

ON THE TRANSFORMATION OF SOME CLASSES OF MARTINGALES BY A CHANGE OF LAW

NORIIHIKO KAZAMAKI AND TAKESHI SEKIGUCHI

(Received January 17, 1978, revised October 2, 1978)

1. Introduction. Let M be a continuous local martingale with $M_0 = 0$, and let us denote by $\langle M \rangle$ the continuous increasing process such that $M^2 - \langle M \rangle$ is also a local martingale. Then the solution Z of the stochastic integral equation:

$$Z_t = 1 + \int_0^t Z_s dM_s$$

is given by the formula $Z_t = \exp(M_t - \langle M \rangle_t/2)$, so that it is a positive local martingale with $Z_0 = 1$. However, it is not always a martingale. The problem of finding sufficient conditions for the process Z to be a martingale, which is proposed by I. V. Girsanov, is important in certain questions concerning the absolute continuity of probability measures of diffusion processes. In Section 3, we shall give a new sufficient condition for the problem of Girsanov. Namely, it will be proved that if M is a BMO-martingale, then Z is an L^p -bounded martingale for some $p > 1$. The theory of H^p and BMO martingales was developed in [3] and [4], and it is well-known nowadays that $(H^1)^* \cong \text{BMO}$, that is, the dual space of H^1 is isomorphic to BMO. In Section 4, Z is assumed to be a uniformly integrable martingale. Then we can define a change of the underlying probability measure dP by the formula $d\hat{P} = Z_\infty dP$. If \mathcal{H} is a class of continuous local martingales, with respect to $d\hat{P}$ we denote by $\hat{\mathcal{H}}$ the class corresponding to \mathcal{H} . Our interest here lies in investigating the relations between \mathcal{H} and $\hat{\mathcal{H}}$. In the section we shall prove that if M is a BMO-martingale, then $\text{BMO} \cong \hat{\text{BMO}}$ and $H^1 \cong \hat{H}^1$. In addition, it is shown that $H^2 \cong \hat{H}^2$ holds in general. In Section 5 we shall give a generalization of the classical inequalities of J. L. Doob.

Let us denote by C a positive constant and by C_x a positive constant depending only on the indicated parameter x . Both letters are not necessarily the same in each occurrence.

2. Preliminaries.

1) **Definitions and notations.** Let (Ω, F, P) be a complete probability

space, and let $(F_t)_{(0 \leq t < \infty)}$ be a non-decreasing right continuous family of sub- σ -fields of F with $F = \bigvee_{t \geq 0} F_t$ such that F_0 contains all null sets. Throughout the paper we shall deal only with continuous local martingales. The reader is assumed to be familiar with the martingale theory as given in [3] and [10]. See Gettoor and Sharpe [4] for the theory of conformal martingales.

For any process $X = (X_t, F_t)$, we denote by X^* the quantity $\sup_t |X_t|$. If T is a stopping time, X^T is the process $(X_{t \wedge T})$ stopped at T . Let \mathcal{L} be the class of all continuous local martingales X over (F_t) with $X_0 = 0$. For X and Y in \mathcal{L} , we define $\langle X, Y \rangle = (\langle X + Y \rangle - \langle X \rangle - \langle Y \rangle)/2$. Then, as is well-known, $XY - \langle X, Y \rangle$ belongs to \mathcal{L} . For $X \in \mathcal{L}$ and a locally bounded previsible process H , $H \circ X$ is the unique element of \mathcal{L} such that for all $Y \in \mathcal{L}$, $\langle H \circ X, Y \rangle_t = \int_0^t H_s d\langle X, Y \rangle_s$. The process $H \circ X$ is called the stochastic integral of H relative to X . We also write $(H \circ X)_t = \int_0^t H_s dX_s$.

DEFINITION 1. For any $X \in \mathcal{L}$ and $0 < p < \infty$, let

$$\|X\|_{H^p} = (E[\langle X \rangle_\infty^{p/2}])^{1/p}.$$

We denote by H^p the class of all $X \in \mathcal{L}$ such that $\|X\|_{H^p} < \infty$. If $1 \leq p < \infty$, H^p is a real Banach space with norm $\| \cdot \|_{H^p}$.

Recall now the inequality of B. Davis:

$$(1/4\sqrt{2})E[X^*] \leq E[\langle X \rangle_\infty^{1/2}] \leq 2E[X^*], \quad X \in \mathcal{L}.$$

For the proof, see [4]. This implies that if $X \in H^1$, X is uniformly integrable. This inequality of Davis is of fundamental importance in the martingale theory.

DEFINITION 2. For any $X \in \mathcal{L}$, let

$$\|X\|_{\text{BMO}} = \sup_t \|(E[\langle X \rangle_\infty - \langle X \rangle_t | F_t])^{1/2}\|_\infty.$$

Let BMO consist of those $X \in \mathcal{L}$ which satisfy $\|X\|_{\text{BMO}} < \infty$. The energy inequalities (see [10]) give

$$E[\langle X \rangle_\infty^n] \leq n! \|X\|_{\text{BMO}}^{2n}, \quad n = 1, 2, \dots$$

Therefore, $\text{BMO} \subset H^p$ for every p . The space BMO, which can be identified with the dual space of H^1 , is complete with norm $\| \cdot \|_{\text{BMO}}$. The following is an example of BMO-martingales.

EXAMPLE 1. Let $B = (B_t, F_t, P_\alpha)_{\alpha \in R}$ be a one dimensional Brownian motion and let $T_\alpha = \inf\{t; |B_t| = \alpha\}$, ($\alpha > 0$). It is easy to see that T_α is a stopping time. Then the BMO-norm of the martingale B^{T_α} with

respect to the measure P_0 is equal to a . In fact, if $|x| < a$, $E_x[T_a] = a^2 - x^2$ because $|B_{T_a}| = a$ and $E_x[B_{T_a}^2 - T_a] = x^2$. Now let θ_t be the shift operators of the process $B = (B_t)$. Then $T_a - t = T_a \circ \theta_t$ on $(t < T_a)$ by the definition of T_a . It is also clear that $\langle B^{T_a} \rangle_t = t \wedge T_a$, P_0 -a.s., so that using the Markovian character, we have

$$\begin{aligned} E_0[T_a - t \wedge T_a | F_t] &= E_0[T_a \circ \theta_t | F_t] I_{(t < T_a)} \\ &= E_{B_t}[T_a] I_{(t < T_a)} = (a^2 - B_t^2) I_{(t < T_a)}. \end{aligned}$$

Therefore we have $\|B^{T_a}\|_{\text{BMO}} = a$.

Now for $M \in \mathcal{L}$, let us consider the process Z defined by the formula

$$Z_t = e^{Mt - \langle M \rangle_t / 2}, \quad t \geq 0.$$

It is a positive supermartingale such that $Z - 1 \in \mathcal{L}$. As $Z_0 = 1$, $E[Z_t] \leq 1$ for every t . Thus Z is a martingale if and only if $E[Z_t] = 1$ for every t . Let $Z_\infty = \lim Z_t$. The existence of this limit is guaranteed by the martingale convergence theorem due to Doob. Fatou's lemma shows that it is finite with probability 1. Similarly, for each real number a , the process $Z^{(a)}$ defined by $Z_t^{(a)} = \exp(aM_t - a^2 \langle M \rangle_t / 2)$ is also a positive local martingale. As $Z_t Z_t^{(-1)} = \exp(-\langle M \rangle_t)$, $Z_\infty = 0$ implies $\langle M \rangle_\infty = \infty$. Conversely, if $\langle M \rangle_\infty = \infty$, then $Z_\infty = 0$, for $Z_t = (Z_t^{(1/2)})^2 \exp(-\langle M \rangle_t / 4)$. We now remark that $Z^{(-1)}$ is not necessarily a martingale even if Z is bounded. Here is an example.

EXAMPLE 2. Let $B = (B_t, F_t)$ be a one dimensional Brownian motion starting at 0, defined on a probability space (Ω, F, P) . We set $T = \inf(t; B_t \geq 1)$, which is a stopping time such that $0 < T < \infty$. Now let $g: [0, 1[\rightarrow [0, \infty[$ be an increasing homeomorphism, and set

$$\tau_t = \begin{cases} g(t) \wedge T & \text{if } 0 \leq t < 1 \\ T & \text{if } 1 \leq t < \infty. \end{cases}$$

Then these τ_t are stopping times with $\tau_0 = 0$ and $\tau_1 = T$ such that for a.e. $\omega \in \Omega$ the sample functions $\tau_t(\omega)$ are non-decreasing and continuous. Thus, the process M defined by $M_t = B_{\tau_t}$ is a continuous local martingale over (F_{τ_t}) . As $\tau_t \leq T$, we have $M_t \leq 1$, so that Z_t is bounded by e . On the other hand, as $M_1 = B_T = 1$, we have $E[Z_1^{(-1)}] \leq E[\exp(-M_1)] < 1$. This implies that $Z^{(-1)}$ is not a martingale.

In what follows, given $M \in \mathcal{L}$, Z denotes the process $(\exp(M_t - \langle M \rangle_t / 2))$, unless otherwise stated.

DEFINITION 3. Let $1 < p < \infty$. We say that Z satisfies the (A_p) condition if

$$\sup_t \|E[(Z_t/Z_\infty)^{1/(p-1)} | F_t]\|_\infty < \infty .$$

If Z satisfies (A_p) , then $Z_\infty > 0$ a.s., so that $\langle M \rangle_\infty < \infty$ a.s.. If $1 < p < r$, (A_p) implies (A_r) by Hölder's inequality. For simplicity, let us say that (A_∞) holds, if Z satisfies (A_p) for some $p > 1$. By Lemma 5, if Z satisfies (A_∞) , then the process $Z^{(a)}$, defined as before, also satisfies the condition. The (A_p) condition has already appeared many times in the literature in connection with several different questions (for example, see [12]).

2) **Preliminary lemmas.** Here we collect several lemmas which are of use in subsequent sections. The following inequality is called Fefferman's inequality.

LEMMA 1. *If $X \in H^1$ and $Y \in \text{BMO}$, then*

$$E\left[\int_0^\infty |d\langle X, Y \rangle_s|\right] \leq \sqrt{2} \|X\|_{H^1} \|Y\|_{\text{BMO}} .$$

PROOF. It is proved in [4], but for the reader's convenience we shall recall briefly the proof.

By using the usual stopping argument, we may assume X in H^2 . Then we have

$$E\left[\int_0^\infty |d\langle X, Y \rangle_s|\right]^2 \leq E\left[\int_0^\infty \langle X \rangle_s^{-1/2} d\langle X \rangle_s\right] E\left[\int_0^\infty \langle X \rangle_s^{1/2} d\langle Y \rangle_s\right] .$$

The first term on the right hand side is smaller than $2\|X\|_{H^1}$. On the other hand, by integration by parts, the second term is

$$\begin{aligned} E\left[\langle X \rangle_\infty^{1/2} \langle Y \rangle_\infty - \int_0^\infty \langle Y \rangle_s d\langle X \rangle_s^{1/2}\right] &= E\left[\int_0^\infty (\langle Y \rangle_\infty - \langle Y \rangle_s) d\langle X \rangle_s^{1/2}\right] \\ &= E\left[\int_0^\infty E[\langle Y \rangle_\infty - \langle Y \rangle_s | F_s] d\langle X \rangle_s^{1/2}\right] , \end{aligned}$$

which is dominated by $\|Y\|_{\text{BMO}}^2 \|X\|_{H^1}$. Thus the lemma is proved.

Fefferman's inequality implies that $\text{BMO} \subset (H^1)^*$. The following result is also proved in [4].

LEMMA 2. *Let $X \in \mathcal{L}$. Then we have*

$$\|X\|_{H^1} \leq \sup \{E[\langle X, Y \rangle_\infty]; Y \in \text{BMO}, \|Y\|_{\text{BMO}} \leq 1\} .$$

PROOF. Let (T_n) be a non-decreasing sequence of stopping times with $\lim_n T_n = \infty$ a.s., such that $X^{T_n} \in H^1$ for each n . In addition, it is easy to see that $\langle X^{T_n}, Y \rangle = \langle X, Y^{T_n} \rangle$, $\|Y^{T_n}\|_{\text{BMO}} \leq \|Y\|_{\text{BMO}}$ and $\lim_n \|X^{T_n}\|_{H^1} = \|X\|_{H^1}$. Therefore we may assume that $X \in H^1$. Let now

ε be an arbitrary positive real number, and define $Y_t = \int_0^t D_{s-} dX_s$, where $D_t = E[(\varepsilon + \langle X \rangle_\infty)^{-1/2} | F_t]$. Then, by an elementary calculation, we get $\|Y\|_{\text{BMO}} \leq 1$. Furthermore, $\langle X \rangle$ being continuous, we have

$$\begin{aligned} E[\langle X, Y \rangle_\infty] &= E\left[\int_0^\infty D_{s-} d\langle X \rangle_s\right] = E\left[\int_0^\infty D_s d\langle X \rangle_s\right] \\ &= E[(\varepsilon + \langle X \rangle_\infty)^{-1/2} \langle X \rangle_\infty], \end{aligned}$$

which increases to $\|X\|_{H^1}$ as $\varepsilon \rightarrow 0$. This completes the proof.

P. A. Meyer proved in [11] the following inequality.

LEMMA 3. *Let $X \in \mathcal{L}$. Then*

$$\|X\|_{\text{BMO}} \leq \sup\{E[\langle X, X \rangle_\infty]; Y \in H^1, \|Y\|_{H^1} \leq 1\}.$$

PROOF. We prove it, following the idea of Meyer. Let us denote by d its right hand side, and T be any stopping time. It is sufficient to show that

$$E[\langle X \rangle_\infty - \langle X \rangle_T; A] \leq d^2 P(A) \quad \text{for } A \in F_T.$$

For simplicity, set $U = \langle X \rangle_\infty - \langle X \rangle_T$. The stopping argument enables us to assume that $X \in \text{BMO}$, and so $E[UI_A] < \infty$. The process H given by $H_t = I_{A \cap (T < t)}$ is a previsible process such that $H^2 = H$. Then we have $\langle H \circ X, X \rangle_\infty = \langle H \circ X \rangle_\infty = UI_A$, so that

$$E[UI_A] \leq d \|H \circ X\|_{H^1} = dE[I_A \sqrt{UI_A}].$$

By Schwarz' inequality the right hand side is smaller than

$$dP(A)^{1/2} E[UI_A]^{1/2}.$$

Consequently we get $E[UI_A] \leq d^2 P(A)$.

The next inequality, which was established by A. M. Garsia for discrete martingales in [3], plays an important role in our investigation.

LEMMA 4. *If $\|X\|_{\text{BMO}} < 1$, then*

$$E[e^{\langle X \rangle_\infty - \langle X \rangle_t} | F_t] \leq (1 - \|X\|_{\text{BMO}}^2)^{-1}.$$

PROOF. For simplicity, let us denote by d the right hand side of this inequality. It suffices to show that for every $A \in F_t$

$$E[e^{\langle X \rangle_\infty - \langle X \rangle_t}; A] \leq dP(A).$$

We may assume that $P(A) > 0$. To show this, let us set $dP' = (I_A/P(A))dP$ and $F'_s = F_{t+s}$. Then it is not difficult to see that for $X \in \text{BMO}$ the process X' defined by $X'_s = X_{t+s} - X_t$ is also a BMO-martingale over (F'_t) with

respect to dP' and that $\langle X' \rangle_s = \langle X \rangle_{t+s} - \langle X \rangle_t$. Therefore we have

$$E[e^{\langle X \rangle_\infty - \langle X \rangle_t}; A] = E'[e^{\langle X' \rangle_\infty}]P(A),$$

where $E'[\]$ denotes the expectation over Ω with respect to dP' . An elementary calculation shows that the BMO-norm of X' is smaller than $\|X\|_{\text{BMO}}$. Then, by the energy inequalities, we have

$$E'[e^{\langle X' \rangle_\infty}] = \sum_{n=0}^{\infty} \frac{1}{n!} E'[\langle X' \rangle_\infty^n] \leq \sum_{n=0}^{\infty} \|X'\|_{\text{BMO}}^{2n} \leq \sum_{n=0}^{\infty} \|X\|_{\text{BMO}}^{2n} = d,$$

completing the proof.

This estimate is the best possible, as the following example shows.

EXAMPLE 3. Firstly, let G^0 be the class of all topological Borel sets in $R_+ = [0, \infty[$, and S be the identity mapping of R_+ onto R_+ . We define a probability measure $d\mu$ on R_+ such that $\mu(S > t) = e^{-t}$. Let G be the completion of G^0 with respect to $d\mu$, and similarly G_t the completion of the Borel field generated by $S \wedge t$. It is clear that S is a stopping time over (G_t) . We now construct in the usual way a probability system $(\Omega, F, P; (F_t))$ by taking the product of the system $(R_+, G, d\mu; (G_t))$ with another system $(\Omega', F', P'; (F'_t))$ which carries a one dimensional Brownian motion $B = (B_t)$ starting at 0. Then S is also a stopping time over (F_t) so that $X = B^S$ is a continuous martingale. As $\langle X \rangle_t = S \wedge t$, we get

$$E[\langle X \rangle_\infty - \langle X \rangle_t | F_t] = e^t \int_t^\infty (x - t)e^{-x} dx I_{(t < S)} = I_{(t < S)},$$

from which $\|X\|_{\text{BMO}} = 1$. Let now $0 < \varepsilon < 1$. Then by Lemma 4

$$E[e^{(1-\varepsilon)\langle X \rangle_\infty}] \leq (1 - (1 - \varepsilon)\|X\|_{\text{BMO}}^2)^{-1} = \varepsilon^{-1}.$$

But the left hand side is

$$\int_{R_+} e^{(1-\varepsilon)S} d\mu = \int_0^\infty e^{-\varepsilon x} dx = \varepsilon^{-1}.$$

Thus the inequality given in Lemma 4 cannot be improved.

We finish this section with the following result obtained by Kazamaki [6]. Quite recently, the extension to right continuous local martingales was given by C. Doléans-Dade and P. A. Meyer [1] and by Kazamaki [8].

LEMMA 5. *Let $M \in \mathcal{L}$. Then M is a BMO-martingale if and only if Z satisfies (A_∞) .*

PROOF. Suppose firstly that $\|M\|_{\text{BMO}} < \infty$, and choose $p > 1$ such that $\|M\|_{\text{BMO}}^2 < 2(\sqrt{p} - 1)^2$. Now we are going to show that Z satisfies

(A_p). Indeed, set $p_0 = \sqrt{p} + 1$. The exponent conjugate q_0 is $(\sqrt{p} + 1)/\sqrt{p}$, so that $1/q_0(\sqrt{p} - 1)^2 - p_0/(p - 1)^2 = 1/(p - 1)$. By Hölder's inequality

$$\begin{aligned} E[(Z_t/Z_\infty)^{1/(p-1)} | F_t] &= E[\exp(-\langle M_\infty - M_t \rangle / (p-1) - p_0(\langle M \rangle_\infty - \langle M \rangle_t) / 2(p-1)^2) \\ &\quad \times \exp(\langle M \rangle_\infty - \langle M \rangle_t) / 2q_0(\sqrt{p} - 1)^2) | F_t] \\ &\leq E[\exp(-p_0(M_\infty - M_t) / (p-1) - p_0^2(\langle M \rangle_\infty - \langle M \rangle_t) / 2(p-1)^2) | F_t]^{1/p_0} \\ &\quad \times E[\exp(\langle M \rangle_\infty - \langle M \rangle_t) / 2(\sqrt{p} - 1)^2) | F_t]^{1/q_0}. \end{aligned}$$

By the supermartingale inequality, the first term on the right hand side is smaller than 1. In addition, according to Lemma 4, the second term is dominated by $(1 - \|M\|_{\text{BMO}}^2 / 2(\sqrt{p} - 1)^2)^{-1}$.

Conversely, let us assume that Z satisfies the (A_{p-1}) condition for some $p > 2$. Let (T_n) be a non-decreasing sequence of stopping times with $\lim_n T_n = \infty$ such that each process M^{T_n} is a uniformly integrable martingale. We now claim that each Z^{T_n} satisfies (A_p). To see this, we apply Hölder's inequality with exponents $(p - 1)/(p - 2)$ and $p - 1$:

$$\begin{aligned} E[(Z_{t \wedge T_n} / Z_{T_n})^{1/(p-1)} | F_{t \wedge T_n}] &= E[(Z_{t \wedge T_n} / Z_\infty)^{1/(p-1)} (Z_\infty / Z_{T_n})^{1/(p-1)} | F_{t \wedge T_n}] \\ &\leq E[(Z_{t \wedge T_n} / Z_\infty)^{1/(p-2)} | F_{t \wedge T_n}]^{(p-2)/(p-1)} \\ &\quad \times E[Z_\infty / Z_{T_n} | F_{t \wedge T_n}]^{1/(p-1)}. \end{aligned}$$

The first term on the right hand side is dominated by some constant C_p because Z satisfies (A_{p-1}). In addition, as Z is a positive supermartingale, the second term is smaller than 1. Consequently, for every n , Z^{T_n} satisfies the (A_p) condition. Then by Jensen's inequality

$$\begin{aligned} E[(Z_{t \wedge T_n} / Z_{T_n})^{1/(p-1)} | F_{t \wedge T_n}] &\geq \exp(E[-M_{T_n} + M_{t \wedge T_n} + (\langle M \rangle_{T_n} - \langle M \rangle_{t \wedge T_n}) / 2 | F_{t \wedge T_n}] / (p - 1)) \\ &= \exp(E[\langle M \rangle_{T_n} - \langle M \rangle_{t \wedge T_n} | F_{t \wedge T_n}] / 2(p - 1)), \end{aligned}$$

from which $\|M^{T_n}\|_{\text{BMO}}^2 \leq 2(p - 1) \log C_p$ for every n . Letting $n \rightarrow \infty$, we get $M \in \text{BMO}$. Thus the lemma is completely established.

By this lemma it is immediate to see that even if Z is bounded, it does not always satisfy (A_∞). See Example 2.

3. On the problem of Girsanov. If $M \in \mathcal{L}$, when can one assert that $Z_t = \exp(M_t - \langle M \rangle_t / 2)$ is a martingale? In 1960 this problem was posed by I. V. Girsanov. A. A. Novikov [13] gave an answer to the effect that if $\exp(\langle M \rangle_t / 2) \in L^1$ for every t , then the process Z is a martingale. Recently, by making a partial modification of Novikov's proof, Kazamaki [7] showed that if $(\exp(M_t / 2))$ is a submartingale, then Z is a

martingale. Note that Kazamaki's condition is weaker than Novikov's, because $E[\exp(M_t/2)] \leq E[\exp(\langle M \rangle_t/2)]^{1/2}$ by Schwarz' inequality. Furthermore, there exists a BMO-martingale M , which does not satisfy Novikov's condition, although $\exp(M_t/2)$ is a submartingale, as the following example shows.

EXAMPLE 4. Let $S, B = (B_t, F_t)$ and (Ω, F, P) be as in Example 3. Then $X_t = \sqrt{2}B_{S \wedge t}$ is a BMO-martingale over (F_t) . By the result of Novikov

$$\int_{\Omega'} \exp(B_u/\sqrt{2} - u/4) dP' = 1$$

for every $u \geq 0$, and so by Fubini's theorem we have

$$\begin{aligned} E[\exp(X_\infty/2)] &= E[\exp(B_S/\sqrt{2})] = \int_0^\infty \exp(u/4) d\mu \int_{\Omega'} \exp(B_u/\sqrt{2} - u/4) dP' \\ &= \int_0^\infty \exp(-3u/4) du < \infty . \end{aligned}$$

Let now (τ_t) be a continuous change of time such that $\tau_0 = 0$ and $\tau_1 = S$, and consider the martingale $M_t = X_{\tau_t}$. It is a BMO-martingale over (F_{τ_t}) , and the process $\exp(M_t/2)$ is a submartingale. But, $\exp(\langle M \rangle_1/2)$ is not integrable because $\langle M \rangle_1 = 2S$.

We now give a new sufficient condition for the problem of Girsanov as follows.

LEMMA 6. *If M is a BMO-martingale, then Z is a uniformly integrable martingale.*

PROOF. We may assume that $0 < \|M\|_{\text{BMO}} < \infty$. Firstly we show that if $\|M\|_{\text{BMO}} < \sqrt{2}$, then Z is uniformly integrable. Let c be a positive number. Then applying Schwarz' inequality we have $E[\exp(cM_t)] \leq E[\exp(2c^2\langle M \rangle_t)]^{1/2}$. Now let $0 < \delta < 1/\sqrt{2}\|M\|_{\text{BMO}} - 1/2$ and $c = 1/2 + \delta$. As $\|\sqrt{2}cM\|_{\text{BMO}} < 1$, it follows from Lemma 4 that

$$E[\exp((1/2 + \delta)M_t)] \leq E[\exp(2c^2\langle M \rangle_t)]^{1/2} \leq (1 - 2c^2\|M\|_{\text{BMO}}^2)^{-1/2} .$$

Namely, $\sup_t E[\exp((1/2 + \delta)M_t)] < \infty$. Set now $p = 1 + 4\delta > 1$. So the exponent conjugate to p is $q = (1 + 4\delta)/4\delta$. Then by Hölder's inequality we get

$$\begin{aligned} E[Z_t] &= E[\exp(\sqrt{r/p}M_t - r\langle M \rangle_t/2) \exp((r - \sqrt{r/p})M_t)] \\ &\leq E[\exp(\sqrt{pr}M_t - pr\langle M \rangle_t/2)]^{1/p} E[\exp((r - \sqrt{r/p})qM_t)]^{1/q} , \quad r > 0 . \end{aligned}$$

The first term on right hand side is bounded by 1, because the process

$\exp(\sqrt{pr}M_t - pr\langle M \rangle_t/2)$ is nothing but the positive local martingale $Z^{(\sqrt{pr})}$. If $r = (1 + 2\delta)^2/(1 + 4\delta) > 1$, by a simple calculation we have $(r - \sqrt{r/p})q = 1/2 + \delta$, so that $\sup_t E[Z_t^r] < \infty$. Therefore Z is a uniformly integrable martingale if $\|M\|_{\text{BMO}} < \sqrt{2}$. Now we are going to deal with the general case. Let us choose a number a such that $0 < a < \text{Min}(1, 2/\|M\|_{\text{BMO}}^2)$. Then, as $\|aM\|_{\text{BMO}} < \sqrt{2}$, the process $Z^{(a)}$ is a uniformly integrable martingale. Therefore, for any stopping time T

$$\begin{aligned} 1 &= E[Z_\infty^{(a)}/Z_T^{(a)} | F_T] \\ &= E[\exp(a(M_\infty - M_T) - a(\langle M \rangle_\infty - \langle M \rangle_T)/2) \exp(a(1-a)(\langle M \rangle_\infty - \langle M \rangle_T)/2) | F_T]. \end{aligned}$$

Applying Hölder’s inequality with exponents $1/a$ and $1/(1 - a)$ to the right hand side we can obtain:

$$1 \leq E[Z_\infty/Z_T | F_T] E[\exp(a(\langle M \rangle_\infty - \langle M \rangle_T)/2) | F_T]^{(1-a)/a}.$$

By Lemma 4 the second term on the right hand side is smaller than

$$(1 - a\|M\|_{\text{BMO}}^2/2)^{-(1-a)/a} = \{(1 - a\|M\|_{\text{BMO}}^2/2)^{-2/a\|M\|_{\text{BMO}}^2}\}^{(1-a)\|M\|_{\text{BMO}}^2/2},$$

which converges to $\exp(\|M\|_{\text{BMO}}^2/2)$ as $a \rightarrow 0$. Consequently, we have

$$Z_T \leq E[Z_\infty | F_T] \exp(\|M\|_{\text{BMO}}^2/2).$$

This implies that Z is a uniformly integrable martingale.

Our aim in this section is to prove the following:

THEOREM 1. *If M is a BMO-martingale, then the “reverse Hölder inequality”*

$$E[Z_\infty^{1+\varepsilon} | F_t] \leq C_\varepsilon Z_t^{1+\varepsilon}$$

holds for every t , with positive constants C_ε and ε .

REMARK. Quite recently, C. Doléans-Dade and P. A. Meyer [2] proved, assuming the uniform integrability of the process Z , that the reverse Hölder inequality holds if Z satisfies (A_∞) . In [2] they make a systematic study of the subject about the (A_p) condition from a more general point of view.

PROOF. Our proof is an adaptation of the proof given in [2]. Now let $M \in \text{BMO}$. Then, by Lemmas 5 and 6, Z is a uniformly integrable martingale which satisfies (A_p) for some $p > 1$. We denote by $d\hat{P}$ the weighted probability measure $Z_\infty dP$ and by $\hat{E}[\]$ the expectation over Ω with respect to $d\hat{P}$. Clearly, if $A \in F_t$, $\hat{P}(A) = \int_A Z_t dP$ so that for every \hat{P} -integrable random variable V we have

$$\hat{E}[V | F_t] = E[Z_\infty V | F_t]/Z_t \text{ a.s., under } dP \text{ and } d\hat{P}.$$

We shall use this formula many times in the sequel. Let K be a constant ≥ 1 depending only on p such that

$$Z_t E[Z_\infty^{-1/(p-1)} | F_t]^{p-1} \leq K,$$

which follows from the definition of (A_p) . Now we set $a = 1/2^p K$ and $b_\varepsilon = 2\varepsilon/(1 + \varepsilon)a^{1+\varepsilon}$ and let us choose $\varepsilon > 0$ such that $b_\varepsilon < 1$. Then we claim that $E[Z_\infty^{1+\varepsilon} | F_t] \leq C_\varepsilon Z_t^{1+\varepsilon}$ where $C_\varepsilon = (3 - b_\varepsilon)/(1 - b_\varepsilon)$.

Firstly, we show that the basic inequality

$$E[Z_\infty; Z_\infty > \lambda] \leq 2\lambda P(Z_\infty > a\lambda)$$

is valid for every $\lambda > 0$. Indeed, let $T = \inf(t; Z_t > \lambda)$, which is a stopping time with $Z_T \leq \lambda$ a.s.. In addition, $Z_T = \lambda$ on $(T < \infty)$ because Z is continuous. Let us consider the martingale X defined by $X_t = P(Z_\infty \leq aZ_t | F_t)$. As $X_T = Z_T \hat{E}[X_\infty/Z_\infty | F_T]$, we apply Hölder's inequality with exponents p and $q = p/(p-1)$ to the right hand side:

$$\begin{aligned} X_T^p &\leq Z_T^p \hat{E}[Z_\infty^{-q} | F_T]^{p-1} \hat{E}[X_\infty^p | F_T] \\ &= Z_T E[Z_\infty^{-1/(p-1)} | F_T]^{p-1} \hat{E}[X_\infty^p | F_T] \leq KE[Z_\infty X_\infty^p | F_T]/Z_T. \end{aligned}$$

But $Z_\infty X_\infty^p \leq aZ_T$ by the definition of X . Thus $X_T \leq (aK)^{1/p} = 1/2$ and so $P(Z_\infty > a\lambda) \geq P(T < \infty)/2$ because $1/2 \leq 1 - X_T = P(Z_\infty > aZ_T | F_T)$ and $(T < \infty) \in F_T$. Consequently we get

$$\begin{aligned} E[Z_\infty; Z_\infty > \lambda] &\leq E[Z_\infty; T < \infty] = E[Z_T; T < \infty] = \lambda P(T < \infty) \\ &\leq 2\lambda P(Z_\infty > a\lambda). \end{aligned}$$

Now let $U_n = \text{Min}(Z_\infty, n)$ for $n \geq 1$. It is clear that $U_n \rightarrow Z_\infty$ as $n \rightarrow \infty$. It is also immediate to see that for each n the inequality

$$E[U_n; U_n > \lambda] \leq 2\lambda P(U_n > a\lambda)$$

is valid. Then, multiplying both sides of this inequality by $\varepsilon\lambda^{\varepsilon-1}$ and integrating on the interval $[1, \infty[$, we find that

$$\int_{(U_n > 1)} (U_n^{1+\varepsilon} - U_n) dP \leq b_\varepsilon \int_{(U_n > a)} U_n^{1+\varepsilon} dP \leq b_\varepsilon \int_{(U_n > 1)} U_n^{1+\varepsilon} dP + b_\varepsilon.$$

As $E[U_n] \leq E[Z_\infty] \leq 1$ and $E[U_n^{1+\varepsilon}] < \infty$, we have

$$(1 - b_\varepsilon) \int_{(U_n > 1)} U_n^{1+\varepsilon} dP \leq b_\varepsilon + 1 \leq 2.$$

That is, $E[U_n^{1+\varepsilon}] \leq 1 + 2/(1 - b_\varepsilon) = C_\varepsilon$. From Fatou's lemma it follows that $E[Z_\infty^{1+\varepsilon}] \leq C_\varepsilon$.

Secondly, let S be a stopping time, and let A be an arbitrary element of F_S such that $P(A) > 0$. As in the proof of Lemma 4, we set $dP' =$

$I_A dP/P(A)$ and $F'_t = F_{S+t}$. $E'[\]$ denotes the expectation over Ω with respect to dP' . Consider now the process Z' defined by $Z'_t = Z_{S+t}/Z_S$. Clearly $0 < Z'_t$ and $E'[Z'_\infty] = 1$. Furthermore, it is a uniformly integrable martingale over (F'_t) relative to dP' such that for the same constant K as before

$$Z'_t E'[(Z'_\infty)^{-1/(p-1)} | F'_t]^{p-1} \leq K, \quad P'\text{-a.s. .}$$

Therefore, by the same argument as above we obtain $E'[(Z'_\infty)^{1+\epsilon}] \leq C_\epsilon$, that is, $E[(Z_\infty/Z_S)^{1+\epsilon}; A] \leq C_\epsilon P(A)$. This is valid for any $A \in F_S$, so that we have the desired inequality. Hence the theorem is established.

In the proof of Proposition 3, we shall show that, if Z is a uniformly integrable martingale satisfying the reverse Hölder inequality, then M is a BMO-martingale.

COROLLARY. *Let a be a real number. If M is a BMO-martingale, then $Z^{(a)}$ is an L^p -bounded martingale for some $p > 1$.*

PROOF. If M is a BMO-martingale, so is aM . Then the conclusion follows immediately from Theorem 1.

Let $M \in \mathcal{L}$. Obviously, if it is bounded from above, then the process $\exp(M_t/2)$ is a submartingale. But there exists a continuous martingale M , bounded from above, which is not a BMO-martingale. See Example 2. We now remark that, even if M is a BMO-martingale, $\exp(M_t/2)$ is not necessarily a submartingale. We end this section with such examples.

EXAMPLE 5. Let $S, B = (B_t, F_t), (\Omega, F, P)$ be as in Example 3, and let (τ_t) be a continuous change of time such that $\tau_0 = 0$ and $\tau_1 = S$. Then $2\sqrt{2}B_{S \wedge t}$ is a BMO-martingale over (F_t) , and so $M_t = 2\sqrt{2}B_{S \wedge \tau_t}$ is a BMO-martingale over (F_{τ_t}) . But it follows from Fubini's theorem that $\exp(M_t/2) = \exp(\sqrt{2}B_S)$ is not integrable. Namely, $\exp(M_t/2)$ is not a submartingale.

EXAMPLE 6. Let $B = (B_t, F_t)$ be a complex Brownian motion starting at 0 and let $T = \inf\{t; |B_t| = 1\}$. Then $\log(1 - B^t)$ is a conformal martingale on $[0, T]$, because $\log(1 - z)$ is analytic in the unit disc $|z| < 1$. Its imaginary part is bounded, so that by the main theorem of R. K. Gettoor and M. J. Sharpe [4] the real part $\log|1 - B^t|$ is a BMO-martingale. Now let $X = -\log|1 - B^t|$. As is well-known, B_T is uniformly distributed on the unit circle $|z| = 1$. Therefore we get

$$\begin{aligned} E[\exp(X_\infty/2)] &= E[\exp(-\log|1 - B_T|)] \\ &= (2\pi)^{-1} \int_0^{2\pi} \{2(1 - \cos \theta)\}^{-1/2} d\theta = \infty . \end{aligned}$$

Let us define a change of time (τ_t) with $\tau_0 = 0$ and $\tau_1 = T$ as in Example 2. Then $M_t = X_{\tau_t}$ is a desired BMO-martingale.

4. Transformation of the spaces BMO and H^1 by a change of law. Let $M \in \mathcal{L}$ and consider the process $Z_t = \exp(M_t - \langle M \rangle_t / 2)$ as usual. In this section, Z is assumed to be a uniformly integrable martingale with $Z_\infty > 0$. $d\hat{P}$ denotes always the weighted probability measure $Z_\infty dP$. It is obvious that the measures dP and $d\hat{P}$ are mutually absolutely continuous. We shall consider the process W defined by $W_t = 1/Z_t$. It is a uniformly integrable martingale with respect to $d\hat{P}$, for $\hat{E}[W_\infty | F_t] = E[Z_\infty W_\infty | F_t] / Z_t = W_t$. Clearly, $0 < W_t$, $W_0 = 1$ and $W_\infty d\hat{P} = dP$. If \mathcal{H} is a subclass of \mathcal{L} , $\hat{\mathcal{H}}$ denotes the class of continuous local martingales relative to $d\hat{P}$, which corresponds to \mathcal{H} . So $\hat{\mathcal{L}}$ is the class of all \hat{P} -continuous local martingales X' over (F_t) with $X'_0 = 0$. Our interest here lies in investigating the relations between \mathcal{H} and $\hat{\mathcal{H}}$. The following lemma plays a very important role in our discussion.

LEMMA 7. *For any $X \in \mathcal{L}$, $\hat{X} = X - \langle X, M \rangle$ belongs to $\hat{\mathcal{L}}$ and $\langle \hat{X} \rangle = \langle X \rangle$ under either probability measure. Furthermore, the mapping $i: X \rightarrow \hat{X}$ is linear and bijective.*

PROOF. To see $\hat{X} \in \hat{\mathcal{L}}$, it is enough to check that $Z\hat{X} \in \mathcal{L}$. \hat{X} is a semi-martingale with respect to dP , and $\langle X, M \rangle_t = \int_0^t Z_s^{-1} d\langle X, Z \rangle_s$ because $M_t = \int_0^t Z_s^{-1} dZ_s$. Then, by Ito's formula we have

$$\begin{aligned} Z_t \hat{X}_t &= Z_0 \hat{X}_0 + \int_0^t \hat{X}_s dZ_s + \int_0^t Z_s d\hat{X}_s + \langle Z, X \rangle_t \\ &= \int_0^t \hat{X}_s dZ_s + \int_0^t Z_s dX_s, \end{aligned}$$

which belongs to \mathcal{L} . Similarly, we can check the equality $\langle \hat{X} \rangle = \langle X \rangle$. From these facts follows the linearity and the injectivity of the mapping i . So it remains to show the surjectivity. As $\hat{M} = M - \langle M \rangle$ and $\langle \hat{M} \rangle = \langle M \rangle$, we have

$$W_t = \exp(-\hat{M}_t - \langle \hat{M} \rangle_t / 2),$$

so that for any $X' \in \hat{\mathcal{L}}$, $X = X' + \langle X, \hat{M} \rangle$ belongs to \mathcal{L} . On the other hand, $\hat{X} = X - \langle X, M \rangle$ is in $\hat{\mathcal{L}}$. Therefore $X' - \hat{X} = \langle X, M \rangle - \langle X', \hat{M} \rangle$ is also a \hat{P} -continuous local martingale with finite variation on each finite interval. This implies that $X' = \hat{X}$. Thus the lemma is proved.

J. H. Van Schuppen and E. Wong [14] tried to extend this transformation to right continuous local martingales, and the generalization

was completely established by E. Lenglart [9]. Note that “the stochastic integral $H \circ \hat{X}$ relative to $d\hat{P}$ ” coincides with “the stochastic integral of H with respect to the semi-martingale \hat{X} relative to dP ”.

PROPOSITION 1. *If $Z^* \in L^1$, then for any $X \in \mathcal{L}$*

$$\|\hat{X}\|_{\hat{H}^2} \leq (2E[Z^*])^{1/2} \|X\|_{\text{BMO}} .$$

PROOF. Let $X \in \text{BMO}$ and choose a non-decreasing sequence (T_n) of stopping times with $\lim_n T_n = \infty$ such that $\hat{X}^{T_n} \in \hat{H}^2$ for every $n \geq 1$. Then for each n we have

$$\begin{aligned} \hat{E}[\langle \hat{X} \rangle_{T_n}] &= E[Z_{T_n} \langle X \rangle_{T_n}] = E\left[\int_0^{T_n} Z_s d\langle X \rangle_s\right] = E[\langle Z \circ X, X \rangle_{T_n}] \\ &\leq \sqrt{2} E\left[\left(\int_0^{T_n} Z_s^2 d\langle X \rangle_s\right)^{1/2}\right] \|X\|_{\text{BMO}} , \end{aligned}$$

which follows from Lemma 1. The expectation on the right hand side is smaller than

$$\begin{aligned} E\left[(Z^*)^{1/2} \left(\int_0^{T_n} Z_s d\langle X \rangle_s\right)^{1/2}\right] &\leq E[Z^*]^{1/2} E\left[\int_0^{T_n} Z_s d\langle X \rangle_s\right]^{1/2} \\ &= E[Z^*]^{1/2} \hat{E}[\langle \hat{X} \rangle_{T_n}]^{1/2} . \end{aligned}$$

Therefore, as $\hat{E}[\langle \hat{X} \rangle_{T_n}] < \infty$, we have $\hat{E}[\langle \hat{X} \rangle_{T_n}]^{1/2} \leq \sqrt{2} E[Z^*]^{1/2} \|X\|_{\text{BMO}}$, for $n \geq 1$. Letting $n \rightarrow \infty$ and using Fatou’s lemma, we are done.

Proposition 1 shows that if $Z^* \in L^1$, then the mapping $i: \text{BMO} \rightarrow \hat{H}^2$ is continuous.

PROPOSITION 2. *$Z^* \in L^1$ if and only if $\hat{M} \in \hat{H}^2$.*

PROOF. We define $\log^+ x$, as usual, as 0 if $x < 1$ and $\log x$ if $x \geq 1$. We begin with the proof of the “if” part. From the definition of $d\hat{P}$ it follows that

$$E[Z_\infty \log^+ Z_\infty] = \hat{E}[\log^+ Z_\infty] = \hat{E}[M_\infty - \langle M \rangle_\infty / 2; Z_\infty \geq 1] .$$

By Lemma 7 the right hand side is

$$\hat{E}[\hat{M}_\infty + \langle \hat{M} \rangle_\infty / 2; Z_\infty \geq 1] \leq \hat{E}[\langle \hat{M} \rangle_\infty]^{1/2} + \hat{E}[\langle \hat{M} \rangle_\infty] / 2 .$$

Therefore, if $\hat{M} \in \hat{H}^2$, we have $Z^* \in L^1$ by the classical inequality of Doob.

To see the “only if” part, we need the inequality:

$$E[Z_\infty \log Z_\infty] \leq 4\sqrt{2\pi}(E[Z^*] + 1) ,$$

which follows from a result given by S. Watanabe [15]. Following his idea, we show this inequality. Firstly, let us choose Y in \mathcal{L} in such a way that $U_t = Z_t + iY_t$ is a conformal martingale; that is, $\langle Z \rangle = \langle Y \rangle$

and $\langle Z, Y \rangle = 0$. Then $V_t = U_t \log U_t$ is also a conformal martingale, for $f(z) = z \log z$ is analytic in $D = \{z; \operatorname{Re} z > 0\}$. Therefore, $\operatorname{Re} V_t = Z_t \log |U_t| - Y_t \arg U_t$ is a continuous local martingale. By using the stopping argument we may assume that $Z_t \log |U_t|$ and Y_t are in H^2 . Then $E[Z_\infty \log |U_\infty|] = E[Y_\infty \arg U_\infty]$. In addition, $U_\infty \in D$, hence $|\arg U_\infty| \leq \pi/2$. We now apply Davis' inequality:

$$\begin{aligned} E[Z_\infty \log Z_\infty] &\leq E[Z_\infty \log |U_\infty|] \leq (\pi/2)E[|Y_\infty|] \\ &\leq 2\sqrt{2\pi}E[\langle Z \rangle_\infty^{1/2}] \leq 4\sqrt{2\pi}E[(Z-1)^*] \leq 4\sqrt{2\pi}(E[Z^*] + 1). \end{aligned}$$

Therefore, if $Z^* \in L^1$, then $E[Z_\infty \log Z_\infty] < \infty$.

Now we are going to show that $\hat{M} \in \hat{H}^2$. The stopping argument enables us to assume that \hat{M} is \hat{P} -uniformly integrable. Then, as $\hat{E}[\hat{M}_\infty] = 0$, we have

$$\hat{E}[\langle \hat{M} \rangle_\infty] = 2\hat{E}[\hat{M}_\infty + \langle \hat{M} \rangle_\infty/2] = 2E[Z_\infty(M_\infty - \langle M \rangle_\infty/2)] = 2E[Z_\infty \log Z_\infty],$$

and we are done.

Now let $\mathcal{N} = \bigcap_{p>0} H^p$. As is well-known, if $1 < p < \infty$, H^p coincides with the class of all L^p -bounded continuous martingales.

PROPOSITION 3. *Assume that $M \in \text{BMO}$. Then $X \in \mathcal{N}$ if and only if $\hat{X} \in \hat{\mathcal{N}}$.*

PROOF. By the corollary to Theorem 1, Z is an L^{p_0} -bounded martingale for some $p_0 > 1$. It follows from Hölder's inequality that for each X

$$\hat{E}[\langle \hat{X} \rangle_\infty^p] = E[Z_\infty \langle X \rangle_\infty^p] \leq \|Z_\infty\|_{p_0} \|\langle X \rangle_\infty^p\|_{q_0},$$

where $1/p_0 + 1/q_0 = 1$. This implies that if $X \in \mathcal{N}$, then $\hat{X} \in \hat{\mathcal{N}}$.

To see the converse, it is enough to show that $\hat{M} \in \text{BMO}$. As $M \in \text{BMO}$, according to Theorem 1, it satisfies the reverse Hölder inequality, that is, $E[Z_t^{1+\varepsilon} | F_t] \leq C_\varepsilon Z_t^{1+\varepsilon}$ for some $\varepsilon > 0$. This can be rewritten as follows:

$$\hat{E}[(W_t/W_\infty)^\varepsilon | F_t] \leq C_\varepsilon.$$

Namely, W satisfies the (A_p) condition relative to $d\hat{P}$ for each $p > 1$ with $1/(p-1) < \varepsilon$. Consequently, using again Lemma 5, we obtain the fact that $\hat{M} \in \text{BMO}^\wedge$. This completes the proof.

It should be noted that Proposition 3 does not hold without the condition " $M \in \text{BMO}$ ". In the following we give such an example.

EXAMPLE 7. Consider a one dimensional Brownian motion $B = (B_t, F_t)$ starting at 0 and defined on a probability space $(\Omega, F, d\mu)$. Let $T = \inf\{t; B_t \geq 1\}$. Then the process B^r stopped at T is a continuous martingale,

which is not uniformly integrable with respect to $d\mu$. Clearly, the process Y given by $Y_t = \exp(B_{t \wedge T} - (t \wedge T)/2)$ is a bounded martingale. So $dP = Y_\infty d\mu$ is a probability measure on Ω . Now let $M = -B^T + \langle B^T \rangle$ and $Z_t = \exp(M_t - \langle M \rangle_t/2)$. The process Z is a P -uniformly integrable martingale with $Z_t = 1/Y_t$, and the weighted probability measure $d\hat{P} = Z_\infty dP$ equals $d\mu$. By Lemma 7, M is a P -local martingale with $\langle M \rangle = \langle B^T \rangle$. Let us consider the P -local martingale $X = M/\sqrt{2}$. Then from the fact $B_T = 1$ follows

$$\begin{aligned} E[\exp(\langle X \rangle_\infty)] &= \int_\Omega \exp(\langle M \rangle_\infty/2) \exp(B_T - \langle B \rangle_T/2) d\mu \\ &= \int_\Omega \exp(B_T) d\mu = e. \end{aligned}$$

That is, $X \in \mathcal{N}$. However, $\hat{X} = \hat{M}/\sqrt{2} = -B^T/\sqrt{2}$ is not uniformly integrable with respect to $d\mu$. It follows from Proposition 3 that M is not a BMO-martingale.

PROPOSITION 4. $\phi: X \rightarrow Z^{-1/2} \circ \hat{X}$ is an isometric isomorphism of H^2 onto \hat{H}^2 .

PROOF. Let $X \in H^2$. Lemma 7 says that \hat{X} is in $\hat{\mathcal{L}}$. Let $T_n \uparrow \infty$ be stopping times such that $\hat{X}^{T_n} \in \hat{H}^2$ for every n . Since $W_t = 1/Z_t$ is a uniformly integrable martingale with respect to $d\hat{P}$, we have

$$\hat{E}[\langle Z^{-1/2} \circ \hat{X} \rangle_{T_n}] = \hat{E}\left[\int_0^{T_n} W_s d\langle \hat{X} \rangle_s\right] = \hat{E}[W_{T_n} \langle \hat{X} \rangle_{T_n}] = E[\langle X \rangle_{T_n}],$$

for $n \geq 1$.

Letting $n \rightarrow \infty$ and using the monotone convergence theorem, we obtain $\hat{E}[\langle Z^{-1/2} \circ \hat{X} \rangle_\infty] = E[\langle X \rangle_\infty] < \infty$, so that $Z^{-1/2} \circ \hat{X} \in \hat{H}^2$. This implies that the mapping $\phi: H^2 \rightarrow \hat{H}^2$ given by $\phi(X) = Z^{-1/2} \circ \hat{X}$ is well-defined. Clearly it is linear and injective. From the above calculation it follows that $\|\phi(X)\|_{\hat{H}^2} = \|X\|_{H^2}$. Thus, it remains to prove the surjectivity. To see this, let $X' \in \hat{H}^2$. By Lemma 7, $\hat{U} = X'$ and $\langle U \rangle = \langle X' \rangle$ for some $U \in \hat{\mathcal{L}}$. We now set $X = Z^{1/2} \circ U$ and choose stopping times $T_n \uparrow \infty$ such that $U^{T_n} \in H^2$ for every n . Then we have

$$\begin{aligned} E[\langle X \rangle_{T_n}] &= E\left[\int_0^{T_n} Z_s d\langle U \rangle_s\right] = E[Z_{T_n} \langle U \rangle_{T_n}] \\ &= \hat{E}[\langle X' \rangle_{T_n}] \leq \hat{E}[\langle X' \rangle_\infty]. \end{aligned}$$

From Fatou's lemma it follows that $X \in H^2$. Moreover, we have

$$\phi(X) = Z^{-1/2} \circ (Z^{1/2} \circ \hat{U}) = \hat{U} = X'.$$

Consequently, the mapping ϕ is surjective.

Let $1 \leq p < \infty$. In particular, if Z_∞ is bounded, then $i: X \rightarrow \hat{X}$ is a continuous linear mapping of H^p into \hat{H}^p . Therefore, it is evident that if $0 < c \leq Z_\infty \leq C$, then the mapping i is an isomorphism of H^p onto \hat{H}^p .

THEOREM 2. *If $M \in \text{BMO}$, then $i: X \rightarrow \hat{X}$ is an isomorphism of BMO onto BMO^\wedge .*

PROOF. Let $M \in \text{BMO}$. By Lemma 5, Z satisfies (A_p) for some $p > 1$. We now need the following inequality due to Kazamaki [8]:

$$\|X\|_{\text{BMO}} \leq C_p \|\hat{X}\|_{\text{BMO}^\wedge}, \quad \text{for } X \in \mathcal{L}.$$

To show this, let us assume that $0 < \|\hat{X}\|_{\text{BMO}^\wedge} < \infty$, and set $a = (2p \|\hat{X}\|_{\text{BMO}^\wedge}^2)^{-1}$. As $\|\sqrt{ap} \hat{X}\|_{\text{BMO}^\wedge}^2 = 1/2$, Lemma 4 yields

$$\hat{E}[\exp(a p (\langle \hat{X} \rangle_\infty - \langle \hat{X} \rangle_t)) | F_t] \leq 2.$$

By using a simple inequality $x \leq e^{ax}/a$ and Hölder's inequality, we have

$$\begin{aligned} E[\langle X \rangle_\infty - \langle X \rangle_t | F_t] &\leq E[(Z_t/Z_\infty)^{1/p} (Z_\infty/Z_t)^{1/p} \exp(a(\langle X \rangle_\infty - \langle X \rangle_t)) | F_t] / a \\ &\leq E[(Z_t/Z_\infty)^{1/(p-1)} | F_t]^{1/q} \\ &\quad \times E[(Z_\infty/Z_t) \exp(a p (\langle X \rangle_\infty - \langle X \rangle_t)) | F_t]^{1/p} / a, \end{aligned}$$

with $1/p + 1/q = 1$. Clearly, $1/a = 2p \|\hat{X}\|_{\text{BMO}^\wedge}^2$. Since Z satisfies (A_p) , the first expectation on the right hand side is smaller than some constant K_p . The second one can be written as $\hat{E}[\exp(a p (\langle \hat{X} \rangle_\infty - \langle \hat{X} \rangle_t)) | F_t]$, which is bounded by 2. Thus, $\|X\|_{\text{BMO}} \leq C_p \|\hat{X}\|_{\text{BMO}^\wedge}$.

As mentioned in the proof of Proposition 3, if $M \in \text{BMO}$, then $\hat{M} \in \text{BMO}^\wedge$. Therefore we get $c \|X\|_{\text{BMO}} \leq \|\hat{X}\|_{\text{BMO}^\wedge} \leq C \|X\|_{\text{BMO}}$ for $X \in \mathcal{L}$. Here, the positive constants c and C do not depend on X . Then, combining this inequality with Lemma 7, we see that the spaces BMO and BMO^\wedge are isomorphic via the mapping i .

We remark that, without the condition " $M \in \text{BMO}$ ", the conclusion of Theorem 2 no longer follows. In the next theorem, let $1 \leq p \leq \infty$ and $H^\infty = \text{BMO}$. We denote by q the exponent conjugate to p ; namely, $q = \infty$ if $p = 1$ and $q = 1$ if $p = \infty$.

THEOREM 3. *$j: \hat{X} \rightarrow X$ is a continuous mapping of \hat{H}^p into H^p if and only if $\psi: X \rightarrow Z^{-1} \circ \hat{X}$ is a continuous mapping of H^q into \hat{H}^q .*

PROOF. We deal only with the case $p = \infty$; the proof for the other cases is similar. Firstly, let us assume that the mapping j is continuous, that is, $\|Y\|_{\text{BMO}} \leq \|j\| \|\hat{Y}\|_{\text{BMO}^\wedge}$ for every $\hat{Y} \in \text{BMO}^\wedge$. Let $X \in H^1$ and $\hat{Y} \in \text{BMO}^\wedge$. Since $W_t = 1/Z_t$ is a uniformly integrable martingale with respect to $d\hat{P}$, we have

$$\widehat{E}[\langle Z^{-1} \circ \widehat{X}, \widehat{Y} \rangle_\infty] = \widehat{E}\left[\int_0^\infty W_s d\langle \widehat{X}, \widehat{Y} \rangle_s\right] = \widehat{E}[W_\infty \langle X, Y \rangle_\infty] = E[\langle X, Y \rangle_\infty].$$

By Lemma 1 this is smaller than $\sqrt{2} \|X\|_{H^1} \|Y\|_{\text{BMO}}$. Therefore, from Lemma 2 follows the inequality

$$\|Z^{-1} \circ \widehat{X}\|_{\widehat{H}^1} \leq \sqrt{2} \|j\| \|X\|_{H^1}$$

for every $X \in H^1$.

Conversely, suppose that $\psi: X \rightarrow Z^{-1} \circ \widehat{X}$ is a continuous mapping of H^1 into \widehat{H}^1 . Let $X \in H^1$ and $\widehat{Y} \in \text{BMO}^\wedge$. By using the stopping argument we may assume that $Y \in \text{BMO}$. Then, by the same calculation as above, we have

$$\begin{aligned} E[\langle X, Y \rangle_\infty] &= \widehat{E}[\langle Z^{-1} \circ \widehat{X}, \widehat{Y} \rangle_\infty] \leq \sqrt{2} \|Z^{-1} \circ \widehat{X}\|_{\widehat{H}^1} \|\widehat{Y}\|_{\text{BMO}^\wedge} \\ &\leq \sqrt{2} \|\psi\| \|X\|_{H^1} \|\widehat{Y}\|_{\text{BMO}^\wedge}. \end{aligned}$$

In addition, by Lemma 3,

$$\begin{aligned} \|Y\|_{\text{BMO}} &\leq \sup\{E[\langle Y, X \rangle_\infty]; X \in H^1, \|X\|_{H^1} \leq 1\} \\ &\leq \sqrt{2} \|\psi\| \|\widehat{Y}\|_{\text{BMO}^\wedge}. \end{aligned}$$

Thus our claim is established.

The mapping j defined above is nothing else but the inverse of the mapping i . Combining Theorems 2 and 3, we get:

COROLLARY. *If $M \in \text{BMO}$, then the spaces H^1 and \widehat{H}^1 are isomorphic via the mapping ψ .*

We remark that it is impossible to remove the condition “ $M \in \text{BMO}$ ”. In other words, $Z^{-1} \circ \widehat{X} \notin \widehat{H}^1$ for some $X \in H^1$. Here is an example.

EXAMPLE 8. Let $S, B = (B_t, F_t)$ and (Ω, F, P) be as in Example 3, except that we use here the distribution $d\mu = I_{[1, \infty)}(u)u^{-2}du$ of S instead. Let $M = B^S$. Then it is immediate to see that $Z_t = \exp(M_t - \langle M \rangle_t/2)$ is a uniformly integrable martingale. As $E[\langle M \rangle_\infty^{1/2}] = \int_1^\infty u^{-3/2}du = 2$, we have $M \in H^1$. But it does not belong to H^2 , for $E[\langle M \rangle_\infty] = \int_1^\infty u^{-1}du = \infty$. By Proposition 3, $M \notin H^2$ if and only if W^* is not integrable with respect to $d\widehat{P}$. In addition, $W = 1 - W \circ \widehat{M}$, and so $W \circ \widehat{M} = Z^{-1} \circ \widehat{M} \notin H^1$.

Finally, we point out the fact that $i: X \rightarrow \widehat{X}$ is not always a continuous mapping of H^2 onto \widehat{H}^2 , even if M is a BMO-martingale. Indeed, if the mapping i were continuous, then by Theorem 3 $\widehat{H}^2 \ni \widehat{X} \rightarrow Z \circ X \in H^2$ must be continuous. This would imply that if $X \in H^2$, then $Z \circ X \in H^2$. However, for the BMO-martingale $M = B^S$ considered in Example 3, $Z \circ M \notin H^2$.

5. **A generalization of Doob's inequalities.** In this section, let us assume that $M \in \text{BMO}$. Then by Theorem 1 the process Z satisfies the reverse Hölder inequality: $E[Z_\infty^{1+\varepsilon} | F_t] \leq C_\varepsilon Z_t^{1+\varepsilon}$ for some $\varepsilon > 0$. By combining this result with Lemma 7, we can give a generalization of the classical inequalities due to J. L. Doob. The inequality (1) given in the following theorem was essentially proved by M. Izumisawa and N. Kazamaki [5].

THEOREM 4. (1) *Let $p > 1 + 1/\varepsilon$. Then the inequality*

$$E\left[\sup_t |X_t - \langle X, M \rangle_t|^p\right] \leq C_{p,\varepsilon} \sup_t E[|X_t - \langle X, M \rangle_t|^p]$$

is valid for all $X \in \mathcal{L}$.

(2) *In particular, if $Z_\infty/Z_t \leq C$, then there exists a constant $c > 0$ such that the inequality*

$$\begin{aligned} cE\left[\sup_t |X_t - \langle X, M \rangle_t|\right] \\ \leq e/(e-1) + (e/(e-1)) \sup_t E[|X_t - \langle X, M \rangle_t| \log^+ |X_t - \langle X, M \rangle_t|] \end{aligned}$$

is valid for all $X \in \mathcal{L}$.

PROOF. We begin with the proof of (1). Let $X \in \mathcal{L}$ and $0 < \delta < p - (1 + 1/\varepsilon)$. Then $1 < p_0 = (p - \delta)/(p - \delta - 1) < 1 + \varepsilon$ and $q_0 = p_0/(p_0 - 1) = p - \delta > 1$. It follows from the assumption that $E[Z_\infty^{p_0} | F_t] \leq C_{p,\varepsilon} Z_t^{p_0}$. Lemma 7 says that $\hat{X} = X - \langle X, M \rangle \in \hat{\mathcal{L}}$. By using the stopping argument we may assume that $\hat{X} \in \hat{H}^p$. Then $\hat{X}_t = \hat{E}[\hat{X}_\infty | F_t] = E[Z_\infty \hat{X}_\infty / Z_t | F_t]$, and so by Hölder's inequality with exponents p_0 and q_0 we obtain:

$$\begin{aligned} |\hat{X}_t|^{p-\delta} &\leq E[(Z_\infty/Z_t)^{p_0} | F_t]^{p-\delta-1} E[|\hat{X}_\infty|^{p-\delta} | F_t] \\ &\leq C_{p,\varepsilon} E[|\hat{X}_\infty|^{p-\delta} | F_t]. \end{aligned}$$

We now apply the classical theorem of Doob to the martingale $E[|\hat{X}_\infty|^{p-\delta} | F_t]$ to obtain

$$\begin{aligned} E\left[\sup_t |\hat{X}_t|^p\right] &\leq C_{p,\varepsilon} E\left[\sup_t E[|\hat{X}_\infty|^{p-\delta} | F_t]^{p/(p-\delta)}\right] \\ &\leq C_{p,\varepsilon} E[|\hat{X}_\infty|^p]. \end{aligned}$$

Finally, we show (2). For simplicity, we may assume that \hat{X} is a uniformly integrable martingale relative to $d\hat{P}$. Then from the assumption it follows that $|\hat{X}_t| = |\hat{E}[\hat{X}_\infty | F_t]| \leq E[Z_\infty |\hat{X}_\infty| / Z_t | F_t] \leq CE[|\hat{X}_\infty| | F_t]$, and so by applying the theorem of Doob to the martingale $E[|\hat{X}_\infty| | F_t]$, we obtain (2).

If $Z_\infty/Z_t \leq C$, then the inequality (1) is valid for any $p > 1$ and M

belongs to the class BMO. The classical inequalities of Doob correspond to the case $M = 0$.

REFERENCES

- [1] C. DOLÉANS-DADE AND P. A. MEYER, Une caractérisation de BMO, Séminaire de Probabilités XI, Université de Strasbourg, Lecture Notes in Math., 581 (1977), Springer-Verlag, 383-389.
- [2] C. DOLÉANS-DADE AND P. A. MEYER, Inégalités de normes avec poids, Séminaire de Probabilités XIII, Université de Strasbourg, Lecture Notes in Math., Berlin-Heidelberg-New York, Springer-Verlag, (to appear).
- [3] A. M. GARSIA, Martingale inequalities, Seminar Notes on Recent Progress, Benjamin, (1973).
- [4] R. K. GETTOOR AND M. J. SHARPE, Conformal martingales, *Inventiones Math.*, 16 (1972), 271-308.
- [5] M. IZUMISAWA AND N. KAZAMAKI, Weighted norm inequalities for martingales, *Tôhoku Math. J.*, 29 (1977), 115-124.
- [6] N. KAZAMAKI, A characterization of BMO-martingales, Séminaire de Probabilités X, Université de Strasbourg, Lecture Notes in Math., 511 (1976), Springer-Verlag, 536-538.
- [7] N. KAZAMAKI, On a problem of Girsanov, *Tôhoku Math. J.*, 29 (1977), 597-600.
- [8] N. KAZAMAKI, On transforming the class of BMO-martingales by a change of law, *Tôhoku Math. J.*, 31 (1979), 117-125.
- [9] E. LENGLART, Transformation des martingales locales par changement absolument continu de probabilités, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 37 (1977), 65-70.
- [10] P. A. MEYER, *Probabilités et Potentiel*, (1966), Hermann, Paris.
- [11] P. A. MEYER, Complément sur la dualité entre H^1 et BMO, Séminaire de Probabilités IX, Université de Strasbourg, Lecture Notes in Math., 465 (1975), Springer-Verlag, 237-238.
- [12] B. MUCKENHOUPT, Weighted norm inequalities for the Hardy maximal functions, *Trans. Amer. Math. Soc.*, 165 (1972), 207-226.
- [13] A. A. NOVIKOV, On an identity for stochastic integrals, *Theory Prob. Appl.*, 17 (1972), 717-720.
- [14] J. H. VAN SCHUPPEN AND E. WONG, Transformation of local martingales under a change of law, *Ann. of Prob.*, 2 (1974), 879-888.
- [15] S. WATANABE, A remark on the integrability of $\sup_t X_t$ for martingales, *Proc. of the second Japan-USSR symposium on Probability Theory*, no. 2 (1972), 147-155.

DEPARTMENT OF MATHEMATICS
TOYAMA UNIVERSITY
TOYAMA, 930 JAPAN
AND
MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, 980 JAPAN

