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ON THE TRANSLOCATION OF MASSES

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We assume that R is a compact metric space, though some of the definitions and results given below can be formulated for more general spaces.

Let $\Phi(e)$ be a mass distribution, i.e., a set function such that: (1) it is defined for Borel sets, (2) it is nonnegative: $\Phi(e) \geq 0$, (3) it is absolutely additive: if $e = e_1 + e_2 + \cdots$; $e_i \cap e_k = 0$ $(i \neq k)$, then $\Phi(e) = \Phi(e_1) + \Phi(e_2) + \cdots$. Let $\Phi'(e')$ be another mass distribution such that $\Phi(R) = \Phi'(R)$. By definition, a translocation of masses is a function $\Psi(e, e')$ defined for pairs of (B)-sets $e, e' \in R$ such that: (1) it is nonnegative and absolutely additive with respect to each of its arguments, (2) $\Psi(e, R) = \Phi(e)$, $\Psi(R, e') = \Phi'(e')$.

Let $r(x, y)$ be a known continuous nonnegative function representing the work required to move a unit mass from x to y.

We define the work required for the translocation of two given mass distributions as

$$W(\Psi, \Phi, \Phi') = \int_R \int_R r(x, x')\Psi(de, de') = \lim_{\lambda \to 0} \sum_{i,k} r(x_i, x'_k)\Psi(e_i, e'_k),$$

where e_i are disjoint and $\sum_1^n e_i = R$, e'_k are disjoint and $\sum_1^m e'_k = R$, $x_i \in e_i$, $x'_k \in e'_k$, and λ is the largest of the numbers $\operatorname{diam} e_i$ $(i = 1, 2, \ldots, n)$ and $\operatorname{diam} e'_k$ $(k = 1, 2, \ldots, m)$.

Clearly, this integral does exist.

We call the quantity

$$W(\Phi, \Phi') = \inf_{\Psi} W(\Psi, \Phi, \Phi')$$

the minimal translocation work. Since the set of all functions $\{\Psi\}$ is compact, there exists a function Ψ_0 realizing this minimun, so that

$$W(\Phi, \Phi') = W(\Psi_0, \Phi, \Phi'),$$

although this function is not unique. We call such a translocation Ψ_0 a minimal translocation.

In what follows, we say that a translocation Ψ from x to y is nonzero and write $x \to y$ if $\Psi(U_x, U_y) > 0$ for any neighborhoods U_x and U_y of the points x and y. We call Ψ a potential translocation if there exists a function $U(x)$ such that (1) $|U(x) - U(y)| \leq r(x, y)$, (2) $U(y) - U(x) = r(x, y)$ if $x \to y$.

Then the following theorem holds.

Theorem. *A translocation Ψ is minimal if and only if it is potential.*

Sufficiency. Let Ψ_0 be a potential translocation with potential U. Then by property (2) of U

$$\begin{aligned} W(\Psi_0, \Phi, \Phi') &= \int_R \int_R r(x, y)\Psi_0(de, de') = \int_R \int_R [U(y) - U(x)]\Psi_0(de, de') \\ &= \int_R \int_R U(y)\Psi_0(de, de') - \int_R \int_R U(x)\Psi_0(de, de') \\ &= \int_R U(y)\Phi'(de') - \int_R U(x)\Phi(de), \end{aligned}$$

while if Ψ is another function, then

$$\begin{aligned} W(\Psi, \Phi, \Phi') &= \int_R \int_R r(x, y)\Psi(de, de') \geq \int_R \int_R [U(y) - U(x)]\Psi(de, de') \\ &= \int_R U(y)\Phi'(de') - \int_R U(x)\Phi(de), \end{aligned}$$

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so that $W(\Psi, \Phi, \Phi') \geq W(\Psi_0, \Phi, \Phi')$, and Ψ_0 is minimal.

Necessity. Let Ψ_0 be a minimal translocation. Take a set of points $\xi_0, \xi_1, \dots$ that is dense in R. Denote by D_n the smallest set containing ξ_n such that if $x \in D_n$ and $x \to y$ or $y \to x$, then $y \in D_n$. Obviously, if $y \in D_n$, then there exists a system of points x_i, y_i such that $\xi_0 = x_0 \to y_1$, $x_1 \to y_1$, $x_1 \to y_2$, $\dots$, $x_{n-1} \to y_n$, $x_n \to y_n$ ($y_n = y$) (or a similar chain with arrows at the beginning or at the end directed differently). In the above case let

$$U(y) = \sum_1^n r(x_{i-1}, y_{i-1}) - \sum_1^n r(x_i, y_i).$$

It is not difficult to check that the value of U does not depend on the choice of the connecting chain and also that properties (1) and (2) of a potential hold for U if $x, y \in D_0$. Namely, we can show that the failure of either of these statements would allow us to replace Ψ_0 by a translocation involving less work, which contradicts the assumed minimality of Ψ_0.

Now suppose that the function U is already defined on domains $D_0, D_1, \dots, D_{n-1}$.

If the point ξ_n belongs to $D_0 + D_1 + \dots + D_{n-1}$, then the function U is already defined for both this point and the whole domain D_n. Otherwise define a function $V(x)$ on the domain D_n in the same way as we have defined U on D_0, except that ξ_n plays now the role of ξ_0. Then choose a number μ within the limits

$$\inf_{\substack{x \in D_0 + \dots + D_{n-1} \\ y \in D_n}} \{U(x) - V(y) - r(x,y)\} \leq \mu \leq \inf_{\substack{x \in D_0 + \dots + D_{n-1} \\ y \in D_n}} \{U(x) - V(y) + r(x,y)\}$$

The existence of such a μ is again established using the minimality of Ψ_0. Now let $U(x) = V(x) + \mu$ for $x \in D_n$. Thus the function U is defined on $D_0 + D_1 + \cdots$, and, since this set is dense in R, the function U can be extended to the whole R thanks to condition (2) and satisfies both (1) and (2), i.e., the translocation is potential.

The theorem just proved provides a convenient method of checking whether a given translocation of masses is minimal. Namely, to check this, it suffices to try and construct the potential for such a translocation by the method outlined in the necessity part of the proof. If this attempt fails, i.e., if the translocation is not minimal, then one will discover a method of lowering the translocation work. This allows one to come gradually to the minimal translocation.

It is interesting to study the space of mass distributions taking the quantity $W(\Phi, \Phi')$ as a metric (where $r(x,y) = \rho(x,y)$ is the distance). This method of metrization seems to be, in a sense, the most natural for this space.

In conclusion, we mention two practical problems to the solution of which our theorem can be applied.

Problem 1. *On the assignment of consumption locations to production locations. A network of railways connects a number of production locations $A_1, A_2, \dots, A_m$ with daily output of $a_1, a_2, \dots, a_m$ carriages of a certain good, respectively, to a number of consumption locations $B_1, B_2, \dots, B_n$ with daily demand of $b_1, b_2, \dots, b_n$ carriages ($\sum a_i = \sum b_k$). Given the cost $r_{i,k}$ involved in moving one carriage from A_i to B_k, find an assignment of consumption locations to production locations such that the total transport expenses be minimal.*

A detailed account of the solution of this and more complicated problems of the same type is given in a paper by L. V. Kantorovich and M. K. Govurin,[1] which is soon to be published.[2]

Problem 2. *Levelling a land area. Given the relief of the locality, i.e., the equations of the earth surface $z = f(x,y)$ and $z = f_1(x,y)$ before and after levelling [with $\iint f(x,y)\,dx\,dy = \iint f_1(x,y)\,dx\,dy$], and the cost of transporting 1 m^3 of earth from (x,y) to (x_1,y_1), find a plan of transporting of earth masses with the minimum total transportation cost.*

Translated by A. N. Sobolevskiĭ.

[1]Before the war, M. K. Gavurin spelled his name with "o." – *Editor's comment.*

[2]The paper by L. V. Kantorovich and M. K. Gavurin was published in 1949. – *Editor's comment.*