Remarque. Soit K un corps cubique de discriminant q^2, où q est premier, et de nombre de classes égal à p, avec p premier ($p \equiv 1 \bmod 3$). L'extension abélienne non ramifiée maximale N de K est cyclique de degré p sur K, et galoisienne non abélienne sur Q; soit alors L une extension intermédiaire de N/Q de degré p. On voit facilement que l'idéal premier $\mathfrak{q}$ au-dessus de q dans K se décompose dans N/K; la considération des groupes de décomposition de ses diviseurs premiers dans N montre que $\Delta_{L/Q} = q^{\frac{2(p-1)}{3}}$. Il en résulte que pour $q = 313$, $p = 7$, nous avons obtenu une extension de degré 7 dont le discriminant, égal à 313^4, s'intercale entre les discriminants 43^6 et 7^{12} des extensions cycliques de degré 7, A_1 non ramifiée en dehors de 43 et A_2 non ramifiée en dehors de 7. Pour $q = 1489$, l'extension de degré 19 et de discriminant 1489^{12} est de discriminant inférieur à celui de toute extension cyclique de degré 19 puisque le plus petit d'entre eux est 191^{18}.

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On the transcendence of certain power series of algebraic numbers

by

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1. Introduction. Let $\sigma(z) = \sum_{k=0}^{\infty} a_k z^{e_k}$ be a power series with complex coefficients a_k, convergence radius $R > 0$ and sufficiently rapidly increasing integers e_k. H. Cohn, [2], constructed certain transcendental numbers using such gap series. He proved that under certain conditions $\sigma(\theta)$ is transcendental for every algebraic argument θ with $0 < |\theta| < R$, if the coefficients a_k of σ be rationals. Or, equivalently, $\sigma(\theta)$ is algebraic with $0 < |\theta| < R$, implies that θ is transcendental. Baron and Braune, [1], used the same method with the assumption that the coefficients a_k of σ are algebraic integers of degree s_k. They proved that $\sigma(\theta)$ is transcendental for every algebraic θ with $0 < |\theta| < R$ under the following conditions:

(i) $D^{-k} < \overline{|a_k|} < D^k$ for some $D \geq 1$ and all k,

(ii) $\lim_{k\to\infty} e_k T_k^2 / e_{k+1} = 0$, where $T_k = \prod_{i=0}^{k} s_i$.

It was noted that these conditions imply $R = 1$.

In this paper we will improve and generalise the result of Baron and Braune using a more suitable auxiliary inequality. As a special case of a more general result we will prove the following property for gap series $\sigma(\theta)$ with algebraic integral coefficients a_k of degree s_k: $\sigma(\theta)$ is transcendental for algebraic θ with $0 < |\theta| < R$ under the conditions:

(i) $\overline{|a_k|} \leq D^{e_k}$ for some $D \geq 1$ and all k,

(ii) $\lim_{k\to\infty} e_k S_k / e_{k+1} = 0$, where S_k is the degree of the field obtained by adjoining $a_0, a_1, \ldots, a_k$ to the field of the rationals.

2. Formulation of results. We denote the conjugates of an algebraic number α of degree s by $\alpha^{(1)} = \alpha, \alpha^{(2)}, \ldots, \alpha^{(s)}$. Further, $\overline{|\alpha|} = \max_{i=1,\ldots,s} |\alpha^{(i)}|$. We mention the inequalities $\overline{|\alpha+\beta|} \leq \overline{|\alpha|} + \overline{|\beta|}$ and $\overline{|\alpha\beta|} \leq \overline{|\alpha|}\cdot\overline{|\beta|}$, for arbitrary algebraic numbers α and β. The *height* of an algebraic number α is defined as the maximum absolute value of the coefficients of its minimal defining polynomial. For general information, see Schneider, [6], and Pollard, [5].

In the subsequent text a_k $(k = 0, 1, 2, \ldots)$ are non-zero algebraic numbers of degree s_k and height h_k $(k = 0, 1, 2, \ldots)$. We put $A_k = \max_{i=0,\ldots,k} \overline{|a_i|}$; S_k will be the degree of $Q(a_0, a_1, \ldots, a_k)$ over Q, hence, $S_k \leqslant \prod_{i=0}^{k} s_i$. Further, M_k will be a positive integer such that $M_k a_i$ is an algebraic integer, $i = 0, \ldots, k$.

The sequence $(e_k)_{k=0}^{\infty}$ will be an increasing sequence of integers, with $e_0 \geqslant 0$. We assume that the radius of convergence R of the power series $\sum_{k=0}^{\infty} a_k z^{e_k}$ is positive.

THEOREM. *Suppose* $\lim_{k\to\infty} (e_k + \log M_k + \log A_k) S_k / e_{k+1} = 0$. *Then* $\sigma(\theta) = \sum_{k=0}^{\infty} a_k \theta^{e_k}$ *is transcendental for every algebraic* θ *with* $0 < |\theta| < R$.

COROLLARY 1. *If the* a_k *are rational fractions* p_k/q_k, *we have* $S_k = 1$, $A_k = \max_{i=0,\ldots,k} |a_i|$ *and we can choose* M_k *as the least common multiple of* $q_0, \ldots, q_k$. *With this choice the theorem gives the result of Cohn, mentioned in the introduction.*

COROLLARY 2. *If all numbers* a_k *belong to a fixed algebraic field of degree* S, *we have* $S_k \leqslant S$ $(k = 0, 1, \ldots)$. *Now the condition in the theorem can be weakened to* $\lim_{k\to\infty} (e_k + \log M_k + \log A_k)/e_{k+1} = 0$.

COROLLARY 3. *In case of algebraic integers* a_k $(k = 0, 1, \ldots)$ *we can use* $M_k = 1$ $(k = 0, 1, \ldots)$. *If, moreover,* $\overline{|a_k|} \leqslant D^{e_k}$ *for some* $D \geqslant 1$ *and all* k, *the estimate* $\log A_k \leqslant e_k \log D$ *holds and the theorem reduces to:*

$\sigma(\theta)$ *is transcendental for all algebraic* θ *with* $0 < |\theta| < R$ *under the condition* $\lim_{k\to\infty} e_k S_k / e_{k+1} = 0$.

3. We need the following lemmas:

LEMMA 1. *Let* a *be algebraic of degree* s *and height* h. *Suppose* m *is a positive integer such that* ma *is an algebraic integer. Then*

$$h \leqslant \left(2m \max(1, \overline{|a|})\right)^s.$$

Proof. Let $Q(z) = q_s z^s + \ldots + q_1 z + q_0$ be the minimal polynomial of a. From

$$Q(z) = q_s (z - a^{(1)}) \ldots (z - a^{(s)})$$

we deduce that q_i/q_s $(i = 0, \ldots, s-1)$ are apart from factors ± 1 the elementary symmetric polynomials in $a^{(1)}, \ldots, a^{(s)}$, and thus

$$|q_i| \leqslant |q_s| \binom{s}{i} \overline{|a|}^i \leqslant |q_s| 2^s \left(\max(1, \overline{|a|})\right)^s.$$

Since ma is an algebraic integer, there exists a polynomial

$$R(z) = (mz)^s + \ldots + r_1 (mz) + r_0$$

for which $R(a) = 0$.

Hence, R is a multiple of Q and $|q^s| \leqslant m^s$. This completes the proof of the lemma.

LEMMA 2. *Let* $P(z) = p_N z^N + \ldots + p_1 z + p_0$ *be a polynomial with integral coefficients, of degree* $N \geqslant 1$ *and height* H; *let* a *be an algebraic number of degree* s *and height* h. *Then* $P(a) = 0$ *or*

$$|P(a)| \geqslant \{H^{s-1} h^N (N+1)^{s-1} (s+1)^N\}^{-1}.$$

Proof. See [3], Theorem 5.

4. Proof of the theorem. Suppose θ is algebraic, $0 < |\theta| < R$. Let θ be of degree n and suppose m is a positive integer such that $m\theta$ is an algebraic integer. We put

$$\sigma_k(\theta) = \sum_{i=0}^{k} a_i \theta^{e_i}$$

and

$$r_k(\theta) = \sigma(\theta) - \sigma_k(\theta).$$

Now $\sigma_k(\theta)$ is an algebraic number of degree $s \leqslant n S_k$ and of height h, which can be estimated according to Lemma 1, since $m^{e_k} M_k \sigma_k(\theta)$ is an algebraic integer. We then obtain:

$$\begin{aligned} h &\leqslant \left\{2 m^{e_k} M_k \max(1, \overline{|\sigma_k(\theta)|})\right\}^s \\ &\leqslant \left\{2 m^{e_k} M_k (k+1) A_k \left(\max(1, \overline{|\theta|})\right)^{e_k}\right\}^{nS_k} \\ &\leqslant \{2 M_k A_k\}^{nS_k} \{2m \max(1, \overline{|\theta|})\}^{ne_k S_k}. \end{aligned}$$

Let P be a fixed polynomial with integer coefficients, of degree $N \geqslant 1$ and of height H. By the convergence of $\sigma_k(\theta)$, $k = 0, 1, \ldots$, the difference $|\sigma_k(\theta) - \sigma_{k+1}(\theta)|$ will be smaller than the minimal distance of the zeros of P for k sufficiently large. Hence $P(\sigma_k(\theta)) = 0$ implies $P(\sigma_{k+1}(\theta)) \neq 0$ if $k > K_1$. Consequently there exists an infinite subsequence k_j $(j = 0, 1, \ldots)$ with $P(\sigma_{k_j}(\theta)) \neq 0$ $(j = 0, 1, \ldots)$. Now from Lemma 2

$$\left|P(\sigma_{k_j}(\theta))\right| \geqslant \left\{(8 M_{k_j} A_{k_j})^{nS_{k_j}(N + \log H)} \left(2m \max(1, \overline{|\theta|})\right)^{Nne_{k_j} S_{k_j}}\right\}^{-1}$$

since $N+1 \leqslant 2^N$ and $nS_k + 1 \leqslant 2^{nS_k}$. For fixed σ, θ and P we thus obtain

$$\left|P(\sigma_{k_j}(\theta))\right| \geqslant e^{-c_1 S_{k_j} \log(8 M_{k_j} A_{k_j}) - c_2 e_{k_j} S_{k_j}},$$

where c_1 and c_2 are positive numbers independent of j. We now estimate $r_k(\theta)$ as follows. Choose ϱ with $|\theta| < \varrho < R$. Since $\limsup_{k\to\infty} |a_k|^{1/e_k} = R^{-1}$ one has for $k > K_2$ the inequality $|a_k| < \varrho^{-e_k}$, and hence,

$$|r_k(\theta)| \leqslant \sum_{i=e_{k+1}}^{\infty} (|\theta| \varrho^{-1})^i \leqslant (|\theta| \varrho^{-1})^{e_{k+1}} (1 - |\theta| \varrho^{-1})^{-1}.$$

It follows that

$$\left|P(\sigma(\theta))-P(\sigma_k(\theta))\right| \leqslant c_3\varrho(\varrho-|\theta|)^{-1}(|\theta|\varrho^{-1})^{e_{k+1}} \leqslant e^{-c_4 e_{k+1}}$$

if $k > K_3$, in which c_3 is an upper bound for $|P'(z)|$ on a bounded neighbourhood of $\sigma(\theta)$, and $c_4 = \frac{1}{2}\log(\varrho|\theta|^{-1}) > 0$. Combining this estimate with that of $\left|P(\sigma_{k_j}(\theta))\right|$ we obtain for $k_j > K_3$

$$\left|P(\sigma(\theta))\right| \geqslant e^{-c_1 S_{k_j}\log(8M_{k_j}A_{k_j})-c_2 e_{k_j}S_{k_j}} - e^{-c_4 e_{k_j+1}}.$$

From the condition in the theorem we know $\lim_{j\to\infty} e_{k_j}S_{k_j}/e_{k_j+1} = 0$, $\lim_{j\to\infty} S_{k_j}\log M_{k_j}/e_{k_j+1} = 0$ and $\lim_{j\to\infty} S_{k_j}\log A_{k_j}/e_{k_j+1} = 0$. Hence, we can choose j such that $k_j > K_3$ and that

$$c_1 S_{k_j}\log(8M_{k_j}A_{k_j}) + c_2 e_{k_j}S_{k_j} < c_4 e_{k_j+1}.$$

It has now been shown that $P(\sigma(\theta)) \neq 0$. Since P is chosen arbitrarily, the theorem is proved.

5. Remarks. (i) Concerning the $\overline{|a_k|}$ the conditions in the theorem and its corollaries can often be checked by the use of the well-known inequality $\overline{|a_k|} \leqslant h_k+1$ (see Schneider, [6], p. 5). Further, suppose m_k is the first coefficient of the minimal polynomial of a_k. Then $m_k a_k$ is an algebraic integer; hence, M_k can be chosen as the least common multiple of $m_0, \ldots, m_k$. From the inequality $m_k \leqslant h_k$, M_k will now be at most $h_0 \ldots h_k$.

(ii) Within the condition of the theorem $R = 0$ and $R = \infty$ are possible; e.g. for $e_k = 2^{2^{2k}}$, $a_k = 2^{2^{2^{2k+1}}}$ and $2^{-2^{2^{2k+1}}}$ respectively.

(iii) In the case of Corollary 3 we have $R \geqslant D^{-1}$, for the condition $\overline{|a_k|} \leqslant D^{e_k}$ implies $\limsup_{k\to\infty} |a_k|^{1/e_k} \leqslant D$.

The possibility exists that $R = D^{-1}$ (take $e_k = k!$, $a_k = 2^{k!}$). An upper bound for R cannot be given, since arbitrary high values of R can be constructed.

(iv) When R is finite, $\sigma(\theta)$ need not be transcendental for θ algebraic with $|\theta| = R$. For instance, take $e_k = k!$, $a_k = 2^{k!-k}$; then $R = \frac{1}{2}$ and, using $\theta = \frac{1}{2}$, we obtain $\sigma(\theta) = \sum_{k=0}^{\infty} 2^{-k} = 2$.

(v) The main condition in the theorem can be relaxed to an analogon of a condition introduced by Mahler in [4].

(vi) It is easy to derive a transcendence measure for the number $\sigma(\theta)$ if σ and θ are explicitly given and satisfy the conditions of the theorem. The principal estimates in this paper can be used for establishing transcendence measures.

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