

Peter Weidemaier

On the trace theory for functions in Sobolev spaces with mixed L_p -norm

Czechoslovak Mathematical Journal, Vol. 44 (1994), No. 1, 7–20

Persistent URL: <http://dml.cz/dmlcz/128455>

Terms of use:

© Institute of Mathematics AS CR, 1994

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ON THE TRACE THEORY FOR FUNCTIONS IN SOBOLEV SPACES
WITH MIXED L_p -NORM

PETER WEIDEMAIER, Weil/Rh.

(Received October 21, 1991)

INTRODUCTION

In this paper we prove a theorem on the trace on $\partial\Omega \times (0, T)$ for functions in the Sobolev space $W_{p,q}^{2,1}(\Omega_T) := \{f \mid \partial_x^\alpha f, \partial_t f \text{ (distr. sense)} \in L_q(0, T, L_p(\Omega)) \forall |\alpha| \leq 2\}$ with $1 < p \leq q < \infty$; here $\Omega_T := \Omega \times (0, T)$ and $\Omega \subset \mathbb{R}^n$ with compact sufficiently smooth boundary. Our results, which seem to be sharp, are applicable to the Dirichlet- and Neumann problem for the heat equation and Navier-Stokes equations with *inhomogeneous* boundary conditions. The corresponding problems with homogeneous boundary conditions have been studied in $L_q(0, T, L_p(\Omega))$ -spaces with q different from p by various authors: compare v. Wahl [7] for parabolic equations and Iwashita [4], v. Wahl [8] for the Navier-Stokes system. Our results, stated in Theorem 1, generalize the classical trace theory developed for $q = p$ only (see Ladyshenskaya [6], chapter II, Lemma 3.4.; Il'in and Solonnikov [3]); an elaboration of part of their work can also be found in Weidemaier [9].

We use the method of integral representation introduced by the Russian school (cf. Appendix A) and some weighted inequalities of Hardy-type (cf. Appendix B).

Let us fix our notation: Γ is the boundary of Ω and $\Gamma_T := \Gamma \times (0, T)$. Moreover $Q^{n+1}(0, T^{\underline{\kappa}}) := \prod_{i=1}^{n+1} (0, T^{\kappa_i})$ for $\underline{\kappa} := (\kappa_1, \dots, \kappa_{n+1})$, $Q^{n-1}(\alpha) := (-\alpha, \alpha)^{n-1}$, $Q_+^n(\alpha, \beta) := Q^{n-1}(\alpha) \times (0, \beta)$ for $\alpha, \beta > 0$. The typical point in $Q_+^n(\alpha, \beta) \times (0, T)$ is denoted (x, t) . The prime characterizes $(n-1)$ -dimensional quantities: thus we write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$. The t -coordinate is sometimes also referred to as the $(n+1)$ -th coordinate. The superscript \sim always indicates the deletion of

a coordinate, for example

$$\dot{y} = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{n+1}) \quad \text{and} \quad \overset{n, n+1}{\tilde{Q}}^{n+1}(0, T^\kappa) := \prod_{i=1}^{n+1} (0, T^{\kappa_i}).$$

The natural norm in $L_q(0, T, L_p(\Omega))$ is denoted by $\|\cdot\|_{p,q,\Omega_T}$. We use the notation c^* to emphasise the non-dependence of the constant c on the quantity T .

MAIN RESULT

For the convenience of the reader we shortly introduce our notation used in the description of the boundary of Ω and some function spaces on it.

For $\Omega \subset \mathbb{R}^n$ with compact boundary “ $\Omega \in C^{1,1}$ ” is defined as in the book by Kufner [5], 6.2.2; this in particular implies that there exist finitely many open subsets $U_i \subset \mathbb{R}^n$ ($i = 1, \dots, M$) and invertible mappings $\Psi_i \in C^{1,1}(\overline{Q_+^n(\alpha, \beta)}, \mathbb{R}^n)$ such that

$$\begin{aligned} \Gamma \cap U_i &= \Psi_i(Q^{n-1}(\alpha) \times \{0\}), \quad \bigcup_{i=1}^M (\Gamma \cap U_i) = \Gamma \\ \Omega \cap U_i &= \Psi_i(Q^{n-1}(\alpha) \times (0, \beta)); \end{aligned}$$

let us remark that Ψ_i equals $A_i^{-1} \circ Q_i$ in the notation of [5], 6.2.9, where $Q_i(x', x_n) := (x', a(x') + x_n)$ with a certain $a(\cdot) \in C^{1,1}(\overline{Q^{n-1}(\alpha)})$ and A_i is linear and invertible. From the explicit form of Q_i it is easy to see that Q_i^{-1} is also $C^{1,1}$. Moreover there exists an open subset $U_0 \subset \mathbb{R}^n$ such that

$$\overline{U_0} \subset \Omega, \quad \bigcup_{i=0}^M (\Omega \cap U_i) = \Omega.$$

Ψ_i^* defined by $\Psi_i^* g(x, t) := g(\Psi_i(x), t)$ is the pullback induced by Ψ_i in the spatial variables. We denote by $(\varphi_i)_i$ a partition of unity on $\overline{\Omega}$ with $\varphi_i \in C^\infty(\mathbb{R}^n)$ and $\text{supp } \varphi_i \subset U_i$ for $i = 0, \dots, M$.

The spaces $L_p(\Gamma)$ ($1 \leq p < \infty$) are defined as in [5], 6.3.2: a function u defined a.e. on Γ belongs to $L_p(\Gamma)$ iff $u \circ \Psi_i(\cdot, 0) \in L_p(Q^{n-1}(\alpha))$ for each $i = 1, \dots, M$; in this case

$$\|u\|_{p,\Gamma}^p := \sum_{i=1}^M \|u \circ \Psi_i(\cdot, 0)\|_{p,Q^{n-1}(\alpha)}^p.$$

The spaces $\dot{W}_p^s(\Gamma)$, $s > 0$, are defined similarly (see [5], 6.7.2 and 6.8.6). Finally we define

$$X_{p,q}^{\alpha,\beta}(\Gamma_T) := L_q(0, T, W_p^\alpha(\Gamma)) \cap \{g \mid |g|_{\mathcal{L}_q^{\alpha,\beta}(\Gamma_T)} < \infty\} \quad \text{for } \alpha > 0, \beta \in (0, 1)$$

with

$$\begin{aligned} \|\cdot\|_{X_{p,q}^{\alpha,\beta}(\Gamma_T)} &:= \|\cdot\|_{L_q(0,T,W_p^\alpha(\Gamma))} + |\cdot|_{\mathcal{L}_q^{\alpha,\beta}(\Gamma_T)}, \\ |g|_{\mathcal{L}_q^{\alpha,\beta}(\Gamma_T)}^q &:= \int_0^T h^{-(1+q\beta)} \|\Delta_{n+1,h} g\|_{L_q(0,T-h,L_p(\Gamma))}^q dh \end{aligned}$$

with $\Delta_{n+1,h} g(\xi, t) := g(\xi, t+h) - g(\xi, t)$ for $\xi \in \Gamma$.

Now we are ready to formulate our main result.

Theorem 1. *Assume that $\Omega \subset \mathbb{R}^n$ has compact boundary and belongs to the class $C^{1,1}$; let $1 < p \leq q < \infty$ and $s(m) = 2 - m - 1/p$.*

(i) *Then for each $k = 1, \dots, n$ and $m = 0, 1$ there is a unique linear continuous map $\gamma_{k,m}: W_{p,q}^{2,1}(\Omega_T) \rightarrow X_{p,q}^{s(m),s(m)/2}(\Gamma_T)$ such that $\gamma_{k,m} f = \partial_k^m f|_{\Gamma_T}$ for $f \in D := W_{p,q}^{2,1}(\Omega_T) \cap \{f \mid f(\cdot, t) \in C^1(\bar{\Omega}) \forall t \in (0, T)\}$.*

(ii) *Moreover the norm of each $\gamma_{k,m}$ is independent of T .*

Remark 2. The space $X_{p,q}^{s,s/2}(\Gamma_T)$ coincides for $q = p$ with $W_p^{s,s/2}(\Gamma_T)$ in Ladyshenskaya [6].

Proof of Theorem 1. The estimate for the spatial regularity follows from the well-known trace theorem $W_p^{2-m}(\Omega) \ni u \mapsto u|_{\Gamma} \in W_p^{2-m-1/p}(\Gamma)$ (cf. Kufner [5], 6.10.3) together with an easy scaling argument in t . In the sequel we shall concentrate on the proof for the time-regularity of the trace: since D defined above is dense in $W_{p,q}^{2,1}(\Omega_T)$ and $X_{p,q}^{s(m),s(m)/2}(\Gamma_T)$ is a complete space (two facts for which the (routine) proofs will be given later in Lemma 3 and Lemma 4), it is sufficient to consider $f \in D$. Moreover, since $f = \sum_{i=0}^M f \cdot \varphi_i$ (the φ_i are the functions of the partition of unity introduced above) and since $\Gamma \cap \text{supp } \varphi_0 = \emptyset$, it is sufficient to consider $f_i := f \cdot \varphi_i$ ($i = 1, \dots, M$). Furthermore we are going to reduce the proof to a situation in half-space by flattening the boundary: for u with support contained in U_i we have (see [5], 6.3.9 Lemma)

$$\|u\|_{p,\Gamma} \leq c^* \cdot \|u(\Psi_i(\cdot, 0))\|_{p,Q^{n-1}(\alpha)}$$

applying the last inequality with $u(\cdot) = \Delta_{n+1,h} \partial_k^m f_i(\cdot, t)$ we see that it is sufficient to prove

$$(1) \quad |(\Psi_i^*(\partial_k^m f_i))|_{x_n=0}|_{\mathcal{L}_q^{\alpha,s(m)/2}(Q^{n-1}(\alpha) \times (0,T))} \leq c^* \cdot \|f_i\|_{W_{p,q}^{2,1}(\Omega_T)},$$

(where $|\cdot|_{\mathcal{L}_q^{\alpha,\beta}(Q^{n-1}(\alpha) \times (0,T))}$ is defined, of course, in the same way as $|\cdot|_{\mathcal{L}_q^{\alpha,\beta}(\Gamma_T)}$, but with Γ replaced with $Q^{n-1}(\alpha)$ everywhere). We further claim that the last inequality follows from

$$(2) \quad |(\partial_{x_j}^m (\Psi_i^* f_i))|_{x_n=0}|_{\mathcal{L}_q^{\alpha,s(m)/2}(Q^{n-1}(\alpha) \times (0,T))} \leq c^* \cdot \|\Psi_i^* f_i\|_{W_{p,q}^{2,1}(Q_+^n(\alpha,\beta) \times (0,T))}$$

($j = 1, \dots, n$). For the proof of this claim we note that by the chain rule for weak derivatives (cf. [5], proof of 5.7.3) and the $C^{1,1}$ -regularity of Ψ_i^{-1} the function $\Psi_i^*(\partial_k f_i)$ is a linear combination of spatial derivatives of $\Psi_i^* f_i$ with L_∞ -coefficients (which do not depend on t). In order to pass from the r.h. side of (2) to the r.h. side of (1), we remark that Ψ_i^* induces an isomorphism $W_{p,q}^{2,1}((U_i \cap \Omega)_T) \rightarrow W_{p,q}^{2,1}(Q_+^n(\alpha, \beta)_T)$ (use again the chain rule, the $C^{1,1}$ -regularity of Ψ_i and Ψ_i^{-1} , the transformation rule for integrals and the fact that the Jacobians of Ψ_i and Ψ_i^{-1} are in L_∞).

A last technical remark: for later use of the integral representation in Appendix A it is useful to consider $\Psi_i^* f_i$ in (2) as being defined on $\mathbb{R}_+^n \times (0, 2T)$. This is possible, since extending $\Psi_i^* f_i$ by zero in its spatial variables and reflecting it (cf. Adams [1], p. 83) in its t -variable yield a linear extension operator E_T , which is continuous with respect to the $W_{p,q}^{2,1}$ -norms and whose operator norm is bounded uniformly in T . Thus, denoting $E_T(\Psi_i^* f_i)$ by f again, it is enough to prove

$$(3) \quad \|(\partial_j^m f)\big|_{x_n=0}\big|_{L^{q, \nu}(\alpha)^{1/2}(Q^{n-1}(\alpha) \times (0, T))}\| \leq c^* \cdot \|f\|_{W_{p,q}^{2,1}(\mathbb{R}_+^n \times (0, 2T))}.$$

In the sequel we shall prove (3). By density it is clearly no restriction to assume that $f \in C^2(\overline{\mathbb{R}_+^n} \times [0, 2T])$ additionally. We start from representation (A.1) for $\partial_j^m f$: splitting $\int_0^T (\dots) dv = \int_0^h (\dots) dv + \int_h^T (\dots) dv$ in the sum in the second line in (A.1) we get

$$\partial_j^m f(\cdot) = H_1(\cdot) + \sum_{i=1}^{n+1} \tilde{B}_i \{H_2^{(i)}(\cdot) + H_3^{(i)}(\cdot)\} \quad \text{for } m = 0, 1,$$

where

$$(4) \quad \begin{aligned} H_1(\cdot) &:= \frac{A}{T^r} \int \dots \int_{Q^{n+1}(0, T^{\underline{\kappa}})} f(\cdot + y) \Pi(y, T) dy, \\ H_2^{(i)}(\cdot) &:= \int_0^h v^{-(1+r)} \int \dots \int_{Q^{n+1}(0, v^{\underline{\kappa}})} \partial_i^{l_i} f(\cdot + y) \cdot K_i(y, v) dy dv, \\ H_3^{(i)}(\cdot) &:= \int_h^T v^{-(1+r)} \int \dots \int_{Q^{n+1}(0, v^{\underline{\kappa}})} \partial_i^{l_i} f(\cdot + y) \cdot K_i(y, v) dy dv. \end{aligned}$$

We choose $\underline{l} := (2, \dots, 2, 1) \in \mathbb{N}^{n+1}$ and $\underline{\kappa} := (\frac{1}{2}, \dots, \frac{1}{2}, 1) \in \mathbb{R}^{n+1}$. Abbreviating $(\gamma H_1)(x', t) := H_1(x', 0, t)$, we find

$$(5) \quad \|\Delta_{n+1, h}(\gamma H_1)\|_{p, q, Q^{n-1}(\alpha) \times (0, T-h)} \leq h \cdot \|\partial_t(\gamma H_1)\|_{p, q, Q^{n-1}(\alpha) \times (0, T)}$$

(use $|\Delta_{n+1,h} g(\tau)| \leq \int_0^h |g'(\tau+s)| ds$ and Minkowski's integral inequality, cf. Wheeden and Zygmund [10], p. 143); now

$$\begin{aligned} |\partial_t(\gamma H_1)(x', t)| &\leq \frac{|A|}{T^r} \cdot \|\Pi(\cdot, T)\|_{\infty, Q^{n+1}(0, T^\varepsilon)} \cdot |Q^{n+1}(0, T^\varepsilon)|^{1/p'} \\ &\quad \cdot \|\partial_t f((x', 0, t) + \cdot)\|_{p, Q^{n+1}(0, T^\varepsilon)} \end{aligned}$$

by (4) and Hölder's inequality; hence

$$\leq c^* \cdot T^{-|\varepsilon| \cdot (1-1/p') - m \cdot \kappa_j} \cdot \|\partial_t f((x', 0, t) + \cdot)\|_{p, Q^{n+1}(0, T^\varepsilon)}$$

by kernel-estimate (A.2). Thus

$$\begin{aligned} \|\partial_t(\gamma H_1)(\cdot, t)\|_{p, Q^{n-1}(\alpha)} &\leq c^* \cdot T^{-|\varepsilon| \cdot (1-1/p') - m \cdot \kappa_j} \cdot \left| \dot{Q}^{n+1}(0, T^\varepsilon) \right|^{1/p} \\ &\quad \times \left(\int_0^{T^{\kappa_{n+1}}} \int_0^{T^{\kappa_n}} \|\partial_t f(\cdot, y_n, t + y_{n+1})\|_{p, \mathbb{R}^{n-1}}^p dy_n dy_{n+1} \right)^{1/p} \end{aligned}$$

and consequently, since $|\dot{Q}^{n+1}(0, T^\varepsilon)| = T^{|\varepsilon| - 3/2}$ and $\kappa_{n+1} = 1$,

$$\begin{aligned} &\left(\int_0^T \|\partial_t(\gamma H_1)(\cdot, t)\|_{p, Q^{n-1}(\alpha)}^q dt \right)^{1/q} \leq c^* \cdot T^{-m \cdot \kappa_j - 3/2p} \\ &\quad \times \left(\int_0^T \left(\int_0^T \int_0^{T^{\kappa_n}} \|\partial_t f(\cdot, y_n, t + y_{n+1})\|_{p, \mathbb{R}^{n-1}}^p dy_n dy_{n+1} \right)^{q/p} dt \right)^{1/q}. \end{aligned}$$

By Minkowski's integral inequality the last integral does not exceed

$$\left(\int_0^T \left(\int_0^T \left(\int_0^{T^{\kappa_n}} \|\partial_t f(\cdot, y_n, t + y_{n+1})\|_{p, \mathbb{R}^{n-1}}^p dy_n \right)^{q/p} dt \right)^{p/q} dy_{n+1} \right)^{1/p},$$

which is majorized by

$$T^{1/p} \left(\int_0^{2T} \left(\int_0^{T^{\kappa_n}} \|\partial_t f(\cdot, y_n, \tau)\|_{p, \mathbb{R}^{n-1}}^p dy_n \right)^{q/p} d\tau \right)^{1/q}$$

after integrating out the y_{n+1} -variable. These estimates imply

$$\text{r.h. side in (5)} \leq c^* \cdot h \cdot T^{-(m \cdot \kappa_j + \frac{1}{2p})} \cdot \|\partial_t f\|_{p, q, \mathbb{R}^{n-1} \times (0, T^{\kappa_n}) \times (0, 2T)};$$

thus, abbreviating $\varrho := \frac{1}{2}(2 - m - \frac{1}{p})$, we see that

$$\begin{aligned} |\gamma H_1|_{L_{\varrho}^q(Q^{n-1}(\alpha) \times (0, T))} &= \left(\int_0^T h^{-(1+q\varrho)} \|\Delta_{n+1,h}(\gamma H_1)\|_{p, q, Q^{n-1}(\alpha) \times (0, T-h)}^q dh \right)^{1/q} \\ &\leq c^* \cdot T^{-(m \cdot \kappa_j + \frac{1}{2p})} \cdot \left(\int_0^T h^{-1+q(1-\varrho)} dh \right)^{1/q} \|\partial_t f\|_{p, q, \mathbb{R}_+^n \times (0, 2T)}; \end{aligned}$$

now $1 - \varrho = \frac{1}{2}(m + \frac{1}{p})$ and the T factors in our last inequality cancel (since $\kappa_j = \frac{1}{2}$), as desired.

Let us turn our attention to $H_2^{(i)}$: trivially, for $h \leq T$,

$$(6) \quad \|\Delta_{n+1, h}(\gamma H_2^{(i)})\|_{p, q, Q^{n-1}(\alpha) \times (0, T-h)} \leq 2 \cdot \|\gamma H_2^{(i)}\|_{p, q, Q^{n-1}(\alpha) \times (0, T)};$$

furthermore, using kernel estimate (A.3) (with $s = 0$), we get

$$(7) \quad |\gamma H_2^{(i)}(x', t)| \leq c^* \cdot \int_0^h v^{-(1+|\underline{x}|+\varepsilon\kappa_n)+\frac{1}{2}(2-m)} \int \dots \int_{Q^{n+1}(0, v^{\underline{x}})} y_n^\varepsilon \cdot |\partial_i^{l_i} f((x', 0, t) + y)| dy dv;$$

we now represent the integrand as

$$\{v^{-\frac{1}{p'}(1+|\underline{x}|)+\frac{1}{2}(\varrho-\varepsilon\kappa_n)}\} \cdot \{v^{-\frac{1}{p'}(1+|\underline{x}|-\frac{1}{2})+\frac{1}{2}(\varrho-\varepsilon\kappa_n)} \cdot y_n^\varepsilon \cdot |\partial_i^{l_i} f((x', 0, t) + y)|\}$$

(note that $\frac{1}{2}(2-m) = \varrho + 1/2p$); we choose $\varepsilon \in (0, \varrho/\kappa_n)$; Hölder's inequality (with p', p) in y - v space then yields

$$(8) \quad \text{r.h. side in (7)} \leq c^* \cdot \left(\int_0^h v^{-1+\frac{p'}{2}(\varrho-\varepsilon\kappa_n)} dv \right)^{1/p'} \cdot I^{1/p}$$

with

$$I := \int_0^h \int \dots \int_{Q^{n+1}(0, v^{\underline{x}})} v^{-(1+|\underline{x}|-\frac{1}{2})+\frac{p}{2}(\varrho-\varepsilon\kappa_n)} \cdot y_n^{\varepsilon \cdot p} \cdot |\partial_i^{l_i} f((x', 0, t) + y)|^p dy dv,$$

where in the first integral we took into account that $|Q^{n+1}(0, v^{\underline{x}})| = v^{|\underline{x}|}$; the first integral clearly is proportional to $h^{\frac{1}{2}(\varrho-\varepsilon\kappa_n)}$. Thus, by (7) and (8),

$$(9) \quad \|\gamma H_2^{(i)}(\cdot, t)\|_{p, Q^{n-1}(\alpha)} \leq c^* \cdot h^{\frac{1}{2}(\varrho-\varepsilon\kappa_n)} \cdot \tilde{I}^{1/p}$$

with

$$\begin{aligned} \tilde{I} &:= \int_0^h v^{-(1+|\underline{x}|-\frac{1}{2})+\frac{p}{2}(\varrho-\varepsilon\kappa_n)} |Q^{n+1}(0, v^{\underline{x}})| \times \\ &\times \int_0^{v^{\kappa_{n+1}}} \int_0^{v^{\kappa_n}} y_n^{\varepsilon \cdot p} \cdot \|\partial_i^{l_i} f(\cdot, y_n, t + y_{n+1})\|_{p, \mathbb{R}^{n-1}}^p dy_n dy_{n+1} dv; \end{aligned}$$

abbreviating $F_i(y_n, \tau) := y_n^{\varepsilon \cdot p} \cdot \|\partial_i^{l_i} f(\cdot, y_n, \tau)\|_{p, \mathbb{R}^{n-1}}^p$, (9) implies

$$(10) \quad \left(\int_0^T \|\gamma H_2^{(i)}(\cdot, t)\|_{p, Q^{n-1}(\alpha)}^q dt \right)^{1/q} \leq c^* \cdot h^{\frac{1}{2}(\varrho-\varepsilon\kappa_n)} \times \\ \times \left(\int_0^T \left(\int_0^h \int_0^{v^{\kappa_{n+1}}} v^{-2+\frac{p}{2}(\varrho-\varepsilon\kappa_n)} \int_0^{v^{\kappa_n}} F_i(y_n, t + y_{n+1}) dy_n dy_{n+1} dv \right)^{q/p} dt \right)^{1/q}.$$

By Minkowski's integral inequality the last integral does not exceed

$$\left(\int_0^h \int_0^{v^{\kappa_n+1}} \left(\int_0^T \left(v^{-2+\frac{\varepsilon}{2}(\varepsilon-\varepsilon\kappa_n)} \int_0^{v^{\kappa_n}} F_i(y_n, t + y_{n+1}) dy_n \right)^{q/p} dt \right)^{p/q} dy_{n+1} dv \right)^{1/p},$$

which is majorized by

$$\left(\int_0^h v^{-1+\frac{\varepsilon}{2}(\varepsilon-\varepsilon\kappa_n)} \left(\int_0^{T+v} \left(\int_0^{v^{\kappa_n}} F_i(y_n, \tau) dy_n \right)^{q/p} d\tau \right)^{p/q} dv \right)^{1/p}$$

and thus also by

$$c^* \cdot h^{\frac{1}{2}(\varepsilon-\varepsilon\kappa_n)} \cdot \left(\int_0^{T+h} \left(\int_0^{h^{\kappa_n}} y_n^{\varepsilon \cdot p} \cdot \|\partial_i^l f(\cdot, y_n, \tau)\|_{p, \mathbb{R}^{n-1}}^p dy_n \right)^{q/p} d\tau \right)^{1/q};$$

here we integrated out the y_{n+1} and v variables successively (recall that $\kappa_{n+1} = 1$). By (6), (10) and the last estimates

(11)

$$\begin{aligned} |\gamma H_2^{(i)}|_{L_q^{\alpha, \varepsilon}(Q^{n-1}(\alpha) \times (0, T))}^q &= \int_0^T h^{-(1+q\varepsilon)} \|\Delta_{n+1, h}(\gamma H_2^{(i)})\|_{p, q, Q^{n-1}(\alpha) \times (0, T-h)}^q dh \\ &\leq c^* \cdot \int_0^T h^{-(1+q\varepsilon\kappa_n)} \int_0^{T+h} \left(\int_0^{h^{\kappa_n}} y_n^{\varepsilon \cdot p} \cdot \|\partial_i^l f(\cdot, y_n, \tau)\|_{p, \mathbb{R}^{n-1}}^p dy_n \right)^{q/p} d\tau dh \\ &\leq c^* \cdot \int_0^{2T} \int_0^T h^{-(1+q\varepsilon\kappa_n)} \left(\int_0^{h^{\kappa_n}} y_n^{\varepsilon \cdot p} \cdot \|\partial_i^l f(\cdot, y_n, \tau)\|_{p, \mathbb{R}^{n-1}}^p dy_n \right)^{q/p} dh d\tau, \end{aligned}$$

the last step by Fubini's theorem and since $h \leq T$. By the Hardy-type inequality in Appendix B, Lemma B.1(i) (applied with $r = q/p \geq 1 = s$, $\gamma = \kappa_n$ and ε replaced with $\varepsilon \cdot p$; note that then indeed $\varepsilon \cdot p \cdot \gamma \cdot r = q \cdot \varepsilon \cdot \kappa_n$) we get for the inner integral in the last line

$$\begin{aligned} \int_0^T h^{-(1+q\varepsilon\kappa_n)} \left(\int_0^{h^{\kappa_n}} y_n^{\varepsilon \cdot p} \cdot \|\partial_i^l f(\cdot, y_n, \tau)\|_{p, \mathbb{R}^{n-1}}^p dy_n \right)^{q/p} dh \\ \leq c^* \cdot \left(\int_0^{T^{\kappa_n}} \|\partial_i^l f(\cdot, y_n, \tau)\|_{p, \mathbb{R}^{n-1}}^p dy_n \right)^{q/p}; \end{aligned}$$

using this estimate in the last line in (11) we get the desired result for $H_2^{(i)}$.

Finally, let us turn to $H_3^{(i)}$; we again use (5) and observe that the correct expression for $\partial_i(\gamma H_3^{(i)})$ is obtained just by replacing K_i (in the definition of $H_3^{(i)}$) by $\partial_{y_{n+1}} K_i$

(integrate by parts); after estimating $\partial_{y_{n+1}} K_i$ according to (A.3) we arrive at

$$(12) \quad |\partial_t(\gamma H_3^{(i)})(x', t)| \\ \leq c^* \cdot \int_h^T v^{-(1+|\underline{x}|+\varepsilon \cdot \kappa_n) - \frac{m}{2}} \int \dots \int_{Q^{n+1}(0, v \underline{x})} y_n^\varepsilon \cdot |\partial_i^l f((x', 0, t) + y)| dy dv$$

(see (7); here the v -exponent is smaller by one, since $\partial_{y_{n+1}} K_i$ entails (in (A.3)) the additional factor v^{-1}); in the last integral we write the integrand in the form (note that $-m/2 = \frac{1}{2}p + \varrho - 1$)

$$\left\{ v^{-\frac{1}{p'}(1+|\underline{x}|) - (1-\varrho-\delta)} \right\} \cdot \left\{ v^{-\frac{1}{p}(1+|\underline{x}| - \frac{1}{2}) - (\varepsilon \kappa_n + \delta)} \cdot y_n^\varepsilon \cdot |\partial_i^l f(\dots)| \right\},$$

where we introduced $\delta \in (0, 1-\varrho) \cap (0, 1/q)$. Now we apply Hölder's inequality (with p', p) in y - v space and get

$$\text{r.h. side in (12)} \leq c^* \cdot \left(\int_h^T v^{-1-p'(1-\varrho-\delta)} dv \right)^{1/p'} \cdot J^{1/p}$$

with

$$J := \int_h^T v^{-(1+|\underline{x}| - \frac{1}{2}) - p \cdot (\varepsilon \cdot \kappa_n + \delta)} \int \dots \int_{Q^{n+1}(0, v \underline{x})} y_n^{\varepsilon \cdot p} \cdot |\partial_i^l f((x', 0, t) + y)|^p dy dv;$$

from this we get (see (10))

$$(13) \quad \left(\int_0^T \|\partial_t(\gamma H_3^{(i)})(\cdot, t)\|_{p, Q^{n-1}(\alpha)}^q dt \right)^{1/q} \leq c^* \cdot h^{-(1-\varrho-\delta)} \times \\ \times \left(\int_0^T \left(\int_h^T \int_0^{v^{\kappa_n+1}} v^{-2-p \cdot (\varepsilon \cdot \kappa_n + \delta)} \int_0^{v^{\kappa_n}} F_i(y_n, t + y_{n+1}) dy_n dy_{n+1} dv \right)^{q/p} dt \right)^{1/q}.$$

After applying Minkowski's integral inequality and integrating out the y_{n+1} variable (as after (10)) we see that the last integral does not exceed

$$\left(\int_h^T v^{-1-p \cdot (\varepsilon \cdot \kappa_n + \delta)} \left(\int_0^{T+v} \left(\int_0^{v^{\kappa_n}} F_i(y_n, \tau) dy_n \right)^{q/p} d\tau \right)^{p/q} dv \right)^{1/p} \\ =: \left(\int_h^T g(v) dv \right)^{1/p};$$

from (5), (13) and the last estimate we get

$$|\gamma H_3^{(i)}|_{\mathcal{L}_q^0, \varrho(Q^{n-1}(\alpha) \times (0, T))}^q = \int_0^T h^{-(1+q\varrho)} \|\Delta_{n+1, h}(\gamma H_3^{(i)})\|_{p, q, Q^{n-1}(\alpha) \times (0, T-h)}^q dh \\ \leq c^* \cdot \int_0^T h^{-1+q\delta} \left(\int_h^T g(v) dv \right)^{q/p} dv;$$

now we apply Lemma B.1'(ii) with $r = s := q/p$ and $a \cdot r := 1 - q\delta < 1$ and get (since the total exponent of the weight on the r.h. side in this Lemma equals $s \cdot (-a + 1/s' + 1/r)$, which equals $q\delta - 1 + q/p$ in our case)

$$\begin{aligned} &\leq c^* \cdot \int_0^T v^{-1+q(\delta+1/p)} g(v)^{q/p} dv \\ &= c^* \cdot \int_0^T v^{-(1+q\epsilon\kappa_n)} \int_0^{T+v} \left(\int_0^{v^{\kappa_n}} y_n^{\epsilon p} \cdot \|\partial_i^l f(\cdot, y_n, \tau)\|_{p, \mathbb{R}^{n-1}}^p dy_n \right)^{q/p} d\tau dv, \end{aligned}$$

where we inserted the definitions of g and F_i . The last line is identical with the second line in (11), so that the desired result for $H_3^{(i)}$ follows.

Thus (3) and with it the Theorem are proved. \square

We still have to prove two auxiliary results:

Lemma 3. *Let the assumptions of Theorem 1 be fulfilled. Then D is dense in $W_{p,q}^{2,1}(\Omega_T)$.*

Proof. Take $f \in W_{p,q}^{2,1}(\Omega_T)$; clearly $f(\cdot, t) \in W_p^2(\Omega)$ for a.e. $t \in (0, T)$, say on $(0, T) \setminus E$, $|E| = 0$. Redefining f by $f(\cdot, t) \equiv 0$ for $t \in E$, we may assume $f(\cdot, t) \in W_p^2(\Omega)$ for all $t \in (0, T)$.

The approximation problem can be localized by considering $f\varphi_i$, φ_i from the partition of unity. Next we will flatten the boundary: fix $i \in \{1, \dots, M\}$ and denote $u := \Psi_i^*(f\varphi_i)$. Then $u \in \mathcal{P}$, which means the following: a function g defined on $Q_+^n(\alpha, \beta) \times (0, T)$ is called *spatially properly supported* and we write $g \in \mathcal{P}$, if there exists $\epsilon > 0$ such that $\text{supp } g(\cdot, t) \subset Q^{n-1}(\alpha - \epsilon) \times [0, \beta - \epsilon]$ for a.e. $t \in (0, T)$. Since Ψ_i^* induces an isomorphism $W_{p,q}^{2,1}((U_i \cap \Omega)_T) \rightarrow W_{p,q}^{2,1}(Q_+^n(\alpha, \beta)_T)$ and since for a $\Phi \in W_{p,q}^{2,1}(Q_+^n(\alpha, \beta)_T) \cap \mathcal{P}$ we may regard $(\Psi_i^*)^{-1}\Phi$ as an element of $W_{p,q}^{2,1}(\Omega_T)$ (by zero continuation), it is sufficient to solve the approximation problem in $W_{p,q}^{2,1}(Q_+^n(\alpha, \beta)_T)$ and in such a way that the approximating functions belong to \mathcal{P} also. Since u_δ with $u_\delta(x', x_n, t) := u(x', x_n + \delta, t)$ tends to u in $W_{p,q}^{2,1}(Q_+^n(\alpha, \beta)_T)$ for $\delta \downarrow 0$ and u_δ has the same properties as u (for δ small), it is sufficient to approximate u_δ . To achieve this, set

$$u_k := \varrho_{1/k} * ((u_\delta)^0 \theta),$$

where $\varrho_{1/k}$ is the usual smooth mollifier with $\|\varrho_{1/k}\|_{1, \mathbb{R}^n} = 1$ and $\text{supp } \varrho_{1/k} \subset B_{1/k}(0)$, $\theta = \theta(x)$ is a smooth function with $\theta \equiv 1$ on $\overline{\mathbb{R}_+^n}$ and $\text{supp } \theta \subset \mathbb{R}^{n-1} \times (-\delta/2, \infty)$ and “ $*$ ” denotes convolution in x and “ 0 ” denotes extension by zero (in x)

to the whole space. By standard arguments we then have for all t

$$\begin{aligned} u_k(\cdot, t) &\rightarrow ((u_\delta)^0 \theta)(\cdot, t) \text{ in } W_p^2(\mathbb{R}^n) \quad (k \rightarrow \infty), \\ \|u_k(\cdot, t)\|_{W_p^2(\mathbb{R}^n)} &\leq c^* \cdot \|((u_\delta)^0 \theta)(\cdot, t)\|_{W_p^2(\mathbb{R}^n)} \\ &\leq c^* \cdot \|(u_\delta)^0(\cdot, t)\|_{W_p^2(\mathbb{R}^{n-1} \times (-\delta/2, \infty))} \leq c^* \cdot \|u(\cdot, t)\|_{W_p^2(Q_+^n(\alpha, \beta))} \end{aligned}$$

for δ small; this implies by Lebesgue's theorem

$$u_k|_{Q_+^n(\alpha, \beta)_T} \rightarrow u_\delta \text{ in } L_q(0, T, W_p^2(Q_+^n(\alpha, \beta))).$$

What remains to be shown is

$$\partial_t u_k|_{Q_+^n(\alpha, \beta)_T} \rightarrow \partial_t u_\delta \text{ in } L_q(0, T, L_p(Q_+^n(\alpha, \beta)));$$

this follows as above, if we show that

$$\partial_t u_k = \varrho_{1/k} * \left(((\partial_t u)_\delta)^0 \theta \right) \text{ in } \mathcal{D}'(\mathbb{R}^{n+1} \times (0, T));$$

the last line follows easily, if we show that

$$(14) \quad \partial_t((u_\delta)^0 \theta) = ((\partial_t u)_\delta)^0 \theta \text{ in } \mathcal{D}'(\mathbb{R}^{n+1} \times (0, T));$$

to prove (14) take $\varphi \in C_0^\infty(\mathbb{R}^{n+1} \times (0, T))$; then we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^{n+1}} \dots \int (u_\delta)^0 \theta \partial_t \varphi \\ &= \int_0^T \int_{-\delta/2}^{\beta-\delta} \int_{Q^{n-1}(\alpha)} \dots \int (u_\delta \theta \partial_t \varphi)(x', x_n, t) dx' dx_n dt \\ &= \int_0^T \int_{\delta/2}^\beta \int_{Q^{n-1}(\alpha)} \dots \int u(x', x_n, t) \theta(x', x_n - \delta) \partial_t \varphi(x', x_n - \delta, t) dx' dx_n dt \\ &= \int_0^T \int_0^\beta \int_{Q^{n-1}(\alpha)} \dots \int \dots dx' dx_n dt \quad (\text{since } \theta \text{ cuts off in } x_n) \\ &= \int_0^T \int_0^\beta \int_{Q^{n-1}(\alpha)} \dots \int \eta(x', x_n) u(x', x_n, t) \theta(x', x_n - \delta) \partial_t \varphi(x', x_n - \delta, t) dx' dx_n dt, \end{aligned}$$

where η is a smooth cut-off function with $\eta \equiv 1$ on $\bigcup_{t \in (0, T)} \text{supp } u(\cdot, t)$ and $\eta \in \mathcal{P}$; the last line can be rephrased as

$$\int_0^T \int_0^\beta \int_{Q^{n-1}(\alpha)} \dots \int u(x', x_n, t) \partial_t \tilde{\varphi}(x', x_n, t) dx' dx_n dt,$$

where $\tilde{\varphi}(x', x_n, t) := \eta(x', x_n) \theta(x', x_n - \delta) \varphi(x' x_n - \delta, t)$ belongs to $C_0^\infty(Q_+^n(\alpha, \beta) \times (0, T))$. Now we may shift the ∂_t from $\tilde{\varphi}$ to u and reverse the above chain of reasoning to end up with

$$- \int_0^T \int_{\mathbb{R}^{n+1}} \dots \int ((\partial_t u)_\delta)^0 \theta \varphi.$$

□

Lemma 4. *Let the assumptions of Theorem 1 be fulfilled. Let $\alpha \in (0, 2)$ and $\beta \in (0, 1)$. Then $X_{p,q}^{\alpha,\beta}(\Gamma_T)$ is complete.*

Proof. Let (g_k) be a Cauchy sequence in $X_{p,q}^{\alpha,\beta}(\Gamma_T)$; then (g_k) is also a Cauchy sequence in $L_q(0, T, W_p^\alpha(\Gamma))$ and by the completeness of this latter space we find a $g \in L_q(0, T, W_p^\alpha(\Gamma))$ such that $g_k \rightarrow g$ in $L_q(0, T, W_p^\alpha(\Gamma))$. This implies $\|\Delta_{n+1,h}(g_k - g_j)\|_{L_q(0, T-h, L_p(\Gamma))} \rightarrow \|\Delta_{n+1,h}(g - g_j)\|_{L_q(0, T-h, L_p(\Gamma))}$ for $k \rightarrow \infty$, so that by Fatou's Lemma we may conclude $\|g - g_j\|_{L_q^{\alpha,\beta}(\Gamma_T)} \rightarrow 0$ for $j \rightarrow \infty$. The proof is complete. □

APPENDIX A

Here we give the details about the integral representation used earlier: for a smooth f we have (cf. Il'in and Solonnikov [3], p. 70, (6) with $m_i = 0$, $k_i = l_i$)

$$\begin{aligned} \partial^z f(x, t) &= \frac{A}{T^r} \int \dots \int_{Q^{n+1}(0, T^z)} f((x, t) + y) \Pi(y, T) dy \\ &+ \sum_{i=1}^{n+1} B_i \int_0^T v^{-(1+r)} \int \dots \int_{Q^{n+1}(0, v^z)} f((x, t) + y) \Pi_i(\dot{y}, v) \partial_i^{l_i} \psi_i(y_i, v) dy dv \end{aligned}$$

for $\nu_j \leq l_j - 1$, where (cf. [3], pp. 69–70)

$$\begin{aligned}\Pi(y, T) &:= \prod_{j=1}^{n+1} \partial_j^{l_j} \chi_j(y_j, T), \\ \chi_j(y_j, T) &:= y_j^{l_j - \nu_j - 1} \int_{y_j}^{T^{\kappa_j}} (T^{\kappa_j} - s)^{\mu_j} s^{\lambda_j} ds, \\ \Pi_i(\dot{y}, v) &:= \prod_{\substack{j=1 \\ j \neq i}}^{n+1} \partial_j^{l_j} \chi_j(y_j, v), \\ \psi_i(y_i, v) &:= y_i^{l_i + \lambda_i - \nu_i} \cdot (v^{\kappa_i} - y_i)^{\mu_i}\end{aligned}$$

with certain parameters $l_j, \mu_j, \lambda_j \in \mathbb{N}$ and certain $A, B_i \in \mathbb{R}$; here $\underline{\kappa} = (\kappa_1, \dots, \kappa_{n+1}) \in \mathbb{R}^{n+1}$ and $r := |\underline{\kappa}| + \underline{\kappa} \cdot (\underline{\lambda} + \underline{\mu})$, where $\underline{\lambda} := (\lambda_1, \dots, \lambda_{n+1})$ etc. We choose the parameters μ_i, λ_i so large that $\partial_i^k \psi_i(y_i, v)$ vanishes for $k = 1, \dots, l_i$ at $y_i = 0$ and $y_i = v^{\kappa_i}$. Hence, integrating by parts and introducing $K_i(y, v) := \Pi_i(\dot{y}, v) \psi_i(y_i, v)$, $0 \leq y_i \leq v^{\kappa_i}$, we have show that

$$\begin{aligned}\text{(A.1)} \quad \partial^{\underline{\nu}} f(x, t) &= \frac{A}{T^r} \int \dots \int_{Q^{n+1}(0, T^{\underline{\kappa}})} f((x, t) + y) \Pi(y, T) dy \\ &+ \sum_{i=1}^{n+1} \tilde{B}_i \int_0^T v^{-(1+r)} \int \dots \int_{Q^{n+1}(0, t^{\underline{\kappa}})} \partial_i^{l_i} f((x, t) + y) K_i(y, v) dy dv.\end{aligned}$$

(The kernels Π, K_i in this representation clearly depend on $\underline{\nu}, \underline{\kappa}, \underline{\lambda}, \underline{\mu}, \underline{l}$, but this dependence is suppressed in our notation.) They satisfy (uniformly in $y \in Q^{n+1}(0, v^{\underline{\kappa}})$)

$$\text{(A.2)} \quad |\partial_y^{\underline{\alpha}} \Pi(y, v)| \leq c \cdot v^{r - |\underline{\kappa}| - \underline{\kappa} \cdot (\underline{\nu} + \underline{\alpha})} \quad \forall |\underline{\alpha}| \leq 2,$$

$$\begin{aligned}\text{(A.3)} \quad |\partial_{y_{n+1}}^s K_i(y, v)| &\leq c \cdot y_n^\varepsilon \cdot v^{r + \kappa_i l_i - |\underline{\kappa}| - \underline{\kappa} \cdot \underline{\nu} - \varepsilon \kappa_n - s \kappa_{n+1}}, \\ &(0 \leq s \leq 1, 1 \leq i \leq n+1, \varepsilon \in (0, \varepsilon_0)).\end{aligned}$$

For the proof of these inequalities, we first note that $\partial_j^{l_j + \alpha_j} \chi_j(y_j, v)$ is a linear combination of terms of the form $(v^{\kappa_j} - y_j)^{\varrho_1} y_j^{\varrho_2}$ with $\varrho_1 + \varrho_2 = \mu_j + \lambda_j - \nu_j - \alpha_j$, $\varrho_2 > 0$ (for λ_j large) and consequently

$$|\partial_j^{l_j + \alpha_j} \chi_j(y_j, v)| \leq c \cdot y_j^\varepsilon \cdot v^{-\kappa_j(\varepsilon + \alpha_j)} \cdot v^{\kappa_j(\mu_j + \lambda_j - \nu_j)} \quad (0 \leq y_j \leq v^{\kappa_j})$$

for $\varepsilon \in (0, \varrho_2)$; this implies (for $k = 1, \dots, n-1$)

$$\begin{aligned}|\partial_{n+1}^s \Pi_k(\dot{y}, v)| &\leq c \cdot y_n^\varepsilon \cdot v^{-\kappa_n \varepsilon - \kappa_{n+1} \cdot s} \cdot v^{\underline{\kappa} \cdot \underline{\delta} - \kappa_k \cdot \delta_k}, \\ |\partial_{n+1}^s \Pi_n(\dot{y}, v)| &\leq c \cdot v^{-\kappa_{n+1} \cdot s} \cdot v^{\underline{\kappa} \cdot \underline{\delta} - \kappa_n \cdot \delta_n}, \\ |\Pi_{n+1}(\dot{y}, v)| &\leq c \cdot y_n^\varepsilon \cdot v^{-\kappa_n \cdot \varepsilon} \cdot v^{\underline{\kappa} \cdot \underline{\delta} - \kappa_{n+1} \cdot \delta_{n+1}},\end{aligned}$$

where $\underline{\delta} := \underline{\mu} + \underline{\lambda} - \underline{\nu}$. The definition of ψ_i easily implies

$$\begin{aligned} |\psi_k(y_k, v)| &\leq v^{\kappa_k \cdot (l_k + \delta_k)}, \\ |\psi_n(y_n, v)| &\leq y_n^\varepsilon \cdot v^{-\kappa_n \cdot \varepsilon} \cdot v^{\kappa_n \cdot (l_n + \delta_n)}, \\ |\partial_{n+1}^s \psi_{n+1}(y_{n+1}, v)| &\leq c \cdot v^{-s \cdot \kappa_{n+1}} \cdot v^{\kappa_{n+1} \cdot (l_{n+1} + \delta_{n+1})}, \end{aligned}$$

since $K_i(y, v) = \Pi_i(\dot{y}, v) \psi_i(y_i, v)$, these formulas yield (A.3). For (A.2) compare II' in and Solonnikov [3], p. 72.

APPENDIX B

We state some basic inequalities.

Lemma B.1. *Suppose that $1 \leq s \leq r < \infty$, $f \in L_s(0, T^\gamma)$, $0 < \varepsilon, \gamma < \infty$, $0 < T \leq \infty$. Then*

- (i) $\|x^{-1/r - \varepsilon \gamma} \cdot \int_0^{x^\gamma} y^{\varepsilon - 1/s'} f(y) dy\|_{L_r(0, T, dx)} \leq c(\dots) \|f\|_{L_s(0, T^\gamma)}$,
 - (ii) $\|x^{-1/r + \varepsilon \gamma} \cdot \int_{x^\gamma}^T y^{-\varepsilon - 1/s'} f(y) dy\|_{L_r(0, T, dx)} \leq c(\dots) \|f\|_{L_s(0, T^\gamma)}$,
- where $c(\dots) = c(\varepsilon, \gamma, r, s) = \gamma^{-1/r} \left(\frac{\mu}{\varepsilon}\right)^\mu$, $\mu = 1 - \frac{1}{s} + \frac{1}{r}$.

Proof. Compare Besov [2], 2.15, p. 28. □

Putting $s = r = 1$ in (i) and reformulating (ii) (for $\gamma = 1$) in an equivalent way, we get a version which is sometimes handier for our purposes:

Lemma B.1'. *Let the assumptions of the preceding Lemma hold. Then*

- (i) $\int_0^T x^{-1-\varepsilon} \int_0^x y^\varepsilon \cdot f(y) dy dx \leq \varepsilon^{-1} \int_0^T f(y) dy$ for all $f \in L_1(0, T)$, $f \geq 0$;
- (ii) If $a \cdot r < 1$, then

$$\|x^{-a} \int_x^T f(y) dy\|_{L_r(0, T, dx)} \leq c(a, r, s) \cdot \|y^{-a+1/s'+1/r} f(y)\|_{L_s(0, T, dy)},$$

for all f with r.h. side finite.

References

- [1] *R. A. Adams: Sobolev Spaces.* Academic Press, New York, San Francisco, London, 1975.
- [2] *O. V. Besov, V. P. Il'in and S. M. Nikol'skii: Integral Representations of Functions and Imbedding Theorems I.* Wiley, New York, London, Toronto, 1978.
- [3] *V. P. Il'in and V. A. Solonnikov: On some properties of differentiable functions of several variables.* Transl. AMS *81* (1969), 67-90; Trudy Mat. Inst. Steklov *66* (1962), 205-226.
- [4] *H. Iwashita: L_q - L_r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problem in L_q spaces.* Math. Ann. *285* (1989), 265-288.
- [5] *A. Kufner, O. John and S. Fučík: Function Spaces.* Noordhoff Int. Publ., Leyden, 1977.
- [6] *O. A. Ladyshenskaya, V. A. Solonnikov and N. N. Ural'eva: Linear and Quasilinear Equations of Parabolic Type.* Am. Math. Soc., Providence, R.I., 1968.
- [7] *W. v. Wahl: The equation $u' + A(t)u = f$ in a Hilbert space and L^p -estimates for parabolic equations.* J. London Math. Soc. *25* (1982), 483-497.
- [8] *W. v. Wahl: Vorlesung über das Außenraumproblem für die instationären Gleichungen von Navier-Stokes.* Vorlesungsreihe No. 11, SFB 256. Universität Bonn, 1989.
- [9] *P. Weidemaier: The trace theorem $W_p^{2,1}(\Omega_T) \ni f \mapsto \nabla_x f \in W_p^{1-1/p, 1/2-1/2p}(\partial\Omega_T)$ revisited.* Comment. Math. Univ. Carolinae *32* (1991), 307-314.
- [10] *R. L. Wheeden, A. Zygmund: Measure and Integral.* Dekker, New York, Basel, 1977.

Author's address: Fraunhofer-Institut für Kurzezeitdynamik, Hauptstr. 18, 79576 Weil/Rh., Germany.