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ON THE TOTAL GRAPH OF A COMMUTATIVE RING WITHOUT THE ZERO ELEMENT

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Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. The total graph of R is the (undirected) graph $T(\Gamma(R))$ with vertices all elements of R, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. In this paper, we study the two (induced) subgraphs $Z_0(\Gamma(R))$ and $T_0(\Gamma(R))$ of $T(\Gamma(R))$, with vertices $Z(R)\backslash\{0\}$ and $R\backslash\{0\}$, respectively. We determine when $Z_0(\Gamma(R))$ and $T_0(\Gamma(R))$ are connected and compute their diameter and girth. We also investigate zero-divisor paths and regular paths in $T_0(\Gamma(R))$.

Keywords: Total graph; zero-divisor graph; total graph without zero.

Mathematics Subject Classification: 13A15, 05C99

1. Introduction

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. In [5], we defined the *total graph* of R to be the (undirected) graph $T(\Gamma(R))$ with all elements of R as vertices, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. Let $Z(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with $Z(R)$ as its set of vertices. Then $Z(\Gamma(R))$ is connected with $\mathrm{diam}(Z(\Gamma(R))) \leq 2$ since $x - 0 - y$ is a path between any two vertices x and y in $Z(\Gamma(R))$. In this paper, we consider the (induced) subgraphs $Z_0(\Gamma(R))$ of $Z(\Gamma(R))$ and $T_0(\Gamma(R))$ of $T(\Gamma(R))$ obtained by deleting 0 as a vertex. Specifically, $Z_0(\Gamma(R))$ (respectively,

$T_0(\Gamma(R)))$ has vertices $Z(R)^* = Z(R)\backslash\{0\}$ (respectively, $R^* = R\backslash\{0\}$), and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. Note that $Z_0(\Gamma(R))$ is a finite nonempty graph if and only if R is a finite ring that is not a field (cf. [6, Theorem 2.2]). In addition to $Z(\Gamma(R))$, the (induced) subgraphs $\mathrm{Reg}(\Gamma(R))$ and $\mathrm{Nil}(\Gamma(R))$ of $T(\Gamma(R))$, with vertices $\mathrm{Reg}(R)$ and $\mathrm{Nil}(R)$, respectively, were studied in [5]. The total graph has also been investigated in [1, 2, 14].

Recently, there has been considerable attention in the literature to associating graphs with algebraic structures (see [13]). Probably the most attention has been to the *zero-divisor graph* $\Gamma(R)$ for a commutative ring R. The set of vertices of $\Gamma(R)$ is $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $xy = 0$. So, in some sense, $Z_0(\Gamma(R))$ is the additive analog of $\Gamma(R)$. The concept of a zero-divisor graph goes back to Beck [7], who let all elements of R be vertices and was mainly interested in colorings. Our definition was introduced by Anderson and Livingston in [6], where it was shown, among other things, that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\mathrm{gr}(\Gamma(R)) \in \{3, 4, \infty\}$. For a recent survey article on zero-divisor graphs, see [4].

In the second section, we determine when $Z_0(\Gamma(R))$ is connected and show that $\mathrm{diam}(Z_0(\Gamma(R))) \in \{0, 1, 2, \infty\}$. In the third section, we show that $\mathrm{gr}(Z_0(\Gamma(R))) \in \{3, \infty\}$ and explicitly calculate $\mathrm{gr}(Z_0(\Gamma(R)))$. In both cases, our answers depend on whether or not R is reduced and on the number of minimal prime ideals of R. In the fourth section, we consider the graph $T_0(\Gamma(R))$, show that $\mathrm{diam}(T_0(\Gamma(R))) = \mathrm{diam}(T(\Gamma(R)))$ when $|R| \geq 4$, and determine its girth. In the final section, we define and investigate zero-divisor paths and regular paths in $T_0(\Gamma(R))$.

Let Γ be a graph. For vertices x and y of Γ, we define $\mathrm{d}(x, y)$ to be the length of a shortest path from x to y ($\mathrm{d}(x, x) = 0$ and $\mathrm{d}(x, y) = \infty$ if there is no path). Then the *diameter* of Γ is $\mathrm{diam}(\Gamma) = \sup\{d(x, y) \,|\, x \text{ and } y \text{ are vertices of } \Gamma\}$. The *girth* of Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ ($\mathrm{gr}(\Gamma) = \infty$ if Γ contains no cycles).

Throughout, R will be a commutative ring with nonzero identity, $Z(R)$ its set of zero-divisors, $\mathrm{Reg}(R) = R\backslash Z(R)$ its set of regular elements, $\mathrm{Idem}(R)$ its set of idempotent elements, $\mathrm{Nil}(R)$ its ideal of nilpotent elements, $U(R)$ its group of units, and total quotient ring $T(R) = R_{\mathrm{Reg}(R)}$. For any $A \subseteq R$, let $A^* = A\backslash\{0\}$. We say that R is *reduced* if $\mathrm{Nil}(R) = \{0\}$ and that R is *quasilocal* if R has a unique maximal ideal. Let $\mathrm{Spec}(R)$ denote the set of prime ideals of R, $\mathrm{Max}(R)$ the set of maximal ideals of R, and $\mathrm{Min}(R)$ the set of minimal prime ideals of R. Any undefined notation or terminology is standard, as in [10, 11], or [8].

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2. The Diameter of $Z_0(\Gamma(R))$

In this section, we show that $Z_0(\Gamma(R))$ is connected unless R is a reduced ring with exactly two minimal prime ideals. Moreover, if $Z_0(\Gamma(R))$ is connected, then

$\mathrm{diam}(Z_0(\Gamma(R))) \leq 2$. The case for $Z(\Gamma(R))$ is much simpler since every nonzero vertex in $Z(\Gamma(R))$ is adjacent to 0. If $Z(R)$ is an ideal of R, then $Z(\Gamma(R))$ is a complete graph [5, Theorem 2.1]; and if $Z(R)$ is not an ideal of R, then $Z(\Gamma(R))$ is connected with $\mathrm{diam}(Z(\Gamma(R))) = 2$ [5, Theorem 3.1].

We begin with a lemma containing several results which we will use throughout this paper.

Lemma 2.1. *Let R be a commutative ring.*

(1) *$Z(R)$ is a union of prime ideals of R.*
(2) *$P \subseteq Z(R)$ for every $P \in \mathrm{Min}(R)$.*
(3) *$Z(R) = \cup\{P \mid P \in \mathrm{Min}(R)\}$ if R is reduced.*
(4) *Let $x \in Z(R)$ and $y \in \mathrm{Nil}(R)$. Then $x + y \in Z(R)$.*
(5) *If P_1, P_2, P_3 are distinct minimal prime ideals of R, then $P_1 \cap P_2 \cap P_3 \subsetneq P_1 \cap P_2$.*

Proof. For (1), see [11, Theorem 2 and Remarks]. Parts (2) and (3) may be found in [10, Theorem 2.1; 10, Corollary 2.4], respectively.

(4) By (1) above, $x \in P \subseteq Z(R)$ for some $P \in \mathrm{Spec}(R)$. Since $y \in \mathrm{Nil}(R) \subseteq P$, it follows that $x + y \in P \subseteq Z(R)$.

(5) If $P_1 \cap P_2 = P_1 \cap P_2 \cap P_3$, then $P_1P_2 \subseteq P_1 \cap P_2 \subseteq P_3$. Thus either $P_1 \subseteq P_3$ or $P_2 \subseteq P_3$, a contradiction. $\square$

We first study the case when R is not reduced.

Theorem 2.2. *Let R be a non-reduced commutative ring. Then $Z_0(\Gamma(R))$ is connected with* $\mathrm{diam}(Z_0(\Gamma(R))) \in \{0, 1, 2\}$.

Proof. Assume that R is not reduced, and let $x, y \in Z(R)^*$ be distinct vertices of $Z_0(\Gamma(R))$. If either $x \in \mathrm{Nil}(R)$ or $y \in \mathrm{Nil}(R)$, then $x + y \in Z(R)$ by Lemma 2.1(4); so $x - y$ is an edge in $Z_0(\Gamma(R))$. Thus we may assume that $x \notin \mathrm{Nil}(R)$, $y \notin \mathrm{Nil}(R)$, and $x + y \notin Z(R)$. Let $0 \neq w \in \mathrm{Nil}(R)$. Then $x - w - y$ is a path in $Z_0(\Gamma(R))$ by Lemma 2.1(4), and hence $\mathrm{diam}(Z_0(\Gamma(R))) \leq 2$. $\square$

Note that $Z_0(\Gamma(R))$ is a complete graph if and only if $Z(R)$ is an ideal of R, and in this case, $\mathrm{diam}(Z_0(\Gamma(R))) \leq 1$. Also, $Z(R)$ is a union of prime ideals of R by Lemma 2.1(1); so $Z(R)$ is an ideal of R if and only if it is a prime ideal of R. Thus a non-reduced ring R has $\mathrm{diam}(Z_0(\Gamma(R))) = 0$ if and only if $|Z(R)^*| = 1$, if and only if $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$. Examples of non-reduced rings R with either $\mathrm{diam}(Z_0(\Gamma(R))) = 1$ or $\mathrm{diam}(Z_0(\Gamma(R))) = 2$ are given in Example 2.9 (also see Theorem 2.8).

We next consider the case when R is reduced. In this case, R is an integral domain if and only if $|\mathrm{Min}(R)| = 1$. If R is an integral domain, then $Z_0(\Gamma(R))$ is the empty graph; so we assume that $|\mathrm{Min}(R)| \geq 2$.

Theorem 2.3. *Let R be a reduced commutative ring with $|\mathrm{Min}(R)| = 2$. Then $Z_0(\Gamma(R))$ is not connected.*

Proof. Suppose that R is reduced and $|\mathrm{Min}(R)| = 2$. Let P and Q be the minimal prime ideals of R. Then $\mathrm{Nil}(R) = P \cap Q = \{0\}$, and $Z(R) = P \cup Q$ by Lemma 2.1(3) since R is reduced. Let $0 \neq x \in P$ and $0 \neq y \in Q$. Then $x + y \notin Z(R)$; so there can be no path in $Z_0(\Gamma(R))$ from any $a \in P^*$ to any $b \in Q^*$. Thus, $Z_0(\Gamma(R))$ is not connected. □

Note that the P^* and Q^* in the proof of Theorem 2.3 are the connected components of $Z_0(\Gamma(R))$, and each component is a complete subgraph of $Z_0(\Gamma(R))$. However, in this case, $Z(R)$ is not an ideal of R; so $Z(\Gamma(R))$ is connected with $\mathrm{diam}(Z(\Gamma(R))) = 2$ when R is reduced and $|\mathrm{Min}(R)| = 2$.

Theorem 2.4. *Let R be a reduced commutative ring that is not an integral domain. Then $Z_0(\Gamma(R))$ is connected if and only if $|\mathrm{Min}(R)| \geq 3$. Moreover, if $Z_0(\Gamma(R))$ is connected, then $\mathrm{diam}(Z_0(\Gamma(R))) \in \{1, 2\}$.*

Proof. Suppose that $Z_0(\Gamma(R))$ is connected and R is reduced, but not an integral domain. Then $|\mathrm{Min}(R)| \geq 3$ by Theorem 2.3. Conversely, suppose that R is reduced and $|\mathrm{Min}(R)| \geq 3$. Let $x, y \in Z(R)^*$ such that $x + y \notin Z(R)$ (thus $x \neq y$). Then there are minimal prime ideals P_1 and P_2 of R with $x \in P_1$ and $y \in P_2$ by Lemma 2.1(3), and $P_1 \neq P_2$ since $x + y \notin Z(R)$. Since $|\mathrm{Min}(R)| \geq 3$, there is a $Q \in \mathrm{Min}(R) \backslash \{P_1, P_2\}$; so $P_1 \cap P_2 \neq \{0\}$ by Lemma 2.1(5). Pick $0 \neq z \in P_1 \cap P_2$. Then $x - z - y$ is a path in $Z_0(\Gamma(R))$ from x to y. Thus $Z_0(\Gamma(R))$ is connected with $\mathrm{diam}(Z_0(\Gamma(R))) \leq 2$, and $\mathrm{diam}(Z_0(\Gamma(R))) \neq 0$ since $|Z(R)^*| \geq 2$. Hence $1 \leq \mathrm{diam}(Z_0(\Gamma(R))) \leq 2$. □

Corollary 2.5. *Let R be a reduced commutative ring with $3 \leq |\mathrm{Min}(R)| < \infty$. Then $\mathrm{diam}(Z_0(\Gamma(R))) = 2$. In particular, $\mathrm{diam}(Z_0(\Gamma(R))) = 2$ when R is a reduced Noetherian ring with $|\mathrm{Min}(R)| \geq 3$.*

Proof. We have $1 \leq \mathrm{diam}(Z_0(\Gamma(R))) \leq 2$ by Theorem 2.4. Also, $\mathrm{diam}(Z_0(\Gamma(R))) \leq 1$ if and only if $Z(R)$ is a prime ideal of R. If R is reduced with $\mathrm{Min}(R)$ finite, then $Z(R)$ is a prime ideal of R if and only if $\mathrm{Min}(R) = \{Z(R)\}$ by Lemma 2.1(3) and the Prime Avoidance Lemma [11, Theorem 81]. But $|\mathrm{Min}(R)| \geq 3$; so $\mathrm{diam}(Z_0(\Gamma(R))) = 2$. The "in particular" statement is clear since $\mathrm{Min}(R)$ is finite when R is Noetherian [11, Theorem 88]. □

Corollary 2.6. *The following statements are equivalent for a commutative ring R.*

(1) *$Z_0(\Gamma(R))$ is not connected.*
(2) *$T(R)$ is a von Neumann regular ring with exactly two maximal ideals.*
(3) *$T(R)$ is isomorphic to $K_1 \times K_2$ for fields K_1 and K_2.*

In particular, if R is a finite ring, then $Z_0(\Gamma(R))$ is connected unless $R \cong K_1 \times K_2$ for finite fields K_1 and K_2.

Proof. This follows directly from Theorems 2.3 and 2.4. The "in particular" statement is clear. □

Let R be a reduced commutative ring with $|\mathrm{Min}(R)| \geq 3$. By Corollary 2.5, $\mathrm{diam}(Z_0(\Gamma(R))) = 2$ if $\mathrm{Min}(R)$ is finite. Note that $\mathrm{diam}(Z_0(\Gamma(R))) = 1$ if and only if $Z(R)$ is an (prime) ideal of R; so if R is reduced with $|\mathrm{Min}(R)| \geq 3$ and $\mathrm{diam}(Z_0(\Gamma(R))) = 1$, then both $\mathrm{Min}(R)$ and $Z(R)$ must be infinite. An example of a reduced quasilocal commutative ring R with nonzero maximal ideal $Z(R)$ is given in [3, Example 3.13] (cf. [12, Example 5.1]). For this ring R, both $\mathrm{Min}(R)$ and $Z(R)$ are infinite, and $Z_0(\Gamma(R))$ is connected with $\mathrm{diam}(Z_0(\Gamma(R))) = 1$.

The next two theorems summarize results about $\mathrm{diam}(Z(\Gamma(R)))$ (mentioned earlier from [5]) and $\mathrm{diam}(Z_0(\Gamma(R)))$ when R is a finite commutative ring. Note that $\mathrm{Max}(R) = \mathrm{Min}(R)$ when R is a finite commutative ring.

Theorem 2.7. *Let R be a finite commutative ring. Then* $\mathrm{diam}(Z(\Gamma(R))) \in \{0, 1, 2\}$. *Moreover,*

(1) $\mathrm{diam}(Z(\Gamma(R))) = 0$ *if and only if R is a field,*
(2) $\mathrm{diam}(Z(\Gamma(R))) = 1$ *if and only if R is local and not a field, and*
(3) $\mathrm{diam}(Z(\Gamma(R))) = 2$ *if and only if R is not local.*

Theorem 2.8. *Let R be a finite commutative ring. Then* $\mathrm{diam}(Z_0(\Gamma(R))) \in \{0, 1, 2, \infty\}$. *Moreover,*
(1) $Z_0(\Gamma(R))$ *is the empty graph if and only if R is a field,*
(2) $\mathrm{diam}(Z_0(\Gamma(R))) = 0$ *if and only if R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$,*
(3) $\mathrm{diam}(Z_0(\Gamma(R))) = 1$ *if and only if R is a local ring with maximal ideal M and $|M| \geq 3$,*
(4) $\mathrm{diam}(Z_0(\Gamma(R))) = 2$ *if and only if either $|\mathrm{Max}(R)| \geq 3$ or R is not reduced with $|\mathrm{Max}(R)| = 2$, and*
(5) $\mathrm{diam}(Z_0(\Gamma(R))) = \infty$ *if and only if R is reduced with $|\mathrm{Max}(R)| = 2$.*

We next illustrate the above results by computing $\mathrm{diam}(Z_0(\Gamma(R)))$ for $R = \mathbb{Z}_n$ and $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. The details are left to the reader; they follow directly from Theorem 2.8.

Example 2.9. (a) ($\mathrm{diam}(Z_0(\Gamma(\mathbb{Z}_n)))$) Let $R = \mathbb{Z}_n$ with $n \geq 2$ and n not prime (note that $Z_0(\Gamma(\mathbb{Z}_n))$ is the empty graph if n is prime). Then $\mathrm{diam}(Z_0(\Gamma(\mathbb{Z}_4))) = 0$; $\mathrm{diam}(Z_0(\Gamma(\mathbb{Z}_{p^m}))) = 1$ if either $p = 2$ and $m \geq 3$, or $p \geq 3$ is prime and $m \geq 2$; $\mathrm{diam}(Z_0(\Gamma(\mathbb{Z}_{pq}))) = \infty$ for distinct primes p and q; and $\mathrm{diam}(Z_0(\Gamma(R))) = 2$ otherwise.

(b) ($\mathrm{diam}(Z_0(\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}))))$ Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2$. Then $\mathrm{diam}(Z_0(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))) = \infty$ for primes $p \leq q$; otherwise $\mathrm{diam}(Z_0(\Gamma(R))) = 2$.

3. The Girth of $Z_0(\Gamma(R))$

In this section, we show that $\mathrm{gr}(Z_0(\Gamma(R))) \in \{3, \infty\}$. If $Z(R)$ is an ideal of R, then it is clear that $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ if $|Z(R)| \leq 3$ and $\mathrm{gr}(Z_0(\Gamma(R))) = 3$ if

$|Z(R)| \geq 4$. Just as for the diameter in Sec. 2, our answer depends on the number of minimal prime ideals of R. If $Z(R)$ is an ideal of R, then, $\mathrm{gr}(Z(\Gamma(R))) = \infty$ if $|Z(R)| \leq 2$ and $\mathrm{gr}(Z(\Gamma(R))) = 3$ if $|Z(R)| \geq 3$. If $Z(R)$ is not an ideal of R, then $\mathrm{gr}(Z(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = \infty$ and $\mathrm{gr}(Z(\Gamma(R))) = 3$ if $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ [5, Theorem 3.14(1)] (also, see Theorem 3.3(1)).

We first handle the case when R is not reduced.

Theorem 3.1. *Let R be a non-reduced commutative ring. Then* $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ *if and only if R has a unique nonzero minimal prime ideal P with* $P = \mathrm{Nil}(R) = Z(R)$ *and* $|P| \leq 3$ (i.e. $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ *if and only if* $\mathrm{Nil}(R) = Z(R)$ *and* $|\mathrm{Nil}(R)| \leq 3$). *Otherwise*, $\mathrm{gr}(Z_0(\Gamma(R))) = 3$. *Moreover*, $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ *if* $|Z(R)| \leq 3$ *and* $\mathrm{gr}(Z_0(\Gamma(R))) = 3$ *if* $|Z(R)| \geq 4$.

Proof. Suppose that $|\mathrm{Min}(R)| \geq 2$. Let P and Q be distinct minimal prime ideals of R. Then $\{0\} \subsetneq P \cap Q \subsetneq P$; so $|P \cap Q| \geq 2$, and thus $|P| \geq 4$. Let $x, y, z \in P^*$ be distinct. Then $x - y - z - x$ is a triangle in $Z_0(\Gamma(R))$; so $\mathrm{gr}(Z_0(\Gamma(R))) = 3$. Now suppose that $\mathrm{Min}(R) = \{P\}$, and thus $\mathrm{Nil}(R) = P$. If $\mathrm{Nil}(R) \subsetneq Z(R)$, then there is a prime ideal Q of R with $\{0\} \neq \mathrm{Nil}(R) = P \subsetneq Q \subseteq Z(R)$ by Lemma 2.1(1). As above, $|Q| \geq 4$; so again $\mathrm{gr}(Z_0(\Gamma(R))) = 3$. If $\mathrm{Nil}(R) = Z(R)$, then $\mathrm{gr}(Z_0(\Gamma(R))) = 3$ if $|\mathrm{Nil}(R)| \geq 4$ and $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ if $|\mathrm{Nil}(R)| \leq 3$. The "moreover" statement follows directly from the above arguments. □

We next consider the case when R is reduced.

Theorem 3.2. *Let R be a reduced commutative ring that is not an integral domain. Then* $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ *if and only if* $\mathrm{Min}(R) = \{P, Q\}$ *with* $\max\{|P|, |Q|\} \leq 3$. *Otherwise*, $\mathrm{gr}(Z_0(\Gamma(R))) = 3$. *In particular*, $\mathrm{gr}(Z_0(\Gamma(R))) = 3$ *when* $|\mathrm{Min}(R)| \geq 3$.

Proof. Suppose that P_1, P_2, P_3 are distinct minimal prime ideals of R. Then $\{0\} \subseteq P_1 \cap P_2 \cap P_3 \subsetneq P_1 \cap P_2 \subsetneq P_1$ by Lemma 2.1(5); so $|P_1 \cap P_2| \geq 2$, and thus $|P_1| \geq 4$. Let $x, y, z \in P_1^*$ be distinct. Then $x - y - z - x$ is a triangle in $Z_0(\Gamma(R))$; so $\mathrm{gr}(Z_0(\Gamma(R))) = 3$ if $|\mathrm{Min}(R)| \geq 3$. Thus we may assume that $|\mathrm{Min}(R)| = 2$; say $\mathrm{Min}(R) = \{P, Q\}$. As in the proof of Theorem 2.3, $P \cap Q = \{0\}$ and $Z(R) = P \cup Q$, and hence no $x \in P^*$ and $y \in Q^*$ are adjacent in $Z_0(\Gamma(R))$. Thus $\mathrm{gr}(Z_0(\Gamma(R))) = 3$ if and only if either $|P| \geq 4$ or $|Q| \geq 4$. Otherwise, $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$. The "in particular" statement is clear. □

Using earlier mentioned results from [5] and Theorems 3.1 and 3.2, we can give explicit calculations for $\mathrm{gr}(Z_0(\Gamma(R)))$ and $\mathrm{gr}(Z(\Gamma(R)))$.

Theorem 3.3. *Let R be a commutative ring. Then* $\mathrm{gr}(Z(\Gamma(R))) \in \{3, \infty\}$ *and* $\mathrm{gr}(Z_0(\Gamma(R))) \in \{3, \infty\}$.

(1) $\mathrm{gr}(Z(\Gamma(R))) = \infty$ *if and only if either R is an integral domain or R is isomorphic to* $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$. *Otherwise*, $\mathrm{gr}(Z(\Gamma(R))) = 3$.

(2) $Z_0(\Gamma(R))$ *is the empty graph if and only if* R *is an integral domain. For* R *not an integral domain,* $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ *if and only if* R *is isomorphic to* $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6$, $\mathbb{Z}_9$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, *or* $\mathbb{Z}_3[X]/(X^2)$. *Otherwise,* $\mathrm{gr}(Z_0(\Gamma(R))) = 3$.

Proof. (1) First, suppose that $Z(R)$ is an ideal of R. If $|Z(R)| = 1$, then R is an integral domain; so $|Z(\Gamma(R))| = 1$, and thus $\mathrm{gr}(Z(\Gamma(R))) = \infty$. If $|Z(R)| = 2$, then R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$; so $|Z(\Gamma(R))| = 2$, and hence $\mathrm{gr}(Z(\Gamma(R))) = \infty$. If $|Z(R)| \geq 3$, then $\mathrm{gr}(Z(\Gamma(R))) = 3$ since $x - 0 - y - x$ is a triangle in $Z(\Gamma(R))$ for distinct $x, y \in Z(R)^*$. If $Z(R)$ is not an ideal of R, then $\mathrm{gr}(Z(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = \infty$ and $\mathrm{gr}(Z(\Gamma(R))) = 3$ if $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ [5, Theorem 3.14(1)]. Part (1) now follows directly from the above two cases.

(2) First, suppose that R is not reduced. Then by Theorem 3.1, $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $\{0\} \neq \mathrm{Nil}(R) = Z(R)$ and $|Z(R)| \leq 3$, and $\mathrm{gr}(Z_0(\Gamma(R))) = 3$ otherwise. So in this case, $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ if and only if R is isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_9$, or $\mathbb{Z}_3[X]/(X^2)$.

Next, suppose that R is reduced and not an integral domain. Then by Theorem 3.2, $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $\mathrm{Min}(R) = \{P, Q\}$ with $\max\{|P|, |Q|\} \leq 3$, and $\mathrm{gr}(Z_0(\Gamma(R))) = 3$ otherwise. In the first case, we have $Z(R) = P \cup Q$ and $P \cap Q = \{0\}$ with $\max\{|P|, |Q|\} \leq 3$. In this case, R is a reduced finite ring with two maximal ideals, each with two or three elements. Thus $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Part (2) now follows directly from the above two cases. □

We end this section with the analog of Example 2.9 for $\mathrm{gr}(Z_0(\Gamma(R)))$ when $R = \mathbb{Z}_n$ or $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. The details are left to the reader; they follow directly from Theorem 3.3(2).

Example 3.4. (a) ($\mathrm{gr}(Z_0(\Gamma(\mathbb{Z}_n)))$) Let $R = \mathbb{Z}_n$ with $n \geq 2$ and n not prime (note that $Z_0(\Gamma(\mathbb{Z}_n))$ is the empty graph if n is prime). Then $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ if either $n = 4$, $n = 6$, or $n = 9$. Otherwise, $\mathrm{gr}(Z_0(\Gamma(R))) = 3$.

(b) ($\mathrm{gr}(Z_0(\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})))$) Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2$. Then $\mathrm{gr}(Z_0(\Gamma(R))) = \infty$ if either $n_1 = n_2 = 2$, $n_1 = 2$ and $n_2 = 3$, or $n_1 = n_2 = 3$. Otherwise, $\mathrm{gr}(Z_0(\Gamma(R))) = 3$.

4. $T_0(\Gamma(R))$

In this section, we study the graph $T_0(\Gamma(R))$. We show that $\mathrm{diam}(T_0(\Gamma(R))) = \mathrm{diam}(T(\Gamma(R)))$ if and only if $|R| \geq 4$. (Note that $|R| \leq 3$ if and only if R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.) We then explicitly compute $\mathrm{gr}(T_0(\Gamma(R)))$. For $x, y \in R^*$, let $\mathrm{d}_T(x, y)$ (respectively, $\mathrm{d}_{T_0}(x, y)$) denote the distance from x to y in $T(\Gamma(R))$ (respectively, $T_0(\Gamma(R))$). We first show that these two distances are always equal.

Lemma 4.1. *Let R be a commutative ring and $x, y \in R^*$. Then x, y are connected by a path in $T_0(\Gamma(R))$ if and only if x, y are connected by a path in $T(\Gamma(R))$. Moreover, $d_{T_0}(x, y) = d_T(x, y)$ and $\text{diam}(T_0(\Gamma(R))) \leq \text{diam}(T(\Gamma(R)))$.*

Proof. If x, y are connected by a path in $T_0(\Gamma(R))$, then clearly x, y are connected by a path in $T(\Gamma(R))$. Conversely, assume that $x - a_1 - \cdots - a_n - y$ is a shortest path from x to y in $T(\Gamma(R))$, and assume that $a_i = 0$ for some i with $1 \leq i \leq n$. Then $a_{i-1}, a_{i+1} \in Z(R)^*$ and $a_{i-1} + a_{i+1} \in \text{Reg}(R)$ (let $a_0 = x$ and $a_{n+1} = y$). Let $z_i = -(a_{i-1} + a_{i+1})$. Then $x - a_1 - \cdots - a_{i-1} - z_i - a_{i+1} - \cdots - a_n - y$ is a shortest path from x to y in $T_0(\Gamma(R))$, and hence x, y are connected by a path in $T_0(\Gamma(R))$. The "moreover" statement is clear. □

Recall that $T(\Gamma(R))$ is not connected if $Z(R)$ is an ideal of R [5, Theorem 2.1]. If $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is connected if and only if $(Z(R)) = R$ (i.e. R is generated by $Z(R)$ as an ideal) [5, Theorem 3.3]. Moreover, in this case, $\text{diam}(T(\Gamma(R))) = n$, where $n \geq 2$ is the least positive integer such that $R = (z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in Z(R)$ [5, Theorem 3.4]. Also, $\text{diam}(T(\Gamma(R))) = \text{d}_T(0, 1)$ [5, Corollary 3.5(1)]. Thus $T(\Gamma(R))$ is connected if and only if $\text{diam}(T(\Gamma(R))) < \infty$.

Theorem 4.2. *Let R be a commutative ring.*

(1) *If $|R| \leq 3$, then $T_0(\Gamma(R))$ is connected, but $T(\Gamma(R))$ is not connected.*
(2) *If $|R| \geq 4$, then $T_0(\Gamma(R))$ is connected if and only if $T(\Gamma(R))$ is connected.*

Proof. (1) If $|R| \leq 3$, then $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$. It is easily verified that (1) holds for these two rings.

(2) If $T(\Gamma(R))$ is connected, then $T_0(\Gamma(R))$ is also connected by Lemma 4.1. Conversely, assume that $T_0(\Gamma(R))$ is connected and $|R| \geq 4$. Then R is not an integral domain; so there is an $x \in Z(R)^*$. Let $y \in R^*$. Then there is a path from x to y in $T_0(\Gamma(R))$. But x is adjacent to 0 in $T(\Gamma(R))$; so there is a path from 0 to y in $T(\Gamma(R))$. Thus $T(\Gamma(R))$ is also connected. □

Corollary 4.3. *Let R be a commutative ring. Then $T_0(\Gamma(R))$ is connected if and only if either $(Z(R)) = R$ or R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Moreover, $T_0(\Gamma(R))$ is connected if and only if $\text{diam}(T_0(\Gamma(R))) < \infty$.*

Proof. This follows directly from Theorem 4.2 and the discussion preceding Theorem 4.2. □

In general, there is no relationship between $\text{diam}(Z_0(\Gamma(R)))$ and $\text{diam}(T_0(\Gamma(R)))$. By Examples 2.9 and 4.6, we have $\text{diam}(Z_0(\Gamma(\mathbb{Z}_8))) = 1 < \infty = \text{diam}(T_0(\Gamma(\mathbb{Z}_8)))$, $\text{diam}(T_0(\Gamma(\mathbb{Z}_6))) = 2 < \infty = \text{diam}(Z_0(\Gamma(\mathbb{Z}_6)))$, and $\text{diam}(Z_0(\Gamma(\mathbb{Z}_{12}))) = 2 = \text{diam}(T_0(\Gamma(\mathbb{Z}_{12})))$.

Our next goal is to show that $\mathrm{diam}(T_0(\Gamma(R))) = \mathrm{diam}(T(\Gamma(R)))$ when $|R| \geq 4$. However, we will need the following lemma.

Lemma 4.4. *Let R be a commutative ring with $\mathrm{diam}(T(\Gamma(R))) = n < \infty$, and let $s \in R^*$ and $u \in U(R)$ be distinct.*

(1) *If $s \in Z(R)^*$, then $d_{T_0}(u, s) = d_T(u, s) \in \{n-1, n\}$.*
(2) *If n is an even integer, then $d_{T_0}(u-s, s) = m = d_T(u-s, s)$ for some even integer $m \leq n$.*
(3) *If n is an odd integer and $u \neq -s$, then $d_{T_0}(u+s, s) = m = d_T(u+s, s)$ for some odd integer $m \leq n$.*
(4) *If n is an even integer, then $d_{T_0}(u-s, s) = n = d_T(u-s, s)$ for every $s \in Z(R)^*$.*
(5) *If n is an odd integer, then $d_{T_0}(u+s, s) = n = d_T(u+s, s)$ for every $s \in Z(R)^*$.*

Proof. Observe that $n \geq 2$ by [5, Theorem 3.4].

(1) Let $s - a_1 - \cdots - a_{m-1} - u$ be a shortest path from s to u in $T_0(\Gamma(R))$ of length m. Then $m = d_{T_0}(x, y) = d_T(x, y) \leq n$ by Lemma 4.1. Since $u \in (s, s+a_1, a_1+a_2, \ldots, a_{m-1}+u)$, we have $R = (s, s+a_1, a_1+a_2, \ldots, a_{m-1}+u)$. Since R is generated by $m+1$ elements of $Z(R)$ and $\mathrm{diam}(T(\Gamma(R))) = n$, we have $n \leq m+1$ by [5, Theorem 3.4]. Thus $m \leq n \leq m+1$; so either $m = n-1$ or $m = n$.
(2) Let n be an even integer. If $u - s = s$, then $\mathrm{d}_{T_0}(u-s, s) = 0$. Thus we may assume that $u - s \neq s$, and hence $\mathrm{d}_{T_0}(u-s, s) \geq 2$ since $(u-s)+s = u \notin Z(R)$. Let $m \geq 2$, and let $s - a_1 - \cdots - a_{m-1} - (u-s)$ be a shortest path from s to $u-s$ in $T_0(\Gamma(R))$ of length m. Thus $m \leq n$. Suppose that m is an odd integer. Since $u = (s+a_1) - (a_1+a_2) + \cdots - (a_{m-2}+a_{m-1}) + (a_{m-1}+(u-s))$, we have $R = (s+a_1, a_1+a_2, a_2+a_3, \ldots, a_{m-1}+(u-s))$ is generated by m elements of $Z(R)$. Hence $n \leq m$ by [5, Theorem 3.4]; so $m = n$, which is a contradiction since n is an even integer. Thus $d_{T_0}(u-s, s) = m = d_T(u-s, s)$ for some even integer $m \leq n$.
(3) Let n be an odd integer and $s \neq -u$; so $u \neq u+s \in R^*$. If $u+2s \in Z(R)$, then $\mathrm{d}_{T_0}(u+s, s) = 1$. Thus we may assume that $u+2s \notin Z(R)$, and hence $\mathrm{d}_{T_0}(u+s, s) \geq 2$. Let $m \geq 2$, and let $s - a_1 - \cdots - a_{m-1} - (u+s)$ be a shortest path from s to $u+s$ in $T_0(\Gamma(R))$ of length m. Thus $m \leq n$. Suppose that m is an even integer. Since $-u = (s+a_1) - (a_1+a_2) + \cdots + (a_{m-2}+a_{m-1}) - (a_{m-1} + (u+s))$, we have $R = (s+a_1, a_1+a_2, a_2+a_3, \ldots, a_{m-1}+(u+s))$ is generated by m elements of $Z(R)$. Hence $n \leq m$ by [5, Theorem 3.4]; so $m = n$, which is a contradiction since n is an odd integer. Thus $\mathrm{d}_{T_0}(u+s, s) = m = d_T(u+s, s)$ for some odd integer $m \leq n$.
(4) Let n be an even integer and $s \in Z(R)^*$. Then $u-s, s \in R^*$ are distinct and $(u-s)+s = u \notin Z(R)$; so $m = d_{T_0}(u-s, s)$ is an even positive integer by part (2) above. Let $s - a_1 - \cdots - a_{m-1} - (u-s)$ be a shortest path from s to $u-s$ in $T_0(\Gamma(R))$ of length m. If $m = n$, then we are done; so assume that $m < n$.

Since $u = 2s - (s + a_1) + (a_1 + a_2) - \cdots - (a_{m-2} + a_{m-1}) + (a_{m-1} + (u - s))$, we have $R = (s, s + a_1, a_1 + a_2, a_2 + a_3, \ldots, a_{m-1} + (u - s))$ is generated by $m+1$ elements of $Z(R)$. Hence $n \leq m+1$ by [5, Theorem 3.4]. Thus $n = m+1$, which is a contradiction since n is an even integer and $m+1$ is an odd integer. Thus $\mathrm{d}_{T_0}(u - s, s) = n = d_T(u - s, s)$.

(5) Let n be an odd integer and $s \in Z(R)^*$. Thus $u + s, s \in R^*$ are distinct and $2s + u \notin Z(R)$ (for if $2s + u \in Z(R)$, then $R = (s, 2s + u)$, and hence $\mathrm{diam}(T(\Gamma(R))) = 2$ by [5, Theorem 3.4]); so $m = d_{T_0}(u + s, s) \geq 3$ is an odd integer by part (3) above. Let $s - a_1 - \cdots - a_{m-1} - (u+s)$ be a shortest path from s to $u + s$ in $T_0(\Gamma(R))$ of length m. If $m = n$, then we are done; so assume that $m < n$. Since $-u = 2s - (s+a_1) + (a_1+a_2) - \cdots + (a_{m-2}+a_{m-1}) - (a_{m-1}+(u+s))$, we have $R = (s, s + a_1, a_1 + a_2, a_2 + a_3, \ldots, a_{m-1} + (u - s))$ is generated by $m+1$ elements of $Z(R)$. Hence $n \leq m+1$ by [5, Theorem 3.4]. Thus $n = m+1$, which is a contradiction since n is an odd integer and $m+1$ is an even integer. Hence $\mathrm{d}_{T_0}(u + s, s) = n = d_T(u + s, s)$. □

Theorem 4.5. *Let R be a commutative ring.*

(1) $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_2))) = 0 < \infty = \mathrm{diam}(T(\Gamma(\mathbb{Z}_2)))$.
(2) $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_3))) = 1 < \infty = \mathrm{diam}(T(\Gamma(\mathbb{Z}_3)))$.
(3) *If $|R| \geq 4$, then* $\mathrm{diam}(T_0(\Gamma(R))) = \mathrm{diam}(T(\Gamma(R)))$.

Proof. Parts (1) and (2) are easily verified; so we may assume that $|R| \geq 4$. Then $T(\Gamma(R))$ is connected if and only if $T_0(\Gamma(R))$ is connected by Theorem 4.2, and $\mathrm{diam}(T_0(\Gamma(R))) \leq \mathrm{diam}(T(\Gamma(R)))$ by Lemma 4.1. Thus $\mathrm{diam}(T(\Gamma(R))) = \infty$ if and only if $\mathrm{diam}(T_0(\Gamma(R))) = \infty$ by Corollary 4.3 and the remarks before Theorem 4.2. Hence we may assume that $\mathrm{diam}(T(\Gamma(R))) = n < \infty$, and thus R is not an integral domain. Let $z \in Z(R)^*$. If n is an odd integer, then $\mathrm{d}_T(1 + z, z) = n = \mathrm{d}_{T_0}(1 + z, z)$ by Lemma 4.4(5), and hence $\mathrm{diam}(T(\Gamma(R))) = \mathrm{diam}(T_0(\Gamma(R))) = n$ by Lemma 4.1. If n is an even integer, then $\mathrm{d}_T(1 - z, z) = \mathrm{d}_{T_0}(1 - z, z) = n$ by Lemma 4.4(4), and thus $\mathrm{diam}(T(\Gamma(R))) = \mathrm{diam}(T_0(\Gamma((R))) = n$ by Lemma 4.1. Hence $\mathrm{diam}(T_0(\Gamma(R))) = \mathrm{diam}(T(\Gamma(R)))$ for all rings R with $|R| \geq 4$. □

The next example follows directly from Theorem 4.5 and the discussion preceding Theorem 4.2.

Example 4.6. (a) ($\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_n)))$) We have observed that $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_2))) = 0$, $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_3))) = 1$, and $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_p))) = \infty$ when $p \geq 5$ is prime. Let $n = p_1^{m_1} \cdots p_k^{m_k}$ for distinct primes p_i and $m_i \geq 1$. If $k = 1$ and $m_1 \geq 2$, then $\mathrm{diam}(T_0(\Gamma(R))) = \infty$. If $k \geq 2$, then $\mathrm{diam}(T_0(\Gamma(R))) = 2$.

(b) ($\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})))$) Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2$. Then $\mathrm{diam}(T_0(\Gamma(R))) = 2$.

The girth of $T_0(\Gamma(R))$ is also easily determined. Recall from [5, Theorem 2.6(3)] that if $Z(R)$ is an ideal of R, then $\mathrm{gr}(T(\Gamma(R))) = 3$ if and only if $|Z(R)| \geq 3$,

$\mathrm{gr}(T(\Gamma(R))) = 4$ if and only if $2 \notin Z(R)$ and $|Z(R)| = 2$, and $\mathrm{gr}(T(\Gamma(R))) = \infty$ otherwise. (Note that if $|Z(R)| = 2$, then R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and $2 \in Z(R)$ in either case. So, "the $\mathrm{gr}(T(\Gamma(R))) = 4$ case" cannot actually happen when $Z(R)$ is an ideal of R.) If $Z(R)$ is not an ideal of R, then $\mathrm{gr}(T(\Gamma(R))) = 4$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathrm{gr}(T(\Gamma(R))) = 3$ otherwise [5, Theorem 3.14]. Thus $\mathrm{gr}(T(\Gamma(R))) \in \{3, 4, \infty\}$. Note that $\mathrm{gr}(T(\Gamma(R))) \leq \mathrm{gr}(T_0(\Gamma(R)))$ since $T_0(\Gamma(R))$ is a (induced) subgraph of $T(\Gamma(R))$.

We next give explicit calculations for $\mathrm{gr}(T(\Gamma(R)))$ and $\mathrm{gr}(T_0(\Gamma(R)))$. These calculations show that $\mathrm{gr}(T_0(\Gamma(R))) = \mathrm{gr}(T(\Gamma(R)))$ unless R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_9$, or $\mathbb{Z}_3[X]/(X^2)$.

Theorem 4.7. *Let R be a commutative ring. Then $\mathrm{gr}(T(\Gamma(R))) \in \{3, 4, \infty\}$. Moreover,*

(1) $\mathrm{gr}(T(\Gamma(R))) = \infty$ *if and only if either R is an integral domain or R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$,*
(2) $\mathrm{gr}(T(\Gamma(R))) = 4$ *if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and*
(3) $\mathrm{gr}(T(\Gamma(R))) = 3$ *otherwise.*

Proof. By [5, Theorem 2.6(3); 5, Theorem 3.14], $\mathrm{gr}(T(\Gamma(R))) = 3$ unless $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $|Z(R)| \leq 2$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathrm{gr}(T(\Gamma(R))) = 4$. If $|Z(R)| \leq 2$, then R is either an integral domain or isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. In each of these cases, $\mathrm{gr}(T(\Gamma(R))) = \infty$. The result now follows. □

Theorem 4.8. *Let R be a commutative ring. Then $\mathrm{gr}(T_0(\Gamma(R))) \in \{3, 4, \infty\}$. Moreover,*

(1) $\mathrm{gr}(T_0(\Gamma(R))) = \infty$ *if and only if either R is an integral domain or R is isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, or $\mathbb{Z}_2 \times \mathbb{Z}_2$,*
(2) $\mathrm{gr}(T_0(\Gamma(R))) = 4$ *if and only if R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$, and*
(3) $\mathrm{gr}(T_0(\Gamma(R))) = 3$ *otherwise.*

Proof. Note that $\mathrm{gr}(T_0(\Gamma(R))) \leq \mathrm{gr}(Z_0(\Gamma(R)))$ since $Z_0(\Gamma(R))$ is a (induced) subgraph of $T_0(\Gamma(R))$. Thus Theorem 4.8 follows directly from Theorem 3.3(2) since one can easily verify that the rings $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6$, $\mathbb{Z}_9$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_3[X]/(X^2)$ have $\mathrm{gr}(T_0(\Gamma(R)))$ equal to ∞, ∞, ∞, 3, 4, 3, and 4, respectively. □

We close this section with the analog of Example 2.9 for $\mathrm{gr}(T_0(\Gamma(R)))$. It follows directly from Theorem 4.8.

Example 4.9. (a) ($\mathrm{gr}(T_0(\Gamma(\mathbb{Z}_n)))$) Let $R = \mathbb{Z}_n$ with $n \geq 2$. Then $\mathrm{gr}(T_0(\Gamma(\mathbb{Z}_n))) = \infty$ if n is prime, $\mathrm{gr}(T_0(\Gamma(\mathbb{Z}_4))) = \infty$, $\mathrm{gr}(T_0(\Gamma(\mathbb{Z}_9))) = 4$, and $\mathrm{gr}(T_0(\Gamma(R))) = 3$ otherwise.

(b) ($\mathrm{gr}(T_0(\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})))$) Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2$. Then $\mathrm{gr}(T_0(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = \infty$, and $\mathrm{gr}(T_0(\Gamma(R))) = 3$ otherwise.

5. Zero-Divisor Paths and Regular Paths in $T_0(\Gamma(R))$

Let R be a commutative ring and $x, y \in R^*$ be distinct. We say that $x - a_1 - \cdots - a_n - y$ is a *zero-divisor path* from x to y if $a_1, \ldots, a_n \in Z(R)^*$ and $a_i + a_{i+1} \in Z(R)$ for every $0 \leq i \leq n$ (let $a_0 = x$ and $a_{n+1} = y$). We define $\mathrm{d}_Z(x, y)$ to be the length of a shortest zero-divisor path from x to y ($\mathrm{d}_Z(x, x) = 0$ and $\mathrm{d}_Z(x, y) = \infty$ if there is no such path) and $\mathrm{diam}_Z(R) = \sup\{\mathrm{d}_Z(x, y) \,|\, x, y \in R^*\}$. Thus $\mathrm{d}_T(x, y) = \mathrm{d}_{T_0}(x, y) \leq \mathrm{d}_Z(x, y)$, for every $x, y \in R^*$. In particular, if $x, y \in R^*$ are distinct and $x+y \in Z(R)$, then $x - y$ is a zero-divisor path from x to y with $\mathrm{d}_Z(x, y) = 1$. For any commutative ring R, we have $\max\{\mathrm{diam}(Z_0(\Gamma(R))), \mathrm{diam}(T_0(\Gamma(R)))\} \leq \mathrm{diam}_Z(R)$. However, if R is a quasilocal reduced ring with $|\mathrm{Min}(R)| \geq 3$, then $\mathrm{diam}(Z_0(\Gamma(R))) \leq 2$ by Theorem 2.4, but $\mathrm{diam}_Z(R) = \infty$ since there is no zero-divisor path from 1 to any $x \in Z(R)^*$ (cf. Theorem 5.1(1)). Also, $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_{1225}))) = 2 < 3 = \mathrm{diam}_Z(\mathbb{Z}_{1225})$ by Examples 4.6 and 5.5. Note that $\mathrm{diam}_Z(\mathbb{Z}_2) = 0$, $\mathrm{diam}_Z(\mathbb{Z}_3) = 1$, and $\mathrm{diam}_Z(R) = \infty$ for any other integral domain R.

We first determine when there is a zero-divisor path between every two distinct elements of R^*.

Theorem 5.1. *Let R be a commutative ring that is not an integral domain. Then there is a zero-divisor path from x to y for every $x, y \in R^*$ if and only if one of the following two statements holds.*

(1) *R is reduced*, $|\mathrm{Min}(R)| \geq 3$, *and $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.*
(2) *R is not reduced and $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.*

Moreover, if there is a zero-divisor path from x to y for every $x, y \in R^*$, then R is not quasilocal and $\mathrm{diam}_Z(R) \in \{2, 3\}$.

Proof. Suppose that there is a zero-divisor path from x to y for every $x, y \in R^*$. First, assume that R is reduced and not an integral domain. Since $Z_0(\Gamma(R))$ is connected if and only if $|\mathrm{Min}(R)| \geq 3$ by Theorem 2.4, we have $|\mathrm{Min}(R)| \geq 3$. Let $y \in Z(R)^*$. Then there is a zero-divisor path $1 - a_1 - \cdots - a_n - y$ from 1 to y for some $a_1, \ldots, a_n \in Z(R)^*$. Thus $z = 1 + a_1 \in Z(R)^*$, and hence $R = (a_1, z)$. If R is not reduced, then a similar argument, as in the reduced case, shows that $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.

Conversely, assume that (1) holds. Thus $1 = w + z$ for some $w, z \in Z(R)^*$. Let $x, y \in R^*$ be distinct. Then $x = xw + xz$ and $y = yw + yz$. We consider two cases. Case one: assume that $x, y \in Z(R)^*$. Then we are done by Theorem 2.4. Case two: assume that $x \notin Z(R)$. Hence $xw, xz \in Z(R)^*$. Suppose that $x + y \notin Z(R)$. Then assume that either $xw = yw$ or $y = \pm yw$. Then $x - (-xw) - y$ is the desired zero-divisor path of length two from x to y. Next, assume that $xw \neq yw$, $yw \neq 0$ and $y \neq \pm yw$. Then $x - (-xw) - (-yw) - y$ is the desired zero-divisor path of length three from x to y. Finally, assume that $yw = 0$. Since $y \neq 0$ and $y = yw + yz$, we have $yz = y \neq 0$. Thus $x - (-xz) - y$ is the desired zero-divisor path of length two from

x to y. Now assume that (2) holds. Since $Z_0(\Gamma(R))$ is connected by Theorem 2.2, an argument similar to that in case two of the reduced case completes the proof.

Assume that there is a zero-divisor path from x to y for every $x, y \in R^*$ and that R is not an integral domain. Then R cannot be quasilocal since $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$ by (1) and (2) above. Clearly $\mathrm{diam}_Z(R) \neq 0$. Let $z \in Z(R)^*$. Then $z, 1 - z \in R^*$ are distinct and $z + (1 - z) = 1 \notin Z(R)$; so $\mathrm{diam}_Z(R) \geq 2$. The "moreover" statement now follows from the above proof. □

Corollary 5.2. *Let R be a commutative ring. Then* $\mathrm{diam}_Z(R) \in \{0, 1, 2, 3, \infty\}$. *Moreover,* $\mathrm{diam}_Z(R) \in \{2, 3, \infty\}$ *except for* $\mathrm{diam}_Z(\mathbb{Z}_2) = 0$ *and* $\mathrm{diam}_Z(\mathbb{Z}_3) = 1$.

Corollary 5.3. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R. Then there is a zero-divisor path from x to y for every $x, y \in T(R)^*$ if and only if either R is reduced with $|\mathrm{Min}(R)| \geq 3$ or R is not reduced.*

Proof. Since $Z(R)$ is not an ideal of R, there are $z_1, z_2 \in Z(R)^*$ such that $z_1 + z_2 \in \mathrm{Reg}(R)$. Thus $T(R) = (z_1, z_2)$; so the corollary follows directly from Theorem 5.1. □

Theorem 5.4. (1) *Let $R = R_1 \times R_2$ for commutative quasilocal rings R_1, R_2 with maximal ideals M_1, M_2, respectively. If there are $a_1 \in U(R_1)$ and $a_2 \in U(R_2)$ with $(2a_1, 2a_2) \in U(R)$ and $(3a_1, 3a_2) \notin Z(R)$, then* $\mathrm{diam}_Z(R) \in \{3, \infty\}$. *Moreover,* $\mathrm{diam}_Z(R) = 3$ *if either R_1 or R_2 is not reduced.*

(2) *Let $R = R_1 \times \cdots \times R_n$ for commutative rings $R_1, \ldots, R_n$ with $n \geq 3$. Then* $\mathrm{diam}_Z(R) = 2$.

Proof. (1) Let $a = (a_1, a_2), b = (2a_1, 2a_2) \in U(R)$. Then $a \neq b$ and $\mathrm{d}_Z(a, b) \neq 1$ since $a + b = (3a_1, 3a_2) \notin Z(R)$. Assume that there is an $f = (m_1, m_2) \in R^*$ such that $a - f - b$ is a zero-divisor path from a to b. Thus $f \in Z(R)^*$; so either $m_1 \in M_1$ or $m_2 \in M_2$. If $m_1 \in M_1$, then $m_1 + a_1, m_1 + 2a_1 \in U(R_1)$. Hence $m_2 + a_2, m_2 + 2a_2 \in M_2$, since $a + f, b + f \in Z(R)$. But then $a_2 = (m_2 + 2a_2) - (m_2 + a_2) \in M_2$, a contradiction. In a similar manner, $m_2 \in M_2$ also leads to a contradiction; so no such f exists. Thus $\mathrm{d}_Z(a, b) \geq 3$; so $\mathrm{diam}_Z(R) \in \{3, \infty\}$. The "moreover" statement now follows from Theorem 5.1.

(2) We have $\mathrm{diam}_Z(R) \in \{2, 3\}$ by Theorem 5.1 since $|\mathrm{Min}(R)| \geq n \geq 3$. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R^*$ with $x + y \notin Z(R)$. We may assume that $x_1 \neq 0$. Let $z = (-x_1, -y_2, 1, \ldots, 1, 0) \in Z(R)^*$. Then $x + z, y + z \in Z(R)$; so $x - z - y$ is the desired zero-divisor path from x to y of length 2. Hence $\mathrm{diam}_Z(R) = 2$. □

The following example shows that all possible values for $\mathrm{diam}_Z(R)$ given in Corollary 5.2 and Theorem 5.4 may be realized. The details are left to the reader.

Example 5.5. (a) ($\mathrm{diam}_Z(\mathbb{Z}_n)$) We have already observed that $\mathrm{diam}_Z(\mathbb{Z}_2) = 0$, $\mathrm{diam}_Z(\mathbb{Z}_3) = 1$, and $\mathrm{diam}_Z(\mathbb{Z}_p) = \infty$ when $p \geq 5$ is prime. Let $R = \mathbb{Z}_n$ with

$n \geq 2$ and n not prime. Let $n = p_1^{m_1} \cdots p_k^{m_k}$ for distinct primes p_i and $m_i \geq 1$. If either $k = 1$, or $k = 2$ and $m_1 = m_2 = 1$, then $\mathrm{diam}_Z(R) = \infty$. If $k = 2, p_1, p_2 \geq 5$, and $m_1 + m_2 \geq 3$, then $\mathrm{diam}_Z(R) = 3$. Otherwise, $\mathrm{diam}_Z(R) = 2$.

(b) ($\mathrm{diam}_Z(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$) Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2$. If $k = 2$ and n_1, n_2 are prime, then $\mathrm{diam}_Z(R) = \infty$. If $k = 2$ and $n_1 = p_1^{m_1}, n_2 = p_2^{m_2}$ for primes $p_1, p_2 \geq 5$ and $m_1 + m_2 \geq 3$, then $\mathrm{diam}_Z(R) = 3$. Otherwise, $\mathrm{diam}_Z(R) = 2$.

Let $x, y \in R^*$ be distinct. We say that $x - a_1 - \cdots - a_n - y$ is a *regular path* from x to y if $a_1, \ldots, a_n \in \mathrm{Reg}(R)$ and $a_i + a_{i+1} \in Z(R)$ for every $0 \leq i \leq n$ (let $a_0 = x$ and $a_{n+1} = y$). We define $\mathrm{d_{reg}}(x, y)$ to be the length of a shortest regular path from x to y ($\mathrm{d_{reg}}(x, x) = 0$ and $\mathrm{d_{reg}}(x, y) = \infty$ if there is no such path), and $\mathrm{diam_{reg}}(R) = \sup\{\mathrm{d_{reg}}(x, y) \mid x, y \in R^*\}$. Thus $\mathrm{d}_T(x, y) = \mathrm{d}_{T_0}(x, y) \leq \mathrm{d_{reg}}(x, y)$ for every $x, y \in R^*$. In particular, if $x, y \in R^*$ are distinct and $x + y \in Z(R)$, then $x - y$ is a regular path from x to y with $\mathrm{d_{reg}}(x, y) = 1$. For any commutative ring R, we have $\max\{\mathrm{diam}(T_0(\Gamma(R))), \mathrm{diam}(\mathrm{Reg}(\Gamma(R)))\} \leq \mathrm{diam_{reg}}(R)$. Note that $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_{60}))) = 2 < \infty = \mathrm{diam_{reg}}(\mathbb{Z}_{60})$ and $\mathrm{diam}(\mathrm{Reg}(\Gamma(\mathbb{Z}_6))) = 1 < 2 = \mathrm{diam_{reg}}(\mathbb{Z}_6)$. However, if R is an integral domain, then $T_0(\Gamma(R)) = \mathrm{Reg}(\Gamma(R))$; so all three diameters are equal. Moreover, $\mathrm{diam_{reg}}(\mathbb{Z}_2) = 0$, $\mathrm{diam_{reg}}(\mathbb{Z}_3) = 1$ and $\mathrm{diam_{reg}}(R) = \infty$ for any other integral domain R. Hence $\mathrm{diam}_Z(R) = \mathrm{diam_{reg}}(R)$ for any integral domain R.

Theorem 5.6. *Let R be a commutative ring with* $\mathrm{diam}(T_0(\Gamma(R))) = n < \infty$.

(1) *Let $u \in U(R)$, $s \in R^*$, and P be a shortest path from s to u of length $n - 1$ in $T_0(\Gamma(R))$. Then P is a regular path from s to u.*
(2) *Let $u \in U(R)$, $s \in R^*$, and $P : s - a_1 - \cdots - a_n = u$ be a shortest path from s to u of length n in $T_0(\Gamma(R))$. Then either P is a regular path from s to u, or $a_1 \in Z(R)^*$ and $a_1 - \cdots - a_n = u$ is a regular path from a_1 to u of length $n - 1 = \mathrm{d}_{T_0}(a_1, u)$.*

Proof. (1) If $n = 2$, then P is a regular path from s to u by definition. Thus we may assume that $n > 2$. Since $\mathrm{d}_{T_0}(z, u)$ is either $n - 1$ or n for every $z \in Z(R)^*$ by Lemma 4.4(1) and $\mathrm{d}_{T_0}(s, u) = n - 1$, we conclude that P must be a regular path.

(2) Suppose that P is not a regular path; so $a_i \in Z(R)^*$ for some $1 \leq i \leq n - 1$. Since $\mathrm{d}_{T_0}(z, u)$ is either $n - 1$ or n for every $z \in Z(R)^*$ by Lemma 4.4(1) and $\mathrm{d}_{T_0}(s, u) = n$, we must have $a_1 \in Z(R)^*$ and $a_i \in \mathrm{Reg}(R)$ for every $2 \leq i \leq n - 1$. Thus $a_1 - \cdots - a_n - u$ is a regular path of length $n - 1 = \mathrm{d}_{T_0}(a_1, u)$. □

We next determine when there is a regular path between every two distinct elements of R^*.

Theorem 5.7. *Let R be a commutative ring.*

(1) *If $s \in \mathrm{Reg}(R)$ and $w \in \mathrm{Nil}(R)^*$, then there is no regular path from s to w.*
(2) *If R is reduced and quasilocal, then there is no regular path from any unit to any nonzero nonunit in R.*

In particular, if there is a regular path from x to y for every $x, y \in R^*$, then either R is reduced and not quasilocal or R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Proof. (1) Let $s \in \mathrm{Reg}(R)$ and $w \in \mathrm{Nil}(R)^*$. Since $a + w \in \mathrm{Reg}(R)$ for every $a \in \mathrm{Reg}(R)$ by Lemma 2.1(4), there is no regular path from s to w.
(2) Let M be the maximal ideal of R, $x \in U(R)$, and $0 \neq y \in M$. Suppose that there is a regular path $x - x_1 - \cdots - x_n - y$. Then $x + x_1 = z_1 \in Z(R) \subseteq M$; so $x_1 = -x + z_1 \in U(R)$. In a similar manner, each $x_i \in U(R)$. But then $x_n + y \in U(R)$, a contradiction.

The "in particular" statement is clear by parts (1) and (2) above and the remarks preceding Theorem 5.6. □

Theorem 5.8. *Let R be a commutative ring. Then there is a regular path from x to y for every $x, y \in R^*$ if and only if R is reduced,* $\mathrm{Reg}(\Gamma(R))$ *is connected, and for every $z \in Z(R)^*$ there is a $w \in Z(R)^*$ such that $\mathrm{d}_Z(z, w) > 1$ (possibly with $\mathrm{d}_Z(z, w) = \infty$).*

Proof. Suppose that there is a regular path from x to y for every $x, y \in R^*$. Then R is reduced by Theorem 5.7, and it is clear that $\mathrm{Reg}(\Gamma(R))$ is connected. Let $z \in Z(R)^*$, and let $z - a_1 - \cdots - 1$ be a regular path from z to 1. Then $a_1 \in \mathrm{Reg}(R)$ and $w = -(z + a_1) \in Z(R)^*$. Thus $z \neq w$ and $z + w \notin Z(R)$; so $\mathrm{d}_Z(z, w) > 1$.

Conversely, suppose that R is reduced, $\mathrm{Reg}(\Gamma(R))$ is connected, and for every $z \in Z(R)^*$ there is a $w \in Z(R)^*$ such that $\mathrm{d}_Z(z, w) > 1$ (possibly with $\mathrm{d}_Z(z, w) = \infty$). Let $x, y \in R^*$. If $x, y \in \mathrm{Reg}(R)$, then there is nothing to prove. First, assume that $x \in Z(R)^*$ and $y \in \mathrm{Reg}(R)$. Since $x \in Z(R)^*$, there is a $w \in Z(R)^*$ such that $\mathrm{d}_Z(x, w) > 1$. Then $x + w \notin Z(R)$; so $x + u = -w \in Z(R)$ for some $u \in \mathrm{Reg}(R)$. Since $\mathrm{Reg}(\Gamma(R))$ is connected, let $u - u_1 - \cdots - y$ be a regular path from u to y. Then $x - u - u_1 - \cdots - y$ is a regular path from x to y. Next, assume that $x, y \in Z(R)^*$. Then again as above, there are $u, v \in \mathrm{Reg}(R)$ such that $x + u \in Z(R)$ and $y + v \in Z(R)$. If $u = v$, then $x - u - y$ is a regular path from x to y. So assume that $u \neq v$. Since $\mathrm{Reg}(\Gamma(R))$ is connected, let $u - \cdots - v$ be a regular path from u to v. Then $x - u - \cdots - v - y$ is a regular path from x to y. □

In view of Theorems 2.3 and 5.8, we have the following result.

Corollary 5.9. *Let R be a reduced commutative ring with $|\mathrm{Min}(R)| = 2$. Then there is a regular path from x to y for every $x, y \in R^*$ if and only if* $\mathrm{Reg}(\Gamma(R))$ *is connected.*

Recall from [9] that a commutative ring R is a *p.p. ring* if every principal ideal of R is projective. For example, a commutative von Neumann regular ring is a p.p. ring, and $\mathbb{Z} \times \mathbb{Z}$ is a p.p. ring that is not von Neumann regular. It was shown in [15, Proposition 15] that a commutative ring R is a p.p. ring if and only if every element of R is the product of an idempotent element and a regular element of R (thus a commutative p.p. ring that is not an integral domain has non-trivial idempotents). We show that a commutative p.p. ring R that is not an integral domain has $\mathrm{diam}_{\mathrm{reg}}(R) = 2$, but first a lemma.

Lemma 5.10. *Let R be commutative ring, $u, v \in \mathrm{Reg}(R)$, and $e \in \mathrm{Idem}(R)$. Then $eu + (1 - e)v \in \mathrm{Reg}(R)$.*

Proof. Let $eu + (1 - e)v = w \in R$, and suppose that $cw = 0$ for some $c \in R$. Then $ew = e[eu + (1 - e)v] = eu$ and $(1 - e)w = (1 - e)[eu + (1 - e)v] = (1 - e)v$. Thus $ceu = cew = 0$ and $c(1 - e)v = c(1 - e)w = 0$, and hence $ce = c(1 - e) = 0$ since $u, v \in \mathrm{Reg}(R)$. Thus $c = ce + c(1 - e) = 0$; so $eu + (1 - e)v = w \in \mathrm{Reg}(R)$. □

Theorem 5.11. *Let R be a commutative p.p. ring that is not an integral domain. Then there is a regular path from x to y for every $x, y \in R^*$. Moreover, $\mathrm{diam}_{\mathrm{reg}}(R) = \mathrm{diam}(T_0(\Gamma(R))) = \mathrm{diam}(T(\Gamma(R))) = 2$.*

Proof. Let $x, y \in R^*$ be distinct, and suppose that $x + y \notin Z(R)$. We consider three cases. Case one: assume that $x, y \in Z(R)^*$. Since $x+y \notin Z(R)$, necessarily $x+y \in \mathrm{Reg}(R)$, and thus $x-(-(x+y))-y$ is the desired regular path of length two from x to y. Case two: assume that $x, y \in \mathrm{Reg}(R)$. Since R is a p.p. ring and not an integral domain, there is an $e \in \mathrm{Idem}(R)\backslash\{0, 1\}$. Hence $w = -[(1-e)x+ey] \in \mathrm{Reg}(R)$ by Lemma 5.10. Since $e(1-e) = 0$ and $e \notin \{0, 1\}$, we have $x+w = ex-ey = e(x-y) \in Z(R)$ and $y+w = (e-1)x-(e-1)y = (e-1)(x-y) \in Z(R)$. Thus $x - w - y$ is the desired regular path of length 2 from x to y. Case three: assume that $x \in \mathrm{Reg}(R)$ and $y \in Z(R)^*$. Hence $y = fu$ for some $f \in \mathrm{Idem}(R)\backslash\{0, 1\}$ and $u \in \mathrm{Reg}(R)$. Then $h = -[(1 - f)x + fu] \in \mathrm{Reg}(R)$ by Lemma 5.10. Since $f(1 - f) = 0$ and $f \notin \{0, 1\}$, we have $x + h = fx - fu = f(x - u) \in Z(R)$ and $y + h = (f - 1)x \in Z(R)$. Thus $x - h - y$ is the desired regular path of length two from x to y; so $\mathrm{diam}_{\mathrm{reg}}(R) \leq 2$.

For the "moreover" statement, we first note that $T(\Gamma(R))$ is connected with $\mathrm{diam}(T(\Gamma(R))) = 2$ by [5, Corollary 3.6] since R has a non-trivial idempotent. Thus $2 = \mathrm{diam}(T(\Gamma(R))) = \mathrm{diam}(T_0(\Gamma(R))) \leq \mathrm{diam}_{\mathrm{reg}}(R) \leq 2$ by Theorem 4.7, since $|R| \geq 4$; so we have the desired equality. □

Corollary 5.12. *Let R be a commutative von Neumann regular ring that is not a field. Then there is a regular path from x to y for every $x, y \in R^*$. Moreover, $\mathrm{diam}_{\mathrm{reg}}(R) = 2$.*

Corollary 5.13. *Let R be a commutative ring. If there is an $e \in \mathrm{Idem}(R)\backslash\{0, 1\}$, then $\mathrm{Reg}(\Gamma(R))$ is connected with $\mathrm{diam}(\mathrm{Reg}(\Gamma(R))) \in \{0, 1, 2\}$.*

Proof. Let $u, v \in \mathrm{Reg}(R)$ be distinct, $u + v \notin Z(R)$, and $e \in \mathrm{Idem}(R)\backslash\{0, 1\}$. Then $w = -eu + (1 - e)v \in \mathrm{Reg}(R)$ by Lemma 5.10; so $u - w - v$ is the desired path from u to v in $\mathrm{Reg}(\Gamma(R))$ of length two. Thus $\mathrm{Reg}(\Gamma(R))$ is connected and $\mathrm{diam}(\mathrm{Reg}(\Gamma(R))) \leq 2$. □

One easily verifies that $\mathrm{diam}(\mathrm{Reg}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = 0$, $\mathrm{diam}(\mathrm{Reg}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))) = 1$, and $\mathrm{diam}(\mathrm{Reg}(\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_5))) = 2$. Thus all possible values for $\mathrm{diam}(\mathrm{Reg}(\Gamma(R)))$ in Corollary 5.13 may be realized.

We next determine $\mathrm{diam}_{\mathrm{reg}}(R)$ for $R = \mathbb{Z}_n$ and $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. The details are left to the reader; they follow directly from Theorem 5.7 and Corollary 5.12.

Example 5.14. (a) ($\mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_n)$) We have already observed that $\mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_2) = 0$, $\mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_3) = 1$, and $\mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_p) = \infty$ when $p \geq 5$ is prime. Let $R = \mathbb{Z}_n$ with $n \geq 2$ and n not prime. Then $\mathrm{diam}_{\mathrm{reg}}(R) = 2$ if n is the product of (at least 2) distinct primes. Otherwise, $\mathrm{diam}_{\mathrm{reg}}(R) = \infty$.

(b) ($\mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$) Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2$. Then $\mathrm{diam}_{\mathrm{reg}}(R) = 2$ if every n_i is prime. Otherwise, $\mathrm{diam}_{\mathrm{reg}}(R) = \infty$.

The rings in Theorem 5.11 and Corollary 5.12 are reduced and not quasilocal. We next give an example of a reduced non-quasilocal ring R that is not an integral domain such that there is no regular path from x to y for some $x, y \in R^*$.

Example 5.15. Let $I = 2X\mathbb{Z}[X]$ be an ideal of $\mathbb{Z}[X]$, and let $R = \mathbb{Z}[X]/I$. Then R is reduced, not quasilocal, and $Z(R) = X\mathbb{Z}[X]/I \cup 2\mathbb{Z}[X]/I$. Note that $R \neq (Z(R))$; so $T_0(\Gamma(R))$ is not connected by Corollary 4.3. Thus there is no regular path from x to y for some $x, y \in R^*$. It may easily be shown that there is no regular path from $x = 1 + I$ to $y = X + I$.

We have $\mathrm{diam}(T_0(\Gamma(R))) \leq \min\{\mathrm{diam}_Z(R), \mathrm{diam}_{\mathrm{reg}}(R)\}$ for any commutative ring R. Examples 4.6, 5.5 and 5.14 show that all three diameters may be different. For $n = 5^2 \cdot 7^2 = 1225$, we have $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_n))) = 2 < 3 = \mathrm{diam}_Z(\mathbb{Z}_n) < \infty = \mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_n)$. For $n = 2^2 \cdot 3 \cdot 5 = 60$, we have $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_n))) = \mathrm{diam}_Z(\mathbb{Z}_n) = 2 < \infty = \mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_n)$. Also, $\mathrm{diam}(T_0(\Gamma(\mathbb{Z}_{35}))) = \mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_{35}) = 2 < \infty = \mathrm{diam}_Z(\mathbb{Z}_{35})$.

We could also define $\mathrm{gr}_Z(R)$ and $\mathrm{gr}_{\mathrm{reg}}(R)$ by only using cycles in $Z(R)^*$ and $\mathrm{Reg}(R)$, respectively. However, this gives nothing new since $\mathrm{gr}_Z(R) = \mathrm{gr}(Z_0(\Gamma(R)))$ and $\mathrm{gr}_{\mathrm{reg}}(R) = \mathrm{gr}(\mathrm{Reg}(\Gamma(R)))$. We have already determined $\mathrm{gr}(Z_0(\Gamma(R)))$ in Theorem 3.3(2), and $\mathrm{gr}(\mathrm{Reg}(\Gamma(R)))$ has been studied in [5, Theorems 2.6 and 3.14]. We end this paper by giving $\mathrm{gr}(\mathrm{Reg}(\Gamma(R)))$ for $R = \mathbb{Z}_n$ and $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$; details are left to the reader.

Example 5.16. (a) ($\mathrm{gr}(\mathrm{Reg}(\Gamma(\mathbb{Z}_n)))$) Let $R = \mathbb{Z}_n$ with $n \geq 2$. Then $\mathrm{gr}(\mathrm{Reg}(\Gamma(R))) = \infty$ if $n = 4, n = 6$, or n is prime; $\mathrm{gr}(\mathrm{Reg}(\Gamma(R))) = 4$ if $n = p^m$ with $p \geq 3$ prime and $m \geq 2$; and $\mathrm{gr}(\mathrm{Reg}(\Gamma(R))) = 3$ otherwise.

(b) (gr(Reg($\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$))) Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2$. Then $\mathrm{gr}(\mathrm{Reg}(\Gamma(R))) = \infty$ if $n_{k-1} = 2$ and $n_k = 2, 3, 4$, or 6. Otherwise, $\mathrm{gr}(\mathrm{Reg}(\Gamma(R))) = 3$.

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