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On the trade-off between control performance and communication cost for event-triggered control over lossy networks

Burak Demirel[1], Vijay Gupta[2] and Mikael Johansson[1]

Abstract— **This paper develops a theoretical framework for quantifying the trade-off between communication cost and control performance in event-triggered control over lossy networks. We consider a system where the communication between the controller and actuator is dictated by a threshold-based event-triggering algorithm, and develop a Markov-chain model that describes the attempted and successful transmissions of control messages over the lossy communication channel. A feature of our model is that it considers retransmissions of unsuccessful messages and that it accounts for the delay associated with such retransmissions. A systematic framework for analyzing the trade-off between the communication rate and control performance and for optimal tuning of the event threshold emanates by combining this model with an analytical model of the closed-loop performance. Numerical examples demonstrate the effectiveness of the proposed framework.**

I. Introduction

Event-triggered algorithms have emerged as an alternative approach to the traditional periodic implementation of estimators and controllers, and are often able to achieve good performance using significantly reduced communication rates. A reduced communication rate decreases the energy at the transmitter side and could also reduce congestion when the communication takes place over a shared medium. For this reason, there are several potential applications of event-triggered algorithms, *e.g.*, control over communication networks [1], [2], multi-agent systems [3], distributed optimization [4], and embedded control systems [5].

The literature on stability and performance of basic event-triggered control algorithms is by now extensive. The works [4]–[8] proposed triggering algorithms which compute control actions whenever it is necessary to ensure a certain decrease in a Lyapunov function. Otanez *et al.* [9] investigated adjustable deadbands for reducing the communication rate while accounting for the system response and also derived stability conditions for this controller. Model-based event-triggered state-feedback control was considered in [1]. There are also many event-triggering approaches which only consider scalar stochastic control systems, see *e.g.*, [10]–[12] and references therein.

Another line of research has focused on obtaining optimal or sub-optimal event-triggering mechanisms for predefined performance [13]–[17]. Xu and Hespanha [14] proposed an event-triggered estimation framework which attempts to minimize the communication rate while guaranteeing a certain bound on the estimation error covariance matrix. Molin and Hirche [15] worked on a joint optimal scheduling and control problem and established the existence of certain equivalences under communication constraints. Cogill *et al.* [16] aimed at scheduling transmissions so as to balance a trade-off between the communication rate and estimation error. Similarly, Xia *et al.* [17] demonstrated the trade-off between the communication rate and estimation error covariance when the information from the sensor to the controller is transmitted over a shared communication link. Distinctively, Antunes *et al.* [18] considered the co-design of transmission schedules and control inputs for event-triggered control systems while providing a better discounted control performance than periodic control strategies.

In this paper, we consider a threshold-based scheme for triggering transmissions from the controller to the actuator over a lossy network. A feature of our model is that we allow packets to be retransmitted (up to a maximum number of times) to mitigate the detrimental effect of packet loss, and these packets are dropped only if all retransmission attempts fail. Communication happens on a faster time scale than sampling, and our model accounts for the randomly varying delay introduced by the retransmission mechanism. Together with an analytical model of the closed-loop performance, this allows us to quantify the trade-off between control performance and the number of attempted and successful transmissions over the communication link.

Outline of the Paper. In Section II, we introduce the model, the controller and the event-triggering algorithms, along with measures of control performance and communication cost. Sections III and IV present our main results. Numerical examples are provided in Section V while Section VI summarizes the paper.

Notation. In this paper, $\mathbb{R}^n$ denotes the n–dimensional Euclidean space, $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) is the set of non-negative (positive) real numbers, $\mathbb{Z}_{\geq 0}$ ($\mathbb{Z}_{>0}$) is the set of all non-negative (positive) integer numbers, and $\mathbb{Z}_{[a,b]}$ is the set of all integer numbers between a and b; *i.e.*, $\mathbb{Z}_{[a,b]} \triangleq \{x \in \mathbb{Z}_{\geq 0} : a \leq x \leq b\}$. Here, $\mathbb{S}^n_{\geq 0}$ ($\mathbb{S}^n_{>0}$) is the set of all real symmetric positive (semi-) definite matrices of dimension n. We write the vector of all zeros as $\mathbf{0}$, and the vector of all ones as $\mathbf{1}$. The $\mathcal{Q}$ function is defined as

$$\mathcal{Q}(\mu, \Sigma, \epsilon) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \times \int_{\epsilon}^{\infty} \cdots \int_{\epsilon}^{\infty} e^{-\frac{1}{2}(x-\mu)^{\intercal}\Sigma^{-1}(x-\mu)} dx_1 \cdots dx_n \,. \quad (1)$$

For an n–dimensional multivariate Gaussian random vari-

[1]B. Demirel and M. Johansson are with the Automatic Control Laboratory, School of Electrical Engineering, KTH Royal Institute of Technology, Stockholm firstname.secondname@ee.kth.se

[2]V. Gupta is with the Department of Electrical Engineering, University of Notre Dame vgupta2@nd.edu

able X with mean vector $\mu \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{S}^n_{>0}$, we denote the generalization of the $\mathcal{Q}$ function as $\Pr(X \geq x) \triangleq \mathcal{Q}(\mu, \Sigma, x)$, where the inequality is interpreted element-wise.

II. Problem formulation

This section summarizes the process model (including its discrete-time equivalence), the control architecture, and the event-triggering rule. Moreover, it introduces the assumptions under which we will develop a performance analysis for the event-triggered control over lossy channels.

A. Process model and discrete-time equivalent

Consider the continuous-time linear stochastic system

$$dx_t = ax_t dt + bu_t dt + dw_t \ , \quad x(0) = x_0 \ , \qquad (2)$$

where a and b are scalars, x_t is the state, u_t is controlled input, and $\{w_t, \ t \in \mathbb{R}_{\geq 0}\}$ is a Wiener process with zero mean and incremental covariance $R_w dt$. Similarly, the initial state x_0 is modeled as a random variable having a normal distribution with zero mean and variance Π_0. The process $\{w_t\}$ is independent of the initial condition x_0. Noise-free samples of the process state are taken every h seconds and are made available immediately to the controller node. The control signal is computed using an event-triggered control strategy (defined below) and transmitted to the actuator over an unreliable communication link that drops packets according to a Bernoulli process with probability p_ℓ as illustrated in Figure 1. A successful packet transmission between the controller and actuator takes τ seconds. In case of transmission failure, we retransmit the packet until success, or until we have reached a maximum limit of R retransmission attempts. In the absence of new information, the actuator applies zero control; cf. [19]. When a new control command arrives, the actuator applies it immediately and holds it until the next sampling instance. The timing diagram of various signals is shown in Figure 2. It is natural to discretize the system over time intervals of length equal to the slot-time τ of the communication medium. We let n_k denote the number of transmissions required until successful transmission in sampling interval k. Using the techniques

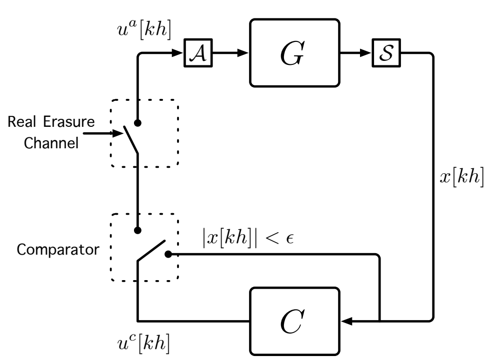

Fig. 1: A block diagram of the event-triggered control system with a (linear) plant G, a dead-beat controller C, a sensor $\mathcal{S}$, an actuator $\mathcal{A}$, a comparator with event-triggering rule $|x[kh]| > \epsilon$, and an unreliable communication link.

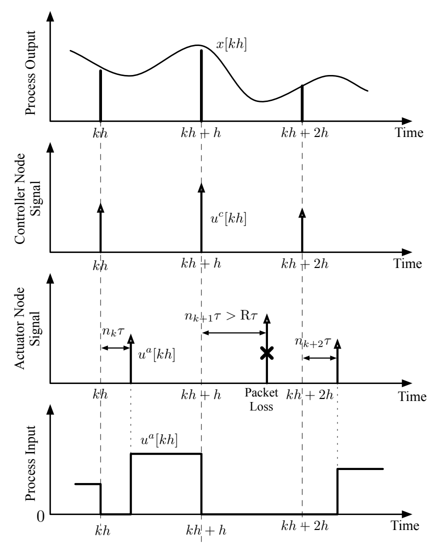

Fig. 2: Timing diagram of signals in the event-triggered control system. The first diagram illustrates the process state and the sampling instants; the second diagram illustrates the signal into the controller node; the third diagram illustrates the signal into the actuator node, and the fourth diagram illustrates the process input. Furthermore, the third diagram shows that if the packet is not delivered at most in R number of retransmissions, then it is dropped. As a consequence, no control signal is applied to the process input.

in [20], we find that the state evolution of (2), for the first n_k time slot intervals following sampling instance k, is given by

$$x[kh + (\tilde{k}+1)\tau] = \Phi(\tau)x[kh + \tilde{k}\tau] + w[kh + \tilde{k}\tau] \ , \quad (3)$$

for all $\tilde{k} \in \mathbb{Z}_{[0,n_k-1]}$ and $k \in \mathbb{Z}_{\geq 0}$ where $\Phi(\tau) \triangleq e^{a\tau}$ (since the actuator applies zero control action during this time). During the remaining time slots until sampling instance $k+1$, the state evolves as

$$\begin{aligned} x[kh + h] = \ & \Phi(h - n_k\tau)x[kh + n_k\tau] \\ & + \Gamma(h - n_k\tau)u^a[kh + n_k\tau] + w[kh + n_k\tau] \ , \quad (4) \end{aligned}$$

where $\Phi(t) \triangleq e^{at}$, $\Gamma(t) \triangleq \int_0^t e^{as} ds \ b = \frac{b}{a}\big(e^{at} - 1\big)$ and $u^a[kh + n_k\tau]$ is the control applied by the actuator at time $kh + n_k\tau$, and computed based on the process state at time kh. The discrete-time equivalent noise processes $\{\omega[k\tau], k \in \mathbb{Z}_{\geq 0}\}$ and $\{\omega[k(h - n_k\tau)], k \in \mathbb{Z}_{\geq 0}\}$ are Gaussian and white with zero means and covariances

$$\Pi_w(h - n_k\tau) = \Pi_w^{h-n_k\tau} \triangleq \int_0^{h-n_k\tau} e^{2as} R_w ds \ , \forall n_k \leq \mathrm{R} \ ,$$

and

$$\Pi_w(\tau) = \Pi_w^{\tau} \triangleq \int_0^{\tau} e^{2as} R_w ds \ ,$$

respectively.

B. Controller and event-triggering rule

We use a deadbeat control strategy, which would transfer the process state at the next sampling instance to the origin in the absence of process noise. Since packet losses and

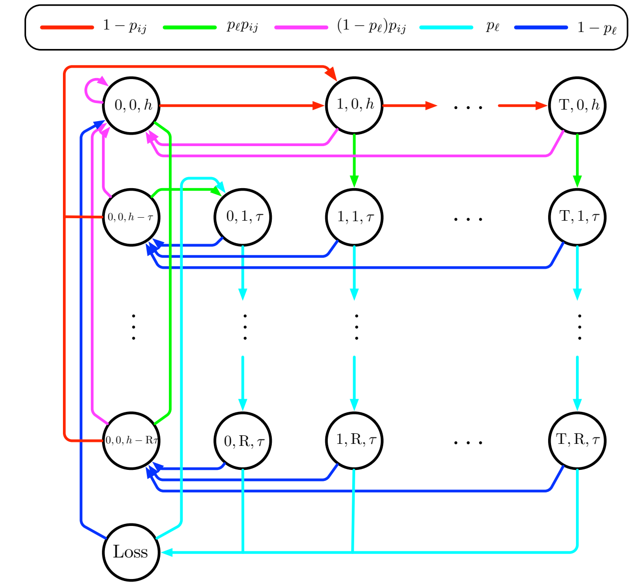

Fig. 3: A multi-dimensional Markov chain which illustrates the packet loss in the network.

retransmissions affect when the control signal is applied at the actuator, such a control law will depend of the number of attempted transmissions and be on the form

$$u^c[kh + n_k\tau] = L(n_k)x[kh] \ , \tag{5}$$

where $L(n_k) \triangleq \Gamma(h - n_k\tau)^{-1}\Phi(h)$.

A threshold-based event-triggering algorithm determines whether or not the computed control signal will be sent to the actuator. Specifically, the information transmission between the controller and actuator will be attempted whenever $|x[kh]| > \epsilon$. In addition, we assume that the controller node transmits at least once every T time steps to guard against the practical concern of no transmissions due to component failure. Considering the unreliable communication channel, there are three possibilities: (a) we choose not to transmit any information at time kh, (b) we choose to transmit a packet but do not succeed in transmission or (c) we choose to and succeed in transmitting the control action to the actuator.

As discussed above, if a transmission attempt fails, the packet is retransmitted until a successful attempt, or a maximum of R attempts have been carried out. After R unsuccessful trials, the packet is dropped. The behavior of the event-triggered communication with packet losses and retransmissions can be described by the Markov chain $\mathcal{M}^\ell$ shown in Figure 3. This discrete-time discrete-state multi-dimensional Markov chain has $(\mathrm{T}+2)\times(\mathrm{R}+1)$ modes. Each mode is represented by three digits: the first digit represents when the last transmission took place; the second one represents the number of retransmissions that have been attempted; and the third one describes the amount of time the system spends in each state.

C. Control Performance

Throughout this paper, for a given set of controllers, we quantify the associated closed-loop performance in terms of its linear quadratic control loss

$$J = \mathbb{E}\left\{ \int_0^{Nh} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q^c_{xx} & Q^c_{xu} \\ (Q^c_{xu})^\mathsf{T} & Q^c_{uu} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t + x^\mathsf{T}(Nh)Q^c_0 x(Nh) \right\} , \tag{6}$$

where Q^c_{xx} and Q^c_0 are symmetric and positive semi-definite matrices, and Q^c_{uu} is positive definite. Here, the expectation is taken over the process noise $\{w[kh],\ k \in \mathbb{Z}_{\geq 0}\}$ and the initial state x_0. When the continuous-time control loss function is discretized over intervals of length h, it has to be modified to account for the number of retransmissions required at each sampling time. The equivalent discrete-time loss function becomes

$$J = \mathbb{E}\left\{ \sum_{k=0}^{N-1} \begin{bmatrix} x[kh] \\ u[kh] \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q^d_{xx} & Q^d_{xu} \\ (Q^d_{xu})^\mathsf{T} & Q^d_{uu} \end{bmatrix} \begin{bmatrix} x[kh] \\ u[kh] \end{bmatrix} + x^\mathsf{T}[Nh]Q^c_0 x[Nh] \right\} ,$$

where

$$\begin{aligned} Q^d_{xx} &\triangleq Q_{xx}(n_k\tau) + \Phi^\mathsf{T}(n_k\tau)Q_{xx}(h-n_k\tau)\Phi(n_k\tau) \ , \\ Q^d_{xu} &\triangleq \Phi^\mathsf{T}(n_k\tau)Q_{xu}(h-n_k\tau) \ , \\ Q^d_{uu} &\triangleq Q_{uu}(h-n_k\tau) \ , \end{aligned}$$

with

$$\begin{aligned} Q_{xx}(t) &= \int_0^t \Phi^\mathsf{T}(s)Q^c_{xx}\Phi(s)\mathrm{d}s \ , \\ Q_{xu}(t) &= \int_0^t \Phi^\mathsf{T}(s)(Q^c_{xx}\Gamma(s) + Q^c_{xu})\mathrm{d}s \ , \\ Q_{uu}(t) &= \int_0^t (\Gamma^\mathsf{T}(s)Q^c_{xx}\Gamma(s) + 2\Gamma^\mathsf{T}(s)Q^c_{xu} + Q^c_{uu})\mathrm{d}s \ . \end{aligned}$$

III. Proposed Method and Main Results

In this section, we will develop an event-triggered control framework to analyze the expected communication rate when there is a reliable or an unreliable channel between the controller and actuator. For both cases, we will provide analytical expressions for communication attempt (resp. success) rates.

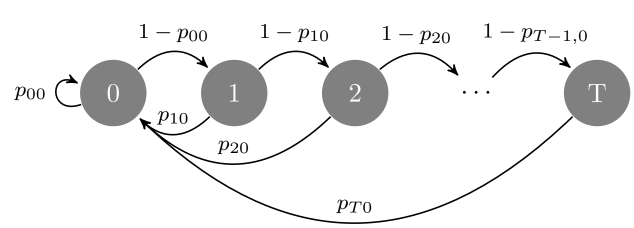

Fig. 4: The transition graph of Markov chain for no packet-loss.

A. Event-triggered channel without packet loss

We first focus on the event-triggered control problem defined in Section II when there is no packet loss. In this case, the Markov chain can be reduced to the chain $\mathcal{M}$ shown in Figure 4. Obviously, we have no modes related to losses or retransmissions; the modes simply describe the number of sample times between two consecutive threshold crossings (bounded by the time-out interval T). Specifically, this Markov chain has $\mathrm{T}+1$ modes and state $\{r[kh]\}_{k\geq 0}$ where $r[kh]=j$ implies that at time kh, the last transmission occurred at time $(k-j)h$. The transition probabilities are

$$p_{ij} = \Pr(r[kh+h]=j \mid r[kh]=i) \; .$$

Since we use the control signal $u^a[kh] = L(0)x[kh]$ and $u^a[kh+jh] = 0,\ \forall j \in \mathbb{Z}_{[1,\mathrm{T}-1]}$, the state trajectory of system (2) is calculated as

$$x[h(k+i)+h] = \Phi(h)^i \underbrace{\big(\Phi(h)+\Gamma(h)L(0)\big)}_{=0} x[kh] + \sum_{j=0}^{i} \Phi(h)^j w[(k+i-j)h] \; . \quad (7)$$

To describe how the transition probabilities depend on the process and the event threshold, we introduce the random variables

$$\delta_i[kh] = \sum_{j=0}^{i} \Phi(h)^j w[(k+i-j)h], \quad \forall i \in \mathbb{Z}_{[0,\mathrm{T}-1]} \; . \quad (8)$$

The probability density function of $\delta_i[kh]$ is time-invariant since the noise $w[kh]$ is independent and identically distributed random variable. Hence, we can drop the time index to simplify notation, and observe that the vector-valued random variable $\Delta_i \triangleq \begin{bmatrix}\delta_0 & \delta_1 & \cdots & \delta_i\end{bmatrix}^{\mathsf{T}}$ has a multi-variate normal distribution with mean $\mathbf{0}$ and covariance matrix (9) (see, next page).

For any $i \in \mathbb{Z}_{[1,\mathrm{T}]}$, we now define the events

$$N_i = \bigcap_{j=0}^{i-1} (|\delta_j| < \epsilon) \quad (10)$$

with the convention that N_0 will eventually occur (*i.e.*, sure event). Then,

$$\Pr(N_i) = \mathcal{Q}(ni, 0, \Xi_i, \epsilon\mathbf{1}), \quad (11)$$

with $\Pr(N_0)=1$, and the transition probabilities are given by the following Lemma.

Lemma 3.1: The transition probabilities p_{ij} of the Markov chain $\mathcal{M}$ defined in Figure 4 are given by

$$p_{ij} = \begin{cases} 1 - \frac{\mathcal{Q}(n(i+1),0,\Xi_{i+1},\epsilon\mathbf{1})}{\mathcal{Q}(ni,0,\Xi_i,\epsilon\mathbf{1})} & \text{if } i \in \mathbb{Z}_{[0,\mathrm{T}-1]}, j=0 \\ 1 & \text{if } i=\mathrm{T}, j=0 \\ 1-p_{i0} & \text{if } i \in \mathbb{Z}_{[0,\mathrm{T}-1]}, j=i+1 \\ 0 & \text{otherwise} \end{cases}$$

which can be evaluated using (1) initiated from $p_{\mathrm{T}0}=1$.

Proof. We focus on the case when $i \in \mathbb{Z}_{[0,T-1]}$, $j=0$ because the other expressions are obvious from the structure of the Markov chain in Figure 4. Let us consider the transition probability p_{00}. Since $r[kh]=0$ is equivalent to $u^a[kh]=u^c[kh]$, we have

$$\begin{aligned} p_{00} &= \Pr\big(r[kh+h]=0 \mid r[kh]=0\big) \\ &= \Pr\big(\mid w[kh] \mid > \epsilon \mid u^a[kh]=u^c[kh]\big) \\ &\overset{(a)}{=} \Pr\big(\mid w[kh] \mid > \epsilon\big) = \Pr\big(\mid \delta_0 \mid > \epsilon\big) \; , \end{aligned}$$

where (a) holds because $u^a[kh]$ is independent of the process noise at time step kh. Similarly, for any $i \in \mathbb{Z}_{[1,T-1]}$

$$\begin{aligned} p_{i0} &= \Pr\big(r[kh+h]=0 \bigm| r[kh]=i\big) \\ &= \Pr\big(r[kh+h]=0 \bigm| r[kh]=i, r[kh-h]=i-1, \\ &\qquad \cdots, r[(k-i)h]=0\big) \\ &= \Pr\big(\mid \delta_i \mid > \epsilon \bigm| \mid \delta_{i-1} \mid < \epsilon, \cdots, \mid \delta_0 \mid < \epsilon, \\ &\qquad u^a[(k-i)h]=u^c[kh]\big) \\ &= \Pr\big(\mid \delta_i \mid > \epsilon \bigm| \mid \delta_{i-1} \mid < \epsilon, \cdots, \mid \delta_0 \mid < \epsilon\big) \\ &= \frac{\Pr\big(\mid \delta_i \mid > \epsilon, N_i\big)}{\Pr\big(N_i\big)} = 1 - \frac{\Pr\big(N_{i+1}\big)}{\Pr\big(N_i\big)} \; . \end{aligned}$$

Finally, combining this expression with (11) yields the desired result. □

Theorem 3.2: In the absence of packet loss, the expected communication rate between the controller and the actuator for the event-triggered algorithm described above is

$$\pi_0^\infty = \frac{1}{1+\sum_{n=1}^{\mathrm{T}} \prod_{m=0}^{n-1} (1-p_{m0})} \; . \quad (12)$$

Proof. The Markov chain $\mathcal{M}$ is irreducible, finite, and aperiodic, so it has a stationary distribution π_i^∞ for all i. We use $\pi_j^\infty = \sum_{j=0}^{T} \pi_j^\infty p_{ji}$ to write

$$\pi_0^\infty = \sum_{j=0}^{\mathrm{T}} \pi_j^\infty p_{j,0}, \quad (13)$$

$$\pi_i^\infty = \big(1-p_{i-1,0}\big)\pi_{i-1}^\infty, \quad i \in \mathbb{Z}_{[1,T]}. \quad (14)$$

Using the balance equation $\sum_{j=0}^{T} \pi_j^\infty = 1$ and the aforementioned equalities, we immediately find

$$\pi_0^\infty = \frac{1}{1+\sum_{n=1}^{\mathrm{T}} \prod_{m=0}^{n-1} (1-p_{m,0})} \; . \quad (15)$$

This concludes our proof. □

B. Event-triggered channel with packet loss

With the basic intuition from analyzing the simpler Markov chain that models the loss-free scenario, we are ready to tackle the more complex case of packet losses and retransmissions. We first investigate whether or not we are able to obtain the similar random variable that we compute in (7). To this end, by applying the control actions $u^a\big[kh+n_k\tau\big] = L(n_k)x\big[kh\big]$ and $u^a[(k+j)h]=0, \forall j \in \mathbb{Z}_{[1,\mathrm{T}-1]}$, the state trajectory of (2) satisfies (16) (see next page).

Note that the variance of (b) equals that of $w[kh]$, *i.e.*, $\mathrm{Cov}\big\{\Phi(h-n_k\tau)\sum_{j=0}^{n_k-1} \Phi^j(\tau)w\big[kh+(n_k-1-j)\tau\big] + w\big[kh+n_k\tau\big]\big\} = \Pi_w^h$. Hence, this expression coincides with (7) and we can use Lemma 3.1 to also calculate the

$$
\Xi_i = \begin{bmatrix} \Pi_w^h & \Pi_w^h \Phi(h) & \dots & \Pi_w^h \Phi(h)^{i-1} & \Pi_w^h \Phi(h)^{i} \\ \star & \sum_{j=0}^{1} \Pi_w^h \Phi(h)^{2j} & \dots & \sum_{j=0}^{1} \Pi_w^h \Phi(h)^{2j+i-2} & \sum_{j=0}^{1} \Pi_w^h \Phi(h)^{2j+i-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \dots & \sum_{j=0}^{i-1} \Pi_w^h \Phi(h)^{2j} & \sum_{j=0}^{i-1} \Pi_w^h \Phi(h)^{2j+1} \\ \star & \star & \dots & \star & \sum_{j=0}^{i} \Pi_w^h \Phi(h)^{2j} \end{bmatrix} \tag{9}
$$

$$
\begin{aligned}
x\big[(k+i)h+h\big] &= \Phi(h)^i \underbrace{\Phi(h-n_k\tau)\Phi(\tau)^{n_k}}_{=e^{h-n_k\tau}e^{n_k\tau}=\Phi(h)} x\big[kh\big] + \Phi(h)^i \Gamma(h-n_k\tau)u^a\big[kh+n_k\tau\big] + \sum_{j=0}^{i-1}\Phi(h)^j\Gamma(h)u^a\big[(k+i-j)h\big] \\
&\quad + \sum_{j=0}^{i-1}\Phi(h)^j w\big[(k+i-j)h\big] + \Phi(h)^i\Big(\Phi(h-n_k\tau)\sum_{j=0}^{n_k-1}\Phi(\tau)^j w\big[kh+(n_k-1-j)\tau\big] + w\big[kh+n_k\tau\big]\Big) \\
&= \Phi(h)^i \underbrace{\big(\Phi(h)+\Gamma(h-n_k\tau)L(n_k)\big)}_{=0} x\big[kh\big] + \sum_{j=0}^{i-1}\Phi(h)^j w\big[(k+i-j)h\big] \\
&\quad + \underbrace{\Phi(h)^i\Big(\Phi(h-n_k\tau)\sum_{j=0}^{n_k-1}\Phi(\tau)^j w\big[kh+(n_k-1-j)\tau\big] + w\big[kh+n_k\tau\big]\Big)}_{(b)}
\end{aligned} \tag{16}
$$

transition probabilities for the Markov chain $\mathcal{M}^\ell$ in Figure 3. Since the only additional parameter is the loss probability p_ℓ, we now have all information necessary to derive the expected rate of attempted and successful transmissions from the Markov chain. We have the following result.

Theorem 3.3: The expected rate of successful reception of control packets at the actuator node $\bar{\pi}^\infty_{\mathcal{SR}}$ under the event-triggered algorithm described in Section II-B is

$$
\bar{\pi}^\infty_{\mathcal{SR}} = \frac{1-\frac{\tau}{h}\Big(\mathrm{R}+\frac{1}{1-p_\ell}-\frac{\mathrm{R}+1}{1-p_\ell^{\mathrm{R}+1}}\Big)}{1+\sum_{n=1}^{\mathrm{T}}\prod_{m=0}^{n-1}(1-p_{m0})+\frac{p_\ell^{\mathrm{R}+1}}{1-p_\ell^{\mathrm{R}+1}}}\ . \tag{17}
$$

Proof. The Markov chain $\mathcal{M}^\ell$ is irreducible, finite, and aperiodic, therefore it has a stationary distribution $\pi^\infty_{i,j,t}$ for all $i,j\in\mathbb{Z}_{\geq 0},\ t\in\mathcal{T}=\{h,h-\tau,\cdots,h-\mathrm{R}\tau,\tau\}$. We use $\pi^\infty_{i',j',t'}=\sum_{i=0}^{\mathrm{T}}\sum_{j=0}^{\mathrm{R}}\sum_{t\in\mathcal{T}}\Pr\big(i,j,t\mid i',j',t'\big)\pi^\infty_{i,j,t}$ to write

$$
\begin{aligned}
\pi^\infty_{0,0,h} &= (1-p_\ell)p_{00}\textstyle\sum_{k=0}^{\mathrm{R}}\pi^\infty_{0,0,h-k\tau} \\
&\quad+\textstyle\sum_{l=1}^{\mathrm{T}}(1-p_\ell)p_{l0}\pi^\infty_{l,0,h} \\
&\quad+(1-p_\ell)\pi^\infty_{\mathrm{loss}}, \\
\pi^\infty_{i,0,h} &= \textstyle\prod_{k=0}^{i-1}(1-p_{k0})\sum_{l=0}^{\mathrm{R}}\pi^\infty_{0,0,h-l\tau}, \quad \forall i\in\mathbb{Z}_{\geq 1}, \\
\pi^\infty_{0,0,h-j\tau} &= (1-p_\ell)\textstyle\sum_{k=0}^{\mathrm{T}}\pi^\infty_{k,j,\tau}, \quad \forall j\in\mathbb{Z}_{\geq 1}, \\
\pi^\infty_{0,j,\tau} &= p_\ell^j\, p_{00}\textstyle\sum_{k=0}^{\mathrm{R}}\pi^\infty_{0,0,h-k\tau}+p_\ell^j\pi^\infty_{\mathrm{loss}}, \quad \forall j\in\mathbb{Z}_{\geq 1}, \\
\pi^\infty_{i,j,\tau} &= p_\ell^j\, p_{i0}\,\pi^\infty_{i,0,h}, \quad \forall i,j\in\mathbb{Z}_{\geq 1}.
\end{aligned}
$$

We also have a stationary distribution for packet-loss:

$$
\pi^\infty_{\mathrm{loss}} = p_\ell\sum_{k=0}^{\mathrm{T}}\pi^\infty_{k,\mathrm{R},\tau}\ .
$$

The balance equation $\sum_{i=0}^{\mathrm{T}}\sum_{j=0}^{\mathrm{R}}\sum_{t\in\mathcal{T}}\pi^\infty_{i,j,t}+\pi^\infty_{\mathrm{loss}}=1$ and the aforementioned equalities imply that

$$
\pi^\infty_{\mathcal{SR}} = \frac{1}{1+\sum_{n=1}^{\mathrm{T}}\prod_{m=0}^{n-1}(1-p_{m0})+\frac{p_\ell}{1-p_\ell}}\ , \tag{18}
$$

where $\pi^\infty_{\mathcal{SR}}\triangleq\sum_{k=0}^{\mathrm{R}}\pi^\infty_{h-k\tau}$. We now need to normalize the invariant distribution for all Markov states corresponding to no transmission, retransmission, successful transmission without retransmission, successful transmission after n retransmission and unsuccessful transmission. The normalization factor is readily found to be

$$
\bar{\pi}^\infty_{\mathcal{SR}} = \frac{\mathrm{H}_1}{\mathrm{H}_1+\mathrm{H}_2+\mathrm{H}_3+\mathrm{H}_4}\ ,
$$

where

$$
\mathrm{H}_1 \triangleq \sum_{i=0}^{\mathrm{R}}(h-i\tau)\pi^\infty_{0,0,h-i\tau}, \quad \mathrm{H}_2 \triangleq\ h\sum_{j=1}^{\mathrm{T}}\pi^\infty_{j,0,h}\ ,
$$

$$
\mathrm{H}_3 \triangleq \tau\sum_{j=1}^{\mathrm{R}}\sum_{i=0}^{\mathrm{T}}\pi^\infty_{i,j,\tau}, \quad \mathrm{H}_4 \triangleq\ (h-\mathrm{R}\tau)\pi^\infty_{\mathrm{loss}}\ ,
$$

with $\bar{\pi}^\infty_{\mathcal{SR}}\triangleq\sum_{k=0}^{\mathrm{R}}\bar{\pi}^\infty_{h-k\tau}$. Hence, the normalized version of (18) is

$$
\bar{\pi}^\infty_{\mathcal{SR}} = \frac{1-\frac{\tau}{h}\Big(\mathrm{R}+\frac{1}{1-p_\ell}-\frac{\mathrm{R}+1}{1-p_\ell^{\mathrm{R}+1}}\Big)}{1+\sum_{n=1}^{\mathrm{T}}\prod_{m=0}^{n-1}(1-p_{m0})+\frac{p_\ell^{\mathrm{R}+1}}{1-p_\ell^{\mathrm{R}+1}}}\ , \tag{19}
$$

which concludes our proof. □

Proposition 3.4: The expected rate of attempted transmissions from the controller node to the actuator under the event-triggered algorithm described in Section II-B is

$$\bar{\pi}^{\infty}_{\mathcal{CA}} = \frac{1}{1 + \sum_{n=1}^{\mathrm{T}} \prod_{m=0}^{n-1} (1 - p_{m0})} \,. \tag{20}$$

Proof. The proof follows similar lines as the proof of Theorem 3.3. □

IV. Linear quadratic performance evaluation

The evolution of the system (2) with control action (5) can be modeled as a Markovian Jump Linear System (MJLS). Many techniques have been developed for analyzing MJLS by reframing the LQ-control problem as the solution to a set of coupled algebraic Riccati equations (see, *e.g.*, [21]). These techniques allow us to state the following result.

Theorem 4.1: Consider the problem formulation in Section II with the event–triggering algorithm and the deadbeat controller described in Section II-B. For a given event threshold ϵ, the linear quadratic control loss is computed as

$$\begin{aligned} J_{\infty} = &\sum_{i=0}^{\mathrm{T}} \sum_{j=0}^{\mathrm{R}} \sum_{t \in \mathcal{T}} \bar{\pi}^{\infty}_{i,j,t} \Big(\mathbb{E}\big\{ x_0^{\intercal} S^{\infty}_{i,j,t} x_0 \big\} + \mathrm{Tr}\big(S^{\infty}_{i,j,t} \Pi^{t}_{\omega} \big) \Big) \\ &+ \bar{\pi}^{\infty}_{\mathrm{loss}} \Big(\mathbb{E}\big\{ x_0^{\intercal} S^{\infty}_{\mathrm{loss}} x_0 \big\} + \mathrm{Tr}\big(S^{\infty}_{\mathrm{loss}} \Pi^{h-\mathrm{R}\tau}_{\omega} \big) \Big) \,, \end{aligned} \tag{21}$$

where $S^{\infty}_{i,j,t}$, $\forall i, j \in \mathbb{Z}_{\geq 0}$, $\forall t \in \mathcal{T} = \{h, h-\tau, \cdots, h - \mathrm{R}\tau, \tau\}$ are positive definite solutions of the coupled Riccati equations

$$\begin{aligned} S^{\infty}_{i,0,h} &= (1 - p_{i0}) \mathcal{E}_{i+1,0,h}\big(S^{\infty}_{i+1,0,h} \big) + p_{\ell} p_{i0} \mathcal{E}_{i,1,\tau}\big(S^{\infty}_{i,1,\tau} \big) \\ &\qquad + (1 - p_{\ell}) p_{i0} \mathcal{E}_{0,0,h}\big(S^{\infty}_{0,0,h} \big), i \in \mathbb{Z}_{\geq 0}, \\ S^{\infty}_{\mathrm{T},0,h} &= (1 - p_{\ell}) \mathcal{E}_{0,0,h}\big(S^{\infty}_{0,0,h} \big) + p_{\ell} \mathcal{E}_{\mathrm{T},1,\tau}\big(S^{\infty}_{\mathrm{T},1,\tau} \big), \\ S^{\infty}_{0,0,h-j\tau} &= (1 - p_{00}) \mathcal{E}_{1,0,h}\big(S^{\infty}_{1,0,h} \big) + p_{\ell} p_{00} \mathcal{E}_{0,j,\tau}\big(S^{\infty}_{0,j,\tau} \big) \\ &\qquad + (1 - p_{\ell}) p_{00} \mathcal{E}_{0,0,h}\big(S^{\infty}_{0,0,h} \big), j \in \mathbb{Z}_{\geq 1}, \\ S^{\infty}_{i,j,\tau} &= (1 - p_{\ell}) \mathcal{E}_{0,0,h-j\tau}\big(S^{\infty}_{0,0,h-j\tau} \big) \\ &\qquad + p_{\ell} \mathcal{E}_{i,j+1,\tau}\big(S^{\infty}_{i,j+1,\tau} \big), i \in \mathbb{Z}_{\geq 0},\ j \in \mathbb{Z}_{[1,\mathrm{R}-1]}, \\ S^{\infty}_{i,\mathrm{R},\tau} &= (1 - p_{\ell}) \mathcal{E}_{0,0,h-\mathrm{R}\tau}(S^{\infty}_{0,0,h-\mathrm{R}\tau}) \\ &\qquad + p_{\ell} \mathcal{E}_{\mathrm{loss}}(S^{\infty}_{\mathrm{loss}}),\ i \in \mathbb{Z}_{\geq 0}, \\ S^{\infty}_{\mathrm{loss}} &= (1 - p_{\ell}) \mathcal{E}_{0,0,h}(S^{\infty}_{0,0,h}) + p_{\ell} \mathcal{E}_{0,1,\tau}(S^{\infty}_{0,1,\tau}), \end{aligned}$$

with

$$\begin{aligned} \mathcal{E}_{0,0,h-i\tau}(X) &= \begin{bmatrix} 1 \\ L(i) \end{bmatrix}^{\intercal} \begin{bmatrix} Q^{h-i\tau}_{xx} & Q^{h-i\tau}_{xu} \\ (Q^{h-i\tau}_{xu})^{\intercal} & Q^{h-i\tau}_{uu} \end{bmatrix} \begin{bmatrix} 1 \\ L(i) \end{bmatrix}, i \in \mathbb{Z}_{\geq 0}, \\ \mathcal{E}_{i,0,h}(X) &= \Phi(h)^2 X + Q^{h}_{xx},\ i \in \mathbb{Z}_{\geq 1}, \\ \mathcal{E}_{i,j,\tau}(X) &= \Phi(\tau)^2 X + Q^{\tau}_{xx},\ i \in \mathbb{Z}_{\geq 0},\ j \in \mathbb{Z}_{\geq 1}, \\ \mathcal{E}_{\mathrm{loss}}(X) &= \Phi(h - \mathrm{R}\tau)^2 X + Q^{h-\mathrm{R}\tau}_{xx} \,. \end{aligned}$$

V. Numerical examples

We will now demonstrate how the analysis techniques developed in Sections III and IV allow us to study the trade-off between communication attempt (resp. success) rate and closed-loop control performance for the even-triggered control scheme detailed in Section II.

To this end, we consider the scalar system (2) with the parameters $a = 1.2$ and $b = 0.8$. The initial state $x(0) = x_0$ has a normal distribution with zero mean and covariance $\Pi_0 = 1$. The process noise w_t is independent of the initial condition x_0, and its incremental variance is $R_w = 1$. The system (2) is periodically sampled with a sampling interval of 500ms. At each time step, the event condition is checked, and if it is necessary, a control signal is generated and transmitted over a lossy network whose packet loss probability is p_{ℓ}. Additionally, we assume that we can retransmit a packet at most five times in case of unsuccessful transmission (*i.e.*, $\mathrm{R} = 5$). If the packet cannot be delivered after five retransmission attempts, the packet is dropped. Furthermore, the duration of each retransmission is $\tau = 20$ms.

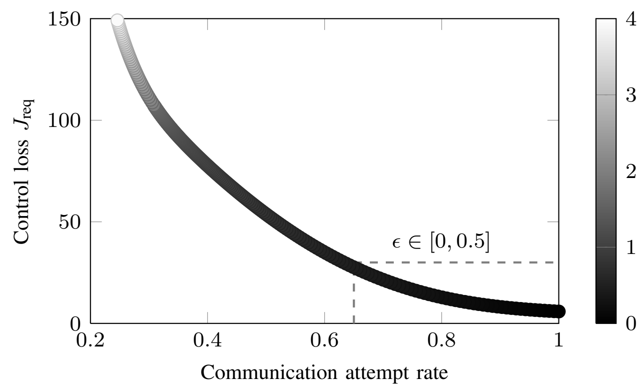

Fig. 5: Control performance for different communication attempt frequency and event thresholds (shown in gray scale).

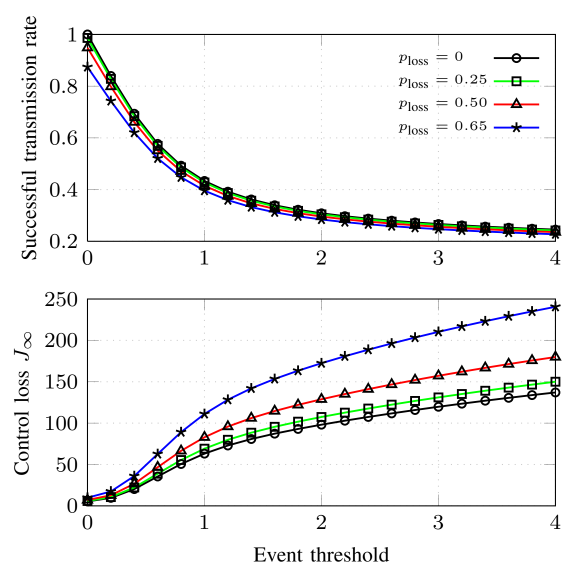

Fig. 6: A comparison of successful reception rate and control performance for several packet loss probabilities.

Figure 5 evaluates the communication attempt rate and the control loss for various event thresholds ϵ ranging from 0 to 4. The communication rate and control loss are calculated using Theorem 3.3 and Theorem 4.1, respectively. Note that the communication rate decreases dramatically with an increased control loss as the threshold ϵ varies between 0 and 1.75 (dark colors). For $\epsilon > 1.75$ (lighter color), both quantities become less sensitive to changes in the threshold value. We can also identify $\epsilon \in [0, 0.5]$ as a particularly attractive region where a large decrease in the communication rate can be obtained for a small loss in control performance.

In Figure 6, we examine the rate of successful reception and the control loss for several different values of the packet loss probability. When the packet loss probability is smaller than 0.25, the control performance and rate of successful reception are quite similar to the case of no packet loss. For $p_\ell = 0.5$ there is a clear performance loss compared to the case of reliable transmission, and the control performance deteriorates significantly when p_ℓ exceeds 0.65.

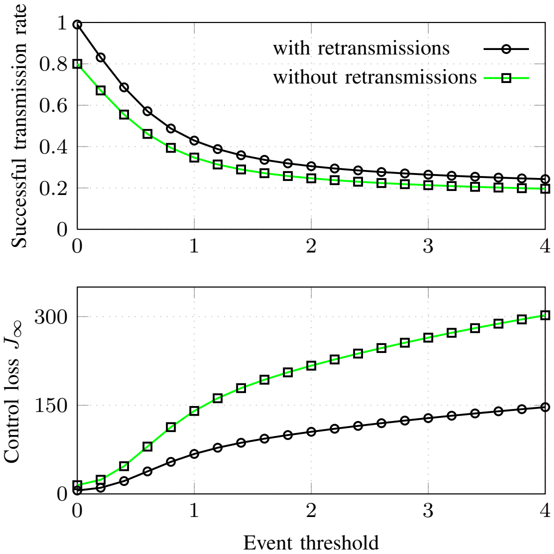

Fig. 7: A comparison of successful reception rate and control performance in the presence and the absence of retransmissions while packet loss probability is 20%.

Finally, Figure 7 compares the closed-loop performance with and without the retransmission mechanism. As shown in Figure 7, the successful reception rate increases in the presence of retransmissions. In the absence of retransmission mechanism, the control loss J_∞ dramatically increases when the threshold ϵ varies between 1 and 4.

VI. Conclusions

In this paper, we developed a theoretical framework to analyze the trade-off between the control performance and the communication cost of the proposed event-triggering algorithm for information transmission over an unreliable network. We assumed that a threshold-based event-triggering algorithm governs the channel used to transmit information from the controller to the actuator. Furthermore, we developed a multi-dimensional Markov chain model, which characterizes the attempted and successful transmissions of control signals over the lossy communication link. Our model also considered retransmission of unsuccessful trials, and it interpreted them as a delay associated with these retransmissions. By combining this communication model with an analytical model of the closed-loop performance, we provided a systematic way to analyze the trade-off between the communication cost and the control performance by appropriately selecting an event-threshold.

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