n-1 \\ \qquad\qquad \Longrightarrow \nu(\mathcal{S}''_\alpha) = \left[\dfrac{p}{2}\right] < 2\left[\dfrac{p}{2}\right]. \end{cases}$$

The equation $[i/2] + 2[(p-i)/2] = 2[p/2]$, for integral i with $1 \le i \le p$, is solvable if and only if p is odd, and in this case we also need to have $i = 1$. Thus $\chi(M) > 0$ if and only if $p = 2h+1$ is odd and $\alpha = \alpha_1$. Then $\mathfrak{k} = \mathfrak{so}(2h+1) \oplus \mathfrak{so}(2n-2h-1)$ and $\mathfrak{k}^s_+ \simeq \mathfrak{so}(2h) \oplus \mathfrak{so}(2n-2h-2)$. Hence in this case we have

$$\chi(\tilde{M}) = \frac{2^h h! \cdot 2^{n-h-1}(n-h-1)!}{2^{h-1} h! \cdot 2^{n-h-2}(n-h-1)!} = 4\,.$$

We also have $\pi_1(M) \simeq \boldsymbol{Z}_2$ (see, e.g., [Wig98]), and hence $\chi(M) = 2$.

[D II] We have $\mathfrak{g} \simeq \mathfrak{so}(1, 2n-1)$, with $n \ge 4$, and $\mathcal{R}$ does not contain any real root, so that Condition (4.4) is trivially fulfilled. We have $\mathfrak{k} \simeq \mathfrak{so}(2n-1)$. When $\alpha = \alpha_1$, we have $\mathfrak{k}^s_+ \simeq \mathfrak{so}(2n-2)$. Thus

$$\chi(M) = \chi(\tilde{M}) = \frac{2^{n-1}(n-1)!}{2^{n-2}(n-1)!} = 2\,.$$

If $\alpha = \alpha_i$ with $2 \le i \le n-2$, we obtain $\mathfrak{k}^s_+ \simeq \mathfrak{su}(i-1) \oplus \mathfrak{so}(2n-2i)$ and

$$\chi(M) = \chi(\tilde{M}) = \frac{2^{n-1}(n-1)!}{(i-1)! 2^{n-i-1}(n-i)!} = 2^i \binom{n-1}{i-1}.$$

When $\alpha \in \{\alpha_{n-1}, \alpha_n\}$, we obtain $\mathfrak{k}^s_+ \simeq \mathfrak{su}(n-1)$ and therefore

$$\chi(M) = \chi(\tilde{M}) = \frac{2^{n-1}(n-1)!}{(n-1)!} = 2^{n-1}\,.$$

The exceptional Lie algebras. We shall discuss the case of the noncompact real forms of the exceptional Lie algebras of type E I and E IV by comparing $\nu = \nu(\mathfrak{g}) = \nu(\mathcal{S})$ with $\nu'' = \nu(\mathfrak{l}) = \nu(\mathcal{S}''_{\{\alpha\}})$. Since the proceeding is straightforward, we limit ourselves to list the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}_+$ and the corresponding value of ν'', for each different choice of $\alpha \in \mathcal{B}$ (see Table 2).

Looking up to the list, we see that $\nu = \nu''$ if, and only if, either:

(i) $\mathfrak{g}$ is of type E I and $\alpha \in \{\alpha_1, \alpha_6\}$, or

(ii) $\mathfrak{g}$ is of type E IV and α is any element of $\mathcal{B}$.

TABLE 2.

type	$\mathfrak{g}$	ν	α	$\mathfrak{l}$	ν''
E I	$\mathfrak{e}_{\mathrm{I}}$	4	α_1, α_6	$\mathfrak{so}(5, 5)$	4
			α_2	$\mathfrak{sl}(6, \boldsymbol{R})$	3
			α_3, α_5	$\mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{sl}(5, \boldsymbol{R})$	3
			α_4	$\mathfrak{sl}(3, \boldsymbol{R}) \oplus \mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{sl}(3, \boldsymbol{R})$	3
E IV	$\mathfrak{e}_{\mathrm{IV}}$	0	α_1, α_6	$\mathfrak{so}(8)$	0
			$\alpha_2, \alpha_3, \alpha_5$	$\mathfrak{su}(4)$	0
			α_4	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	0

In case (i), $\mathfrak{k}_+^s \simeq \mathfrak{so}(5) \oplus \mathfrak{so}(5)$ and hence $|\mathbf{W}(\mathbf{K}_+)| = 64 = 2^6$. We have $\mathfrak{k} = \mathfrak{sp}(4)$ and hence $|\mathbf{W}(\mathbf{K})| = 384 = 2^4 4!$. Thus $\chi(\tilde{M}) = 384/64 = 6$. Finally, since $\pi_1(M) \simeq \mathbf{Z}_2$ (see, e.g., [Wig98]), the manifold $\tilde{M}$ is a two-fold covering of M, and we obtain that $\chi(M) = 3$.

In the case (ii), we have $\mathfrak{k} = \mathfrak{f}_{\mathrm{III}}$ (the compact form of the complex simple Lie algebra of type F_4), so that $|\mathbf{W}(\mathbf{K})| = 1{,}152 = 2^7 3^2$. We need to distinguish the different cases:

(1) If $\alpha = \alpha_1, \alpha_6$, then $\mathfrak{k}_+^s \simeq \mathfrak{so}(9)$, so that $|\mathbf{W}(\mathbf{K}_+)| = 384 = 2^4 4!$ and $\chi(M) = \chi(\tilde{M}) = 1{,}152/384 = 3$.

(2) If $\alpha = \alpha_2, \alpha_3, \alpha_5$, then $\mathfrak{k}_+^s = \mathfrak{l} = \mathfrak{su}(3)$. Hence $|\mathbf{W}(\mathbf{K}_+)| = 6 = 3!$ and $\chi(M) = \chi(\tilde{M}) = 1{,}152/6 = 192$.

(3) If $\alpha = \alpha_4$, then $\mathfrak{k}_+^s = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Hence $|\mathbf{W}(\mathbf{K}_+)| = 8 = 2^3$ and $\chi(M) = \chi(\tilde{M}) = 1{,}152/8 = 144$. □

The computation of the Euler characteristic in the general case reduces to the previous theorem and to the well-known formula $\chi(M) = \chi(M') \cdot \chi(M'')$, that is valid for a smooth fiber bundle $M \to M'$ with typical fiber M''.

In particular, we obtain the following

THEOREM 5.2. *Let M be the minimal orbit associated to the effective pair $(\mathfrak{g}, \mathfrak{q})$ of the real semisimple Lie algebra $\mathfrak{g}$ and the complex parabolic subalgebra $\mathfrak{q}$ of its complexification $\hat{\mathfrak{g}}$. Let $\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ into the direct sum of its simple ideals. For each $i = 1, \ldots, m$, consider the pair $(\mathfrak{g}_i, \mathfrak{q}_i)$ for $\mathfrak{q}_i = \mathfrak{q} \cap \hat{\mathfrak{g}}_i$. Then the Euler characteristic $\chi(M)$ of M is always nonnegative and is positive if and only if each $\mathfrak{g}_i$ is one of the following*:

(1) *of the complex type*;

(2) *compact*;

(3) *of the real types* A II, D II, E IV;

(4) *of the real type* A I, *with* $\mathfrak{g}_i \simeq \mathfrak{sl}(n, \boldsymbol{R})$ *and* $\Phi_i = \{\alpha_{j_1}, \dots, \alpha_{j_r}\}$ *for a sequence of integers* $\{j_h\}_{0 \le h \le r+1}$ *with*

$$0 = j_0 < j_1 < \cdots < j_r < j_{r+1} = n \quad \textit{and} \quad \sum_{h=0}^{r} [(j_{h+1} - j_h)/2] = [n/2]\,;$$

(5) *of the real type* D I, *with* $\mathfrak{g}_i \simeq \mathfrak{so}(p, 2n-p)$ *with* p *odd*, $3 \le p \le n$ *and* $\Phi_i = \{\alpha_1\}$;

(6) *of the real type* E I, *with* $\Phi_i \subset \{\alpha_1, \alpha_6\}$.

PROOF. We recall that $\chi(M) \ge 0$ by [Wan49], because M is the homogeneous space of a compact group.

With the notation of Proposition 3.3, $\chi(M) = \chi(M_1) \cdots \chi(M_m)$, where M_i is a minimal orbit associated to the pair $(\mathfrak{g}_i, \mathfrak{q}_i)$. Therefore it suffices to prove the theorem under the additional assumption that $\mathfrak{g}$ is simple.

Let $\mathfrak{q} = \mathfrak{q}_\Phi$ for a set Φ of simple roots contained in a basis $\mathcal{B}$, that corresponds to the nodes of the Satake diagram $\mathcal{S}$ of $\mathfrak{g}$.

If $\alpha \in \Phi$ and M' is the minimal orbit associated to $(\mathfrak{g}, \mathfrak{q}_{\{\alpha\}})$, then we have a $\mathbf{G}$-equivariant fibration $M \to M'$, say with fiber M'', and $\chi(M) = \chi(M') \cdot \chi(M'')$. The condition $\chi(M') > 0$ is then necessary in order that $\chi(M) \ne 0$.

Thus, by Theorem 5.1, the conditions of the theorem are necessary.

Since $\nu(\mathcal{S}) = 0$ when $\mathfrak{g}$ is either of the complex type, or compact, or of one of the real types A II, D II, E IV, all $\emptyset \ne \Phi \subset \mathcal{B}$ lead in these cases to $\chi(M) > 0$. Also, the case (5) is clear, because, by Theorem 5.1, in that case we may have $\chi(M) > 0$ only with $\Phi = \{\alpha_1\}$.

Thus we only need to consider the cases (4) and (6).

(4) When $\mathfrak{g} \simeq \mathfrak{sl}(n, \boldsymbol{R})$ and $\Phi = \{\alpha_{j_1}, \dots, \alpha_{j_r}\}$, the Levi subalgebra of $\mathfrak{g}_+$ is $\mathfrak{l} \simeq \bigoplus_{h=0}^r \mathfrak{sl}(j_{h+1} - j_h, \boldsymbol{R})$. Hence $\nu'' = \sum_{h=0}^r [(j_{h+1} - j_h)/2]$ and the condition $\nu'' = \nu = [n/2]$ is necessary and sufficient for having $\chi(M) > 0$.

Since $\mathfrak{k}_+^s \simeq \bigoplus_{h=0}^r \mathfrak{so}(j_{h+1} - j_h)$, we obtain $\chi(M) = [n/2]!/\prod_{h=0}^r [(j_{h+1} - j_h)/2]!$.

(6) By Theorem 5.1, it only remains to consider the case where $\Phi = \{\alpha_1, \alpha_6\}$. Let M' be the minimal orbit corresponding to $(\mathfrak{e}_{\mathrm{I}}, \mathfrak{q}_{\{\alpha_6\}})$ and M'' the fiber of the fibration $M \to M'$. By Proposition 3.4, M'' is the minimal orbit associated to $(\mathfrak{so}(5,5), \mathfrak{q}_{\{\alpha_1\}})$. We know from Theorem 5.1 that $\chi(M') = 3$ and $\chi(M'') = 2$. Thus $\chi(M) = \chi(M') \cdot \chi(M'') = 6$. □

6. Some examples.

EXAMPLE 6.1. The method outlined above can also be applied in the classical cases. Let for instance $\mathfrak{g} = \mathfrak{so}(2n)$, with $n \ge 3$. We can assume that $\mathfrak{q} = \mathfrak{q}_\Phi$ with $\Phi = \{\alpha_{j_1}, \dots, \alpha_{j_r}\}$, for a sequence of integers $0 = j_0 < j_1 < \cdots < j_r \le j_{r+1} = n$ with $j_r \ne n-1$. Then we obtain: $\mathfrak{l} = \mathfrak{k}_+^s = \mathfrak{so}(2(n - j_r)) \oplus \bigoplus_{h=0}^{r-1} \mathfrak{su}(j_{h+1} - j_h)$. Hence, for the corresponding $M = M^{\mathrm{D}_n}_{j_1,\dots,j_r}$ we obtain

$$\chi(M^{\mathrm{D}_n}_{j_1,\dots,j_r}) = \begin{cases} \dfrac{2^{j_r} n!}{\prod_{h=0}^{r}(j_{h+1}-j_h)!} & \text{if } j_r \le n-2\,, \\ \dfrac{2^{n-1} n!}{\prod_{h=0}^{r-1}(j_{h+1}-j_h)!} & \text{if } j_r = n\,. \end{cases}$$

EXAMPLE 6.2. Let us turn now to the case D II. Let $\mathfrak{g} \simeq \mathfrak{so}(1, 2n-1)$, with $n \ge 4$ and $\mathfrak{q} = \mathfrak{q}_\Phi$ with $\Phi = \{\alpha_{j_1}, \dots, \alpha_{j_r}\}$, where again we assume that $0 = j_0 < j_1 < \cdots < j_r \le j_{r+1} = n$, and $j_r \ne n-1$. We note that, by Lemma 3.6, $M = M^{[\mathrm{D\,II}]_n}_{j_1,\dots,j_r}$ is diffeomorphic to the minimal orbit associated to $(\mathfrak{g}, \mathfrak{q}_{\Phi'})$, where $\Phi' = \Phi \cup \{\alpha_1\}$. Thus we can as well assume that $\alpha_1 \in \Phi$, i.e., that $j_1 = 1$. Since for the minimal orbit M' associated to $(\mathfrak{g}, \mathfrak{q}_{\{\alpha_1\}})$ we have $\chi(M') = 2$, we can apply Proposition 3.4 to the **G** equivariant fibration $M \to M'$. Since the fiber M'' is the complex flag manifold $M^{\mathrm{D}_{n-1}}_{j_2-1,\dots,j_r-1}$, we conclude that

$$\chi(M^{[\mathrm{D\,II}]_n}_{1,j_2,\dots,j_r}) = 2 \cdot \chi(M^{\mathrm{D}_{n-1}}_{j_2-1,\dots,j_r-1})\,.$$

EXAMPLE 6.3. Assume that $\mathfrak{g} \simeq \mathfrak{sl}(n, \boldsymbol{H})$ is of the real type $[\mathrm{A\,II}]_{2n-1}$ and that $\mathfrak{q} = \mathfrak{q}_\Phi$ with $\Phi = \{\alpha_{2j_1-1}, \dots, \alpha_{2j_r-1}\}$ for a sequence of integers satisfying $0 = j_0 < j_1 < \cdots < j_r < j_{r+1} = n+1$. Consider the minimal orbit $M^{[\mathrm{A\,II}]_{2n-1}}_{2j_1-1,\dots,2j_r-1}$. Since $\bar\alpha_{2h} = \alpha_{2h-1} + \alpha_{2h} + \alpha_{2h+1}$ for $1 \le h \le n-1$, we obtain that the Levi subalgebra of $\mathfrak{g}_+$ is $\mathfrak{l} = \bigoplus_{h=0}^{r} \mathfrak{sl}(j_{h+1} - j_h - 1, \boldsymbol{H})$. Hence we obtain

$$\begin{aligned} \chi(M^{[\mathrm{A\,II}]_{2n-1}}_{2j_1-1,\dots,2j_r-1}) &= \frac{2^n n!}{\prod_{h=0}^{r}\left(2^{j_{h+1}-j_h-1}(j_{h+1}-j_h-1)!\right)} \\ &= \frac{2^r n!}{\prod_{h=0}^{r}(j_{h+1}-j_h-1)!}\,. \end{aligned}$$

EXAMPLE 6.4. Consider the case where $\mathfrak{g} = \mathfrak{e}_{\mathrm{IV}}$. We have already discussed the case where $\Phi \subset \{\alpha_1, \alpha_6\}$. Assume therefore that $\Phi \cap \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\} \ne \emptyset$. We observe that, by Lemma 3.6, the minimal orbit associated to $(\mathfrak{e}_{\mathrm{IV}}, \mathfrak{q}_\Phi)$ is diffeomorphic to the minimal orbit associated to $(\mathfrak{e}_{\mathrm{IV}}, \mathfrak{q}_{\Phi\cup\{\alpha_1,\alpha_6\}})$. Thus we can proceed as in the discussion of the case [D II]. Indeed, we can assume that $\Phi = \{\alpha_1, \alpha_{j_1}, \dots, \alpha_{j_r}, \alpha_6\}$ with $r \ge 1$. By considering the **G**-equivariant fibration over $M' = M^{\mathrm{E\,IV}}_{1,6}$ associated to $(\mathfrak{e}_{\mathrm{IV}}, \mathfrak{q}_{\{\alpha_1,\alpha_6\}})$, we obtain by Proposition 3.4 that the fiber is $M'' = M^{\mathrm{D}_4}_{j_1-1,\dots,j_r-1}$. Hence, since

$$\chi(M^{\mathrm{E\,IV}}_{1,6}) = \chi(M^{\mathrm{E\,IV}}_{6}) \cdot \chi(M^{[\mathrm{D\,II}]_5}_{1}) = 3 \cdot 2 = 6\,,$$

we obtain

$$\chi(M^{\mathrm{E\,IV}}_{1,j_1,\dots,j_r,6}) = 6 \cdot \chi(M^{\mathrm{D}_4}_{j_1-1,\dots,j_r-1})\,.$$

7. Appendix. In Table 3 we give, for each noncompact simple Lie algebra of the real type, a linear representation $\mathfrak{g}$, its maximal compact subalgebra $\mathfrak{k}$, the order of the Weyl group

TABLE 3.

type	$\mathfrak{g}$	$\mathfrak{k}$	$\lvert\mathbf{W}(\mathbf{K})\rvert$	ν	l
A I	$\mathfrak{sl}(n, \boldsymbol{R})$	$\mathfrak{so}(n)$	$2^{[\frac{n+1}{2}]-1} \cdot \left[\frac{n}{2}\right]!$	$\left[\frac{n}{2}\right]$	$n-1$
A II	$\mathfrak{sl}(n, \boldsymbol{H})$	$\mathfrak{sp}(n)$	$2^n \cdot n!$	0	$2n-1$
A III	$\mathfrak{su}(p, q)$ $2 \le p \le q$	$\mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(q))$	$p! \cdot q!$	p	$p+q-1$
A IV	$\mathfrak{su}(1, q)$ $q \ge 1$	$\mathfrak{u}(q)$	$q!$	1	q
B I	$\mathfrak{so}(p, 2n+1-p)$ $2 \le p \le n$	$\mathfrak{so}(p) \oplus \mathfrak{so}(2n+1-p)$	$2^{n-1} \left[\frac{p}{2}\right]! \left[\frac{2n+1-p}{2}\right]!$	p	n
B II	$\mathfrak{so}(1, 2n)$ $n \ge 1$	$\mathfrak{so}(2n)$	$2^{n-1} n!$	1	n
C I	$\mathfrak{sp}(2n, \boldsymbol{R})$	$\mathfrak{u}(n)$	$n!$	n	n
C II	$\mathfrak{sp}(p, q)$ $0 < p \le q$	$\mathfrak{sp}(p) \oplus \mathfrak{sp}(q)$	$2^{p+q} p! \cdot q!$	p	$p+q$
D I	$\mathfrak{so}(p, 2n-p)$ $n \ge 4$ $2 \le p \le n$	$\mathfrak{so}(p) \oplus \mathfrak{so}(2n-p)$	$2^{n-p+2[\frac{p-1}{2}]}$ $\times \left[\frac{p}{2}\right]! \left[\frac{2n-p}{2}\right]!$	$2\left[\frac{p}{2}\right]$	n
D II	$\mathfrak{so}(1, 2n-1)$ $n \ge 4$	$\mathfrak{so}(2n-1)$	$2^{n-1}(n-1)!$	0	n
D III	$\mathfrak{so}^*(2n)$ $n \ge 2$	$\mathfrak{u}(n)$	$n!$	$\left[\frac{n}{2}\right]$	n
E I	$\mathfrak{e}_{\mathrm{I}}$	$\mathfrak{sp}(4)$	384	4	6
E II	$\mathfrak{e}_{\mathrm{II}}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(6)$	1,440	4	6
E III	$\mathfrak{e}_{\mathrm{III}}$	$\mathfrak{so}(10) \oplus \boldsymbol{R}$	1,920	2	6
E IV	$\mathfrak{e}_{\mathrm{IV}}$	$\mathfrak{f}_4$	1,152	0	6
E V	$\mathfrak{e}_{\mathrm{V}}$	$\mathfrak{su}(8)$	40,320	7	7
E VI	$\mathfrak{e}_{\mathrm{VI}}$	$\mathfrak{su}(2) \oplus \mathfrak{so}(12)$	46,080	4	7
E VII	$\mathfrak{e}_{\mathrm{VII}}$	$\mathfrak{e}_6 \oplus \boldsymbol{R}$	51,840	3	7
E VIII	$\mathfrak{e}_{\mathrm{VIII}}$	$\mathfrak{so}(16)$	5,160,960	8	8
E IX	$\mathfrak{e}_{\mathrm{IX}}$	$\mathfrak{su}(2) \oplus \mathfrak{e}_7$	5,806,080	4	8
F I	$\mathfrak{f}_{\mathrm{I}}$	$\mathfrak{su}(2) \oplus \mathfrak{sp}(3)$	96	4	4
F II	$\mathfrak{f}_{\mathrm{II}}$	$\mathfrak{so}(9)$	384	1	4
GI	$\mathfrak{g}_{\mathrm{I}}$	$\mathfrak{so}(3) \oplus \mathfrak{so}(3)$	16	2	2

of the maximal compact subgroup $\mathbf{K}$ of a connected Lie group with Lie algebra $\mathfrak{g}$, the number ν of the elements of a maximal system of strongly orthogonal real roots of $\mathcal{R}$, the dimension l of a Cartan subalgebra of $\mathfrak{g}$. The numbers ν are essentially computed in [Sug59]. However, since the computation is rather implicit there, we also give an explicit list of maximal systems of strongly orthogonal roots for each listed case.

We denote by Γ a maximal system of strongly orthogonal real roots in $\mathcal{R}$. For each case of the list we describe below an explicit Γ.

[A_l]. $\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \le i < j \le n\}$.

A I	$\nu = [n/2]$	$\Gamma = \{e_i - e_{n+1-i} \mid 1 \le i \le \nu\}$
A II	$\nu = 0$	$\Gamma = \emptyset$
A III	$\nu = p$	$\Gamma = \{e_i - e_{n+1-i} \mid 1 \le i \le \nu\}$
A IV	$\nu = 1$	$\Gamma = \{e_1 - e_n\}$

[B_l]. $\mathcal{R} = \{\pm e_i \mid 1 \le i \le l\} \cup \{\pm e_i \pm e_j \mid 1 \le i < j \le l\}$.

B I	$\nu = p$ even	$\Gamma = \{e_{2j-1} \pm e_{2j} \mid 1 \le j \le p/2\}$
B I	$\nu = p$ odd	$\Gamma = \{e_{2j-1} \pm e_{2j} \mid 1 \le j \le (p-1)/2\} \cup \{e_p\}$
B II	$\nu = 1$	$\Gamma = \{e_1\}$

[C_l]. $\mathcal{R} = \{\pm 2e_i \mid 1 \le i \le l\} \cup \{\pm e_i \pm e_j \mid 1 \le i < j \le l\}$.

C I	$\nu = l$	$\Gamma = \{2e_i \mid 1 \le i \le l\}$
C II	$\nu = p$	$\Gamma = \{e_{2i-1} + e_{2i} \mid 1 \le i \le p\}$

[D_l]. $\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \le i < j \le l\}$.

D I	$\nu = 2[p/2]$	$\Gamma = \{e_{2h-1} \pm e_{2h} \mid 2h \le p\}$
D II	$\nu = 0$	$\Gamma = \emptyset$
D III	$\nu = [l/2]$	$\Gamma = \{e_{2h-1} + e_{2h} \mid 1 \le h \le [l/2]\}$.

[E_6, E_7, E_8] Following [Bou68], we set

$$\mathcal{R}(\mathfrak{e}_8) = \{\pm e_i \pm e_j) \mid 1 \le i < j \le 8\} \cup \left\{ \frac{1}{2}\sum_{i=1}^{8}(-1)^{k_i} e_i \,\middle|\, k_i \in \mathbf{Z},\ \sum_{i=1}^{8} k_i \in 2\mathbf{Z} \right\},$$

$$\mathcal{R}(\mathfrak{e}_7) = \mathcal{R}(\mathfrak{e}_8) \cap \{e_7 + e_8\}^{\perp},$$

$$\mathcal{R}(\mathfrak{e}_6) = \mathcal{R}(\mathfrak{e}_8) \cap \{e_6 + e_8, e_7 + e_8\}^{\perp},$$

so that in particular $\mathcal{R}(\mathfrak{e}_6) \subset \mathcal{R}(\mathfrak{e}_7) \subset \mathcal{R}(\mathfrak{e}_8)$.

Likewise, the basis of simple roots can be considered as included one into the other: $\mathcal{B}(\mathfrak{e}_6) = \{\alpha_1, \dots, \alpha_6\} \subset \mathcal{B}(\mathfrak{e}_7) = \{\alpha_1, \dots, \alpha_7\} \subset \mathcal{B}(\mathfrak{e}_8) = \{\alpha_1, \dots, \alpha_8\}$ for $\alpha_1 = \frac{1}{2}(e_1 -$

$e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$, $\alpha_2 = e_1 + e_2$, $\alpha_3 = e_2 - e_1$, $\alpha_4 = e_3 - e_2$, $\alpha_5 = e_4 - e_3$, $\alpha_6 = e_5 - e_4$, $\alpha_7 = e_6 - e_5$, $\alpha_8 = e_7 - e_6$.

Set

$$\beta_i = \begin{cases} e_i - e_{i+1} & \text{for odd } i\,, \\ e_{i-1} + e_i & \text{for even } i\,, \end{cases}$$

so that $\beta_i \in \mathcal{R}(\mathfrak{e}_6)$ for $i \leq 4$, $\beta_i \in \mathcal{R}(\mathfrak{e}_7)$ for $i \leq 7$, $\beta_i \in \mathcal{R}(\mathfrak{e}_8)$ for $i \leq 8$. Note that $\{\beta_i \mid 1 \leq i \leq 8\}$ is a system of eight strongly orthogonal roots in $\mathcal{R}(\mathfrak{e}_8)$.

Since the conjugation in the non-split forms are better expressed in terms of the simple roots α_i, to describe the maximal sets Γ, we also found it convenient to introduce other roots γ_i, defined as linear combinations of the simple roots α_i:

$$\gamma_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \begin{smallmatrix}1&2&3&2&1\\&&2&&\end{smallmatrix} \in \mathcal{R}(\mathfrak{e}_6)\,,$$
$$\gamma_2 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = \begin{smallmatrix}1&1&1&1&1\\&&0&&\end{smallmatrix} \in \mathcal{R}(\mathfrak{e}_6)\,,$$
$$\gamma_3 = \alpha_3 + \alpha_4 + \alpha_5 = \begin{smallmatrix}0&1&1&1&0\\&&0&&\end{smallmatrix} \in \mathcal{R}(\mathfrak{e}_6)\,,$$
$$\gamma_4 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 = \begin{smallmatrix}1&2&2&1&0&0\\&&1&&&\end{smallmatrix} \in \mathcal{R}(\mathfrak{e}_7)\,,$$
$$\gamma_5 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 = \begin{smallmatrix}1&2&2&2&2&1\\&&1&&&\end{smallmatrix} \in \mathcal{R}(\mathfrak{e}_7)\,,$$
$$\gamma_6 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \begin{smallmatrix}1&2&4&3&2&1\\&&2&&&\end{smallmatrix} \in \mathcal{R}(\mathfrak{e}_7)\,,$$
$$\gamma_7 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 = \begin{smallmatrix}0&1&2&2&2&1\\&&1&&&\end{smallmatrix} \in \mathcal{R}(\mathfrak{e}_7)\,,$$
$$\gamma_8 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \begin{smallmatrix}2&3&4&3&2&1\\&&2&&&\end{smallmatrix} \in \mathcal{R}(\mathfrak{e}_7)\,,$$
$$\gamma_9 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \begin{smallmatrix}2&4&6&5&4&3&2\\&&3&&&&\end{smallmatrix} \in \mathcal{R}(\mathfrak{e}_8)\,.$$

We can describe a system Γ of strongly orthogonal roots for the real simple Lie algebra of the exceptional types E_6, E_7, E_8 by

E I	$\nu = 4$	$\Gamma = \{\beta_1, \beta_2, \beta_3, \beta_4\}\,,$
E II	$\nu = 4$	$\Gamma = \{\alpha_4, \gamma_1, \gamma_2, \gamma_3\}\,,$
E III	$\nu = 2$	$\Gamma = \{\gamma_1, \gamma_2\}\,,$
E IV	$\nu = 0$	$\Gamma = \emptyset\,,$
E V	$\nu = 7$	$\Gamma = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7\}\,,$
E VI	$\nu = 4$	$\Gamma = \{\alpha_1, \gamma_4, \gamma_5, \gamma_6\}\,,$
E VII	$\nu = 3$	$\Gamma = \{\alpha_7, \gamma_7, \gamma_8\}\,,$
E VIII	$\nu = 8$	$\Gamma = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8\}\,,$
E IX	$\nu = 4$	$\Gamma = \{\alpha_7, \gamma_7, \gamma_8, \gamma_9\}\,.$

[F_4]. We have

$$\mathcal{R} = \{\pm e_i \mid 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\} \cup \left\{ \frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2} \right\}\,.$$

Then we have

$$\begin{array}{lll} \mathrm{F\,I} & \nu = 4 & \Gamma = \{e_1 \pm e_2, e_3 \pm e_4\}\,, \\ \mathrm{F\,II} & \nu = 1 & \Gamma = \{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4\} = \{e_1\}\,. \end{array}$$

[G_2]. We have

$$\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \le i < j \le 3\} \cup \{\pm(2e_i - e_j - e_k) \mid \{i, j, k\} = \{1, 2, 3\}\}\,.$$

$$\mathrm{G\,I} \qquad \nu = 2 \qquad \Gamma = \{e_1 - e_2, 2e_3 - e_1 - e_2\}\,.$$

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