

ON THE TOTAL ABSOLUTE CURVATURE OF MANIFOLDS IMMERSED IN RIEMANNIAN MANIFOLD, II¹⁾

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In [3], [4] and [8], Chern, Lashof, Kuiper and Ōtsuki studied the total absolute curvature of an oriented compact manifold immersed in a euclidean space, and obtained some interesting results.

In [11], Willmore and Saleemi defined the total absolute curvature for compact oriented manifolds immersed in riemannian manifolds. In [1], the author used the Levi-Civita parallelism to define the total absolute curvature of compact manifolds immersed in a simply-connected riemannian manifold with non-positive sectional curvature, and proved that many results due to Chern-Lashof, Kuiper hold. In 1967, Kuiper [5] proposed to study the total absolute curvature for the surfaces immersed in euclidean 3-sphere.

In this present paper, we consider the total absolute curvature of manifolds immersed in arbitrary riemannian manifold, in particular, the surfaces in real space forms.

1. Preliminaries.

In the following, we assume throughout that M^n is an n -dimensional manifold, and Y^{n+N} is an oriented riemannian manifold of dimension $n+N$.

Let

$$(1) \quad f: M^n \rightarrow Y^{n+N}$$

be an immersion of M^n into Y^{n+N} .

In the following, by a frame x, e_1, \dots, e_{n+N} in Y^{n+N} we mean a point x and an ordered set of mutually perpendicular tangent unit vectors e_1, \dots, e_{n+N} at x , such that their orientation is coherent with that of Y^{n+N} . Unless otherwise stated, we agree on the following ranges of the indices:

$$(2) \quad 1 \leq i, j, k \leq n, \quad n+1 \leq r, s, t \leq n+N, \quad 1 \leq A, B, C \leq n+N.$$

Let $F(Y^{n+N})$ be the bundle of the frames on Y^{n+N} . In $F(Y^{n+N})$, we introduce the linear differential forms θ_A, θ_{AB} by the equations:

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$$(3) \quad dx = \sum_A \theta_A e_A, \quad de_A = \sum_B \theta_{AB} e_B, \quad \theta_{AB} + \theta_{BA} = 0.$$

The exterior derivative satisfies the following equations of structure:

$$(4) \quad d\theta_A = \sum_B \theta_B \wedge \theta_{BA}, \quad d\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB} + \Omega_{AB}.$$

Let B be the set of elements $b = (p, e_1, \dots, e_{n+N})$ such that $(f(p), e_1, \dots, e_{n+N})$ in $F(Y^{n+N})$; $p \in M^n$, e_1, \dots, e_n are tangent vectors and e_{n+1}, \dots, e_{n+N} are normal vectors at $f(p)$. Let ω_A, ω_{AB} be the 1-forms on B induced from the natural immersion $B \rightarrow F(Y^{n+N})$; $(p, e_1, \dots, e_{n+N}) \mapsto (f(p), e_1, \dots, e_{n+N})$. Then we have

$$(5) \quad \omega_r = 0.$$

Hence the first equation of (4) gives

$$(6) \quad \sum_i \omega_i \wedge \omega_{ir} = 0.$$

From this it follows that

$$(7) \quad \omega_{ir} = \sum_j A_{rj} \omega_j, \quad A_{rj} = A_{rji}.$$

We define the normal bundle B_v by

$$(8) \quad B_v = \{(p, e): p \in M^n, e \text{ being unit normal vector at } f(p)\}.$$

We call

$$(9) \quad K(p, e_r) = (-1)^n \det(A_{rji})$$

the *Lipschitz-Killing curvature* at $(p, e_r) \in B_v$.

We call the integral

$$(10) \quad TA(f) = \frac{1}{c_{n+N-1}} \int_{B_v} |K(p, e)| dW$$

the *total absolute curvature* of the immersion f , if the right hand side of (10) exists, where c_{n+N-1} denotes the volume of the unit $(n+N-1)$ -sphere and dW denotes the volume element of the normal bundle B_v .

REMARK. In the special case: Y^{n+N} is euclidean, then the definitions of the total absolute curvature in [1], [3] and this present paper are all equivalent.

2. Minimal flat torus in S^3 with $TA(f) = \pi$.

Let

$$(11) \quad f: M^n \rightarrow S^{n+N}$$

be an immersion from a compact manifold M^n into a euclidean $(n+N)$ -sphere S^{n+N} with radius a . For any $e \in S^{n+N}$, we define the height function h_e on M^n as

follows:

$$(12) \quad h_e: M^n \rightarrow R; \quad h_e(p) = f(p) \cdot \left(\frac{e}{a} \right),$$

where “ \cdot ” denotes the inner product naturally induced by S^{n+N} . Then, by Sard’s theorem, we know that for almost all $e \in S^{n+N}$, the height function h_e have only non-degenerate critical points. In the following, let $\mathcal{F}(M^n)$ be the set of real-valued functions on M^n with only non-degenerate critical points. For any f in $\mathcal{F}(M^n)$, let $m_i(f)$ denote the number of critical points of index i of f on M^n . Let $m(f) = \sum_i m_i(f)$. The immersion (11) is called to be *tight* if

$$(13) \quad m(h_e) = \sum_i \beta_i(M^n)$$

for almost all $e \in S^{n+N}$, where $\beta_i(M^n)$ denotes the i -th betti number of M^n .

Thank a theorem of Ôtsuki [8], we have the following theorem:

THEOREM 1. *Let $f: M^2 \rightarrow S^3$ be an isometric immersion from a closed riemannian surface M^2 into a euclidean 3-sphere S^3 with radius a . Then the following three statements are equivalent:*

(a) M^2 is imbedded as a minimal flat torus with total absolute curvature $TA(f) = \pi$.

(b) M^2 is imbedded as a tight flat torus with total absolute curvature $TA(f) = \pi$.

(c) $f(M^2)$ is equivalent to the standard flat torus

$$(14) \quad \frac{a}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v)$$

under the action of the orthogonal group $O(4)$ on S^3 .

Proof. (a) implies (b): If M^2 is imbedded in S^3 as a minimal flat torus with total absolute curvature $TA(f) = \pi$. Then we have

$$\text{Trace}(A_{3ij}) = 0 \quad \text{and} \quad \det(A_{3ij}) = -\frac{1}{a^2}.$$

Hence for some suitable cross-section $(x, \bar{e}_1, \bar{e}_2, e_3)$ from M^2 into B , the matrix (A_{3ij}) is given in the following form:

$$(A_{3ij}) = \begin{pmatrix} 1/a & 0 \\ 0 & -1/a \end{pmatrix}.$$

Therefore, if we define the immersion

$$(15) \quad f': M^2 \rightarrow E^4$$

by $f'(p) = f(p)$, for all p in M^2 , then the Lipschitz-Killing curvature $K'(p, e)$ of the immersion (15) is given by

$$K'(p, e) = \det \begin{pmatrix} (1/a)(\cos \theta + \sin \theta) & 0 \\ 0 & (1/a)(\cos \theta - \sin \theta) \end{pmatrix}$$

where $e = (\sin \theta)e_3 + ((1/a)\cos \theta)p$. Thus we get

$$(16) \quad K'(p, e) = \frac{1}{a^2} \cos 2\theta.$$

Let

$$\lambda(p) = \max_{e \in S_p^1} K'(p, e), \quad \mu(p) = \min_{e \in S_p^1} K'(p, e)$$

where S_p^1 is the fibre of the normal bundle of the immersion (15) over p . Then by (16), we have

$$(17) \quad \lambda(p) = \frac{1}{a^2}, \quad \mu(p) = -\frac{1}{a^2}.$$

Hence the integral of $\lambda(p)$ over M^2 satisfies

$$(18) \quad \int_{M^2} \lambda(p) dV = 2\pi^2.$$

Therefore, by a result due to Ōtsuki [8], we know that the immersion $f: M^2 \rightarrow S^8$ is tight.

(b) implies (c): If M^2 is imbedded as a tight flat torus with total absolute curvature $TA(f) = \pi$. Then by a result due to Ōtsuki [8], we have

$$(19) \quad \int_{M^2} \lambda(p) dV = 2\pi^2.$$

On the other hand, by the assumption, we can easily verify that

$$(20) \quad \lambda(p) \geq \frac{1}{a^2}, \quad v(M^2) = 2a^2\pi^2 \text{ (volume of } M^2\text{)}.$$

Therefore, by (19) and (20), we can easily find that

$$\lambda(p) = \frac{1}{a^2}, \quad \mu(p) = -\frac{1}{a^2}, \quad \text{for all } p \in M^2.$$

Furthermore, if we set

$$e'_1 = \bar{e}_1, \quad e'_2 = \bar{e}_2, \quad e'_3 = e_3, \quad e'_4 = \frac{1}{a}p,$$

then we have

$$K'(p, e'_3) = -\frac{1}{a^2}, \quad K'(p, e'_4) = \frac{1}{a^2}.$$

This shows that $(p, e'_1, e'_2, e'_3, e'_4)$ is a Frenet frame in the sense of Ōtsuki [8] for the immersion (15). Thus, we have

$$(21) \quad \omega'_{13} \wedge \omega'_{23} = -\frac{1}{a^2} \omega'_1 \wedge \omega'_2, \quad \omega'_{14} \wedge \omega'_{24} = \frac{1}{a^2} \omega'_1 \wedge \omega'_2,$$

$$(22) \quad \omega'_{13} \wedge \omega'_{24} + \omega'_{14} \wedge \omega'_{23} = 0.$$

Furthermore, by the definition of e'_4 , we have

$$(23) \quad \omega'_{41} = \omega'_1, \quad \omega'_{42} = \omega'_2, \quad \omega'_{43} = \omega'_{44} = 0.$$

Therefore, by (19), (21), (22) and (23), we know that $f(M^2)$ is equivalent to a flat torus of the following form [8]:

$$(24) \quad (c \cos u, c \sin u, d \cos v, d \sin v), \quad c^2 + d^2 = a^2,$$

under the action of $O(4)$ on S^3 . Furthermore, by (21), we have

$$(25) \quad 2cd = a^2.$$

Hence, by (24) and (25), we get $c = d = a/\sqrt{2}$. This proves that $f(M^2)$ is equivalent to the standard flat torus (14) under the action of $O(4)$.

(c) implies (a): This step is trivial. This completes the proof of the Theorem.

3. Total absolute curvature for surfaces in real space forms.

Throughout this section, we assume that Y^N is one of the following complete simply connected riemannian manifolds of dimension N :

(I) An N -sphere S^N of radius a (or of curvature $1/a^2$).

(II) A euclidean N -space E^N .

(III) A hyperbolic N -space H^N of curvature $-1/a^2$.

Let $f: M^2 \rightarrow Y^N$ be an immersion from a surface M^2 into Y^N . Then we have the following equation:

$$(26) \quad d\omega_{12} = -\sum_r \omega_{1r} \wedge \omega_{2r} - \frac{\delta}{a^2} \omega_1 \wedge \omega_2,$$

where from now on δ takes the value:

$$(27) \quad \delta = \begin{cases} 1 & \text{if } Y^N = S^N, \\ 0 & \text{if } Y^N = E^N, \\ -1 & \text{if } Y^N = H^N. \end{cases}$$

By (26), we have

$$d\omega_{12} = -\sum_r K(p, e_r) \omega_1 \wedge \omega_2 - \frac{\delta}{a^2} \omega_1 \wedge \omega_2.$$

Hence, the riemannian sectional curvature $S(p)$ of M^2 at p with the induced metric is given by

$$(28) \quad S(p) = \sum_r K(p, e_r) + \frac{\delta}{a^2}.$$

PROPOSITION 2. *Let $f: M^2 \rightarrow Y^3$ be an isometric immersion from a compact surface M^2 into a real space form Y^3 with constant riemannian sectional curvature δ/a^2 . Then the total absolute curvature $TA(f)$ satisfies the following inequality:*

$$(29) \quad TA(f) \geq \left| \frac{\delta v(M^2)}{2a^2\pi} + (2|\delta| - 2)\beta_1(M^2) - e(M^2) \right|$$

where $e(M^2)$ denotes the Euler characteristic of M^2 . In particular, if $\delta \neq 0$, then the equality of (29) holds when and only when the Lipschitz-Killing curvature of the immersion $f: M^2 \rightarrow Y^3$ is of constant sign.

Proof. By the assumption, we know that for every p in M^2 , there exist two elements of B_p over p , said (p, e) and $(p, -e)$. By (7) and (9), we have $K(p, e) = K(p, -e)$. Thus, by equation (28), we get

$$(30) \quad 2\pi e(M^2) = \int_{M^2} K(p, e) dV + \delta v(M^2)/a^2.$$

Thus, by (30) and a result due to Chern-Lashof [3], we can verify that

$$(31) \quad TA(f) \geq |e(M^2) - \delta v(M^2)/2a^2\pi| \quad \text{if } \delta \neq 0,$$

and

$$(32) \quad TA(f) \geq e(M^2) + 2\beta_1(M^2) \quad \text{if } \delta = 0.$$

From (31) and (32), we can easily deduce that the inequality (29) holds. Furthermore, if $\delta \neq 0$, then by (30), we can easily find that the equality of (29) holds when and only when the Lipschitz-Killing curvature of the immersion f is of constant sign. This completes the proof of the Proposition.

THEOREM 3. *Let $f: M^2 \rightarrow S^3$ be an isometric immersion from an oriented compact surface M^2 into a 3-sphere S^3 with radius a . If the total absolute curvature satisfies the following inequality:*

$$(33) \quad TA(f) < \frac{v(M^2)}{2a^2\pi},$$

then M^2 is diffeomorphic to a 2-sphere. Furthermore, there exists an immersion from a torus into S^3 with the total absolute curvature $v(M^2)/2a^2\pi$.

Proof. Let g denote the genus of the oriented surface M^2 . Then by Proposition 2, we have the following inequality:

$$(34) \quad TA(f) \geq \left| \frac{v(M^2)}{2a^2\pi} - 2 + 2g \right|.$$

Hence, if (33) holds, then by (34), we get $g=0$. This means that M^2 is diffeomorphic to a 2-sphere.

Now, let $\gamma: T^2 \rightarrow S^3$ be the inclusion mapping of the flat torus:

$$(35) \quad (c \cos u, c \sin u, d \cos v, d \sin v), \quad c^2 + d^2 = a^2,$$

into S^3 . Then, by (35), we can find that

$$(36) \quad TA(\gamma) = \frac{2cd\pi}{a^2}, \quad v(M^2) = 4cd\pi^2.$$

Thus, by (36), we have $TA(\gamma) = v(M^2)/2a^2\pi$. This completes the proof of the Theorem.

Similarly, we can prove that

THEOREM 4. *Let $f: M^2 \rightarrow S^3$ be an isometric immersion from an oriented compact surface M^2 into a 3-sphere S^3 with radius a . If the total absolute curvature $TA(f)$ satisfies the following inequality:*

$$(37) \quad TA(f) < \frac{v(M^2)}{2a^2\pi} + 2k, \quad k=0, 1, 2, \dots,$$

then the genus of M^2 is one of the following integers: $0, 1, 2, \dots, k$. In particular, if $k=1$, then M^2 is either diffeomorphic to a 2-sphere or a 2-torus.

REMARK. Use Proposition 2. We can get an analogue statement of Theorem 4 for the oriented compact surfaces in hyperbolic space, and also for non-orientable case.

4. Surfaces in real space forms with $TA(f)=0$.

THEOREM 5. *Let $f: M^2 \rightarrow Y^N$ be an isometric immersion from a complete surface M^2 into Y^N . Then we have the following:*

Case (I): $Y^N = S^N$; If the total absolute curvature $TA(f)=0$, then M^2 is isometric to a 2-sphere with radius a or to a projective plane of constant riemannian sectional curvature $1/a^2$. Furthermore, M^2 is immersed as a totally geodesic submanifold of Y^N if and only if $TA(f)=0$ and f is minimal. In particular, if $N=3$, then $TA(f)=0$ if and only if f is totally geodesic.

Case (II) $Y^N = E^N$; The total absolute curvature $TA(f)=0$ if and only if M^2 is immersed as a cylinder in E^N .

Case (III) $Y^N = H^N$; If $TA(f)=0$, then M^2 is a complete surface with constant negative riemannian sectional curvature $-1/a^2$. In particular, if M^2 is compact, then there exists no immersion from M^2 into H^3 such that the total absolute curvature is equal to zero.

Proof. Case (I) $Y^N=S^N$; If the total absolute curvature $TA(f)=0$, then by (28), we find that the riemannian sectional curvature of M^2 is equal to $1/a^2$. Hence, by the completeness of M^2 , we know that M^2 is either isometric to a 2-sphere with radius a or isometric to a real projective plane of constant riemannian sectional curvature $1/a^2$.

If the total absolute curvature $TA(f)=0$ and f is minimal, then we have $\det(A_{r,i,j})=\text{trace}(A_{r,i,j})=0$, for all r . Hence, by the fact that $\dim M^2=2$, we have $A_{r,i,j}=0$ for all r, i, j . Thus the second fundamental form vanishes. This means that M^2 is immersed as a totally geodesic submanifold of Y^N . The converse of this is trivial.

Now, suppose that $N=3$, and the total absolute curvature $TA(f)=0$. Then M^2 is isometric to a 2-sphere S^2 with radius a . Furthermore, by the assumption $TA(f)=0$, we have

$$(38) \quad A_{312}^2 - A_{311}A_{322} = 0.$$

Now let G be the open subset of M^2 such that $A_3=(A_{3i,j})\neq 0$. If $G\neq\emptyset$, we take a local cross-section $(p, \bar{e}_1, \bar{e}_2, e_3)$ from M^2 into the bundle B , such that

$$A_{311}=A_{312}=A_{321}=0, \quad A_{322}=\varphi(p)\neq 0,$$

then, from $\omega_{13}=0$, we get $d\omega_{13}=\omega_{12}\wedge\omega_{23}=\varphi\omega_{12}\wedge\omega_2=0$, hence $\omega_{12}=0 \pmod{\omega_2}$. Therefore the integral curve of the local field \bar{e}_1 is a geodesic in M^2 and S^3 . Now, putting $\omega_{12}=\rho\omega_2$, we have $d\omega_{23}=d\varphi\wedge\omega_2+\varphi\omega_1\wedge\omega_{12}=\omega_{21}\wedge\omega_{13}=0$, hence along this geodesic we have $d\log\varphi+\rho\omega_1=0$, that is $\varphi=\exp(-\int\rho\omega_1)$. By means of the completeness of M^2 , this local field of frame can be extended as possible as far in G . But, the above equality shows that this process is endless. Thus we see that the above geodesic is a great circle in M^2 and S^3 (We may consider $M^2\approx S^2(a)$). Therefore G is an open set such that through any point there exists one and only one closed great circle in it. This is impossible for M^2 . Hence it must be $G=\emptyset$. Thus the immersion f is totally geodesic. This completes the proof of Part (I).

Case (II) $Y^N=E^N$; If the total absolute curvature $TA(f)=0$, then, by (10), we know that the Lipschitz-Killing curvature $K(p, e)$ is identically zero. Hence, M^2 is a complete flat surface in E^N . Therefore, M^2 must be isometric to one of the following surfaces: Euclidean planes, Cylinders, Tori, Möbius bands and Klein bottles. Furthermore, by a result due to Chern-Lashof [3], we know that every closed surface cannot be immersed into euclidean spaces with vanishing total absolute curvature. Hence, by the above results and the fact that the Möbius band in euclidean spaces has positive total absolute curvature, we know that M^2 is either isometric to a euclidean plane or isometric to a complete cylinder. On the other hand, Shiohama [10] proved that the only complete orientable surface in euclidean spaces with vanishing Lipschitz-Killing curvature is cylinder (or plane). Hence, we know that M^2 is immersed as a cylinder (or plane) in E^N . The converse of this is trivial.

Case (III) $Y^N=H^N$; If the total absolute curvature $TA(f)=0$, then, by equation (28), we know that the riemannian sectional curvature is given by

$$(39) \quad S(\rho) = -\frac{1}{a^2}.$$

Thus, M^2 is a complete surface with constant negative riemannian sectional curvature $-1/a^2$. Furthermore, by a result due to O'Neill [7], we know that every compact n -dimensional riemannian manifold with riemannian sectional curvature K cannot be isometrically immersed in a complete simply connected $(n+N)$ -dimensional riemannian manifold with riemannian sectional curvature K' if $N \leq n$ and $K \leq K' \leq 0$. Consequently, we have proved that if M^2 is compact, then there exists no isometric immersion of M^2 into Y^3 with vanishing total absolute curvature. This completes the proof of the Theorem.

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