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Numerische Mathematik

On the Transport-Diffusion Algorithm and Its Applications to the Navier-Stokes Equations

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Summary. This paper deals with an algorithm for the solution of diffusion and/or convection equations where we mixed the method of characteristics and the finite element method. Globally it looks like one does one step of transport plus one step of diffusion (or projection) but the mathematics show that it is also an implicit time discretization of the *PDE* in Lagrangian form. We give an error bound ($h+\Delta t+h\times h/\Delta t$ in the interesting case) that holds also for the Navier-Stokes equations even when the Reynolds number is infinite (Euler equation).

Subject Classifications: AMS(MOS): 65M25, 65N30; Cr: 5.17.

Introduction

So far two methods were available to stabilize algorithms for the solution of the Navier-Stokes equation at high Reynolds number: upwinding (see [7, 9, 10, 15, 17]) and leastsquares (see for instance [15]).

Recently a new method, or rather a revived old method (see [13], [5]), has been considered in [6] which seems to have good stability properties.

The method takes into account the fact that $\frac{\partial u}{\partial t}+u\nabla u$ can be written as Du/Dt, the total derivative of u in the direction of the flow u.

Schematically it could be stated as a fractional step method:

$$\frac{Du^{n+\frac{1}{2}}}{Dt}=0 \tag{1}$$

$$\frac{u^{n+1}}{\Delta t}-\nu\Delta u^{n+1}+\nabla p^{n+1}=f+\frac{u^{n+\frac{1}{2}}}{\Delta t}, \qquad \nabla\cdot u^{n+1}=0 \text{ in } \Omega; \quad u^{n+1}|_\Gamma=u_\Gamma. \tag{2}$$

Physically this means that the velocity at the next time step is obtained by a transport of the velocity at the previous time step plus a diffusion of the result.

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However it is not easy to discretize and one does not know with what precision (1) should be integrated. In this paper we shall give mathematical justifications to the previous algorithm while following the ideas of [1, 14]; i.e. the implementation of the method of characteristics in a finite element context.

Briefly speaking to compute the solution of

$$\frac{\partial\rho}{\partial t}+u\nabla\rho-\nu\Delta\rho=f\ \rho|_\Gamma=0 \tag{3}$$

for a given velocity field u, one first rewrites (3) as

$$\frac{D\rho}{Dt}-\nu\Delta\rho=f \quad \rho|_\Gamma=0 \tag{4}$$

and then discretizes (4) by an implicit scheme:

$$\frac{\rho^{n+1}(x)-\rho^n(X^n(x))}{\Delta t}-\nu\Delta\rho^{n+1}(x)=f^{n+1}; \qquad \rho^{n+1}|_\Gamma=0 \tag{5}$$

where $X^n(x)$ is the solution at $\tau=n\Delta t$ of

$$\frac{dX}{d\tau}=u(X,\tau); \qquad X((n+1)\Delta t)=x \tag{6}$$

((6) is integrated backward in time).

When $\nu=0$ this is the well known method of characteristics for the transport equation. It is also close to the Lagrangian methods studied in [8].

Now if (5) is discretized in x by a finite element method then we can show the following properties:

- unconditionnal stability for all ν and Δt
- conservativity up to quadrature errors when $\nu=0$:

$$\int_\Omega \rho^{n+1}\,dx=\int_\Omega \rho^n\,dx \qquad \forall n \tag{7}$$

- error estimates of order $h+\Delta t+h^2/\Delta t$ with finite elements of degree 1, independent of ν,
- only *symmetric* linear systems to solve.

Therefore the only possible problem with this method is the numerical dissipation. This aspect is discussed with numerical tests for a monodimensional transport equation and for the Navier-Stokes equations in [2] and in a future publication. Our applications to the computation of 3-*D* viscous incompressible flows is done with a non conforming finite element of degree 1 which has $\nabla\cdot u=0$ exactly. This element, developed in [11] is convenient but the method is by no means restricted to that element.

Notations

$$|\varphi|_0=\Big(\int_\Omega|\varphi|^2\,dx\Big)^{\frac{1}{2}},$$

$$|\varphi|_{\infty,Q}=\operatorname*{essup}_{[x,t]\in Q}|\varphi(x,t)|,$$

$$(\varphi, \psi) = \int_\Omega \varphi\, \phi\, dx,$$

$$\rho^m(x) = \rho(x, m\Delta t),$$

$$X_h^m(x) = X_h(x, (m+1)\Delta t;\ m\Delta t).$$

1. Transport Without Dissipation

In this paragraph we shall consider the simple transport (hyperbolic) equation:

$$\frac{\partial\rho}{\partial t} + u\nabla\rho = f \quad \text{in } \Omega\times]0,T[$$

$$\rho(t=0) = \rho^0 \quad \text{in } \Omega. \tag{1.1}$$

Here Ω is a bounded (open) subset of R^2 or R^3 which could be, for example, the region occupied by a fluid; $u(x,t)$ is a vector field, say the velocity of the flow at $x\in\Omega$ and time t. Although it is not essential, we shall assume for simplicity that the two conditions hold:

$$\nabla\cdot u = 0 \quad \text{in } \Omega\times]0,T[\tag{1.2}$$

$u(x,t)$ is tangent to the boundary Γ of Ω for all

$$\{x,t\}\in\Gamma\times]0,T[\cdot \text{i.e. } u\cdot n|_\Gamma = 0. \tag{1.3}$$

Equation (1.1) is the governing equation of a macroscopic state variable of the flow, for example the temperature or the density, which was $\rho^0(x)$ at time $t=0$ and which is transported by the flow without diffusion; f is the external source per unit volume.

We recall that if $D\rho/Dt$ denotes the total derivative of ρ in the flow u then (1.1) is really

$$\frac{D\rho}{Dt} = f \tag{1.4}$$

$$\rho(t=0) = \rho^0$$

so that an explicit solution of (1.1) is

$$\rho(x,t) = \rho^0(X(x,t;0)) + \int_0^t f(X(x,t;\tau),\tau)\, d\tau \tag{1.5}$$

where $X(.) = X(x,t;.)$ is the solution of the ordinary differential equation:

$$\frac{dX}{d\tau} = u(X,\tau); \qquad X(t) = x. \tag{1.6}$$

When it is obvious we shall sometimes omit to write the parameters x,t in X. In (1.6) x and t are parameters while τ is the auxiliary time variable; if u is the

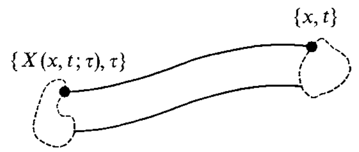

Fig. 1. The method of characteristics. To study the dispersion of a pollutant one can compute the trajectories of the flow particles and then deduce the shape of the polluted region at time t from its shape at time 0

vector field of a fluid then $X(x,t;\tau)$ is the position of a fluid molecule at time τ which will be in x at time t (see Fig. 1).

This result is precisely stated in the following proposition:

Proposition 1. Assume that u satisfies (1.2) and (1.3) and is uniformly lipschitz in x in Ω. Assume furthermore that ρ^0, u and f are in $L^\infty(\Omega)$, $L^\infty(\Omega\times]0,T[)$; then the solution of (1.1) satisfies (1.5), (1.6).

Several numerical methods are based upon this principle; the so called method of characteristics and to a certain extent the vortex methods. It was shown in [1] that the *methods of characteristics could be implemented in a finite element context by simply replacing u by one of its finite element approximation* u_h.

Of course one could integrate (1.6) by any numerical technique like the Runge-Kutta algorithm. But for our application it is easier to approximate u by u_h.

For example if u_h is piecewise constant in x and t on a finite element triangulation of $\Omega\times]0,T[$ then (1.6) has a computable exact solution and (1.5) allows the computation of the exact solution of

$$\frac{\partial\rho}{\partial t}+u_h\nabla\rho=f_h$$
$$\rho(t=0)=\rho_h^0 \tag{1.7}$$

where f_h *and* ρ_h^0 are any interpolates of f and ρ^0.

Indeed when u_h is piecewise constant in $\Omega\times]0,T[$ then X_h solution of

$$\frac{dX_h}{d\tau}=u_h(X_h,\tau); \qquad X_h(t)=x \tag{1.8}$$

is a broken straight line; i.e. $X_h(\tau)$ describes a straight segment inside each finite element when τ varies. In $\Omega\times]0,T[$ each element is a prism.

Therefore to compute $\rho(x,t)$ we get the following algorithm which assumes f_h *to be piecewise constant:*

Algorithm 1 (transport)

0. Set $\{x^0,t^0\}=\{x,t\}$; $i=0$ and $\rho=0$.
1. Find in which element K of $\Omega\times]0,T[$ lies $\{x^i,t^i\}$.
2. Compute $\mu<0$ such that $\{x^i,t^i\}+\mu\{u_{h|K},1\}$ belongs to ∂K.
3. Set $t^{i+1}=t^i+\mu$, $x^{i+1}=x^i+\mu u_{h|K}$. If $t^{i+1}>0$ set $\rho=\rho-f_h|_K\mu$ and go back to 1 with $i=i+1$ else set $\rho=\rho-f_h|_K\mu+\rho_h^0(x^{i+1})$ and stop.

If one wants a piecewise linear approximation of the solution of (1.1) at T then it suffices to apply algorithm 1 with each node of the triangulation of Ω for x and $t=T$. It is shown in [1] that such an algorithm is statistically non dissipative (the integral of ρ over Ω is preserved), unconditionally stable in h and Δt and of order $O(h)+O(\Delta t)$. Since this result is of interest for the understanding of the following methods we reproduce the essential part of the proof.

Proposition 2. Let $\rho(.,.)$ be the solution of (1.1) (i.e. ρ is given by (1.5). Let ρ_h be the solution of (1.7) (i.e. $\rho_h(x,t)$ can be computed by algorithm 1 for all $\{x,t\}\in Q$), then

$$|\rho_h(x,t)-\rho(x,t)|\leqq T|f_h-f|_{\infty,Q}+|\rho_h^0-\rho^0|_{\infty,\Omega}+c_1|u_h-u|_{\infty,Q}\,(|\nabla\rho^0|_{\infty,\Omega}+|\nabla f|_{\infty,Q}) \tag{1.9}$$

($|.|_{\infty,Q}$ stands for the sup norm on $Q=\Omega\times]0,T[$).

Proof. From (1.5) and step 3 of Algorithm 1 we have

$$\rho_h(x,t)-\rho(x,t)=\rho_h^0(X_h(x,t;0))-\rho^0(X(x,t;0))+\int_0^t(f_h(X_h(x,t;\tau),\tau)-f(X(x,t;\tau),\tau))\,d\tau.$$

Hence

$$|\rho_h(x,t)-\rho(x,t)|\leqq|\rho_h^0-\rho^0|_{\infty,\Omega}+|\nabla\rho^0|_{\infty,\Omega}|X_h(x,t;0)-X(x,t;0)|+\int_0^t|f_h(X_h(\tau))-f(X_h(\tau))|\,d\tau+|\nabla f|_{\infty,Q}\int_0^t|X_h(\tau)-X(\tau)|\,d\tau. \tag{1.10}$$

On the other hand according to (1.6) and (1.8)

$$\delta(\tau)=X_h(x,t;\tau)-X(x,t;\tau)$$

satisfies $\delta(t)=0$ and

$$|\dot{\delta}|=|u_h(X_h,\tau)-u(X,\tau)|\leqq|u_h-u|_{\infty,Q}+|\nabla u|_{\infty,Q}|\delta|$$

and from the Bellman-Gronwall lemma this yields

$$|\delta(\tau)|\leqq\frac{|u_h-u|}{|\nabla u|_{\infty,Q}}[\exp(|\nabla u|_{\infty,Q}(t-\tau))-1]. \tag{1.11}$$

Finally (1.10) and (1.11) imply (1.9). □

Important Remark. Proposition 1 does not insure that the solution of (1.7) exists when u_h is piecewise constant because the hypothesis of Lipschitz continuity is not met. And indeed this is seen in step 2 where μ may be zero if at the intersection of two elements both velocities point outward both elements. However if u_h derives from a stream function, or vector, ψ_h:

$$u_h=\nabla\times\psi_h \tag{1.12}$$

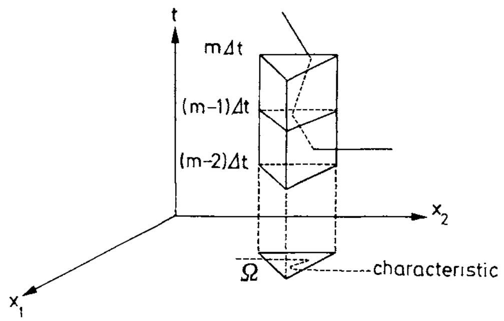

Fig. 2. Finite elements and characteristics. If Ω is divided into non overlapping triangles and $]0,T[$ is divided into intervals of length Δt then $\Omega\times]0,T[$ is triangulated by prisms and the characteristics are broken lines made of straight segments inside each prism

and if ϕ_h is piecewise linear in x and continuous then this trouble cannot happen; the normal component of u_h is then continuous at the interfaces of the elements. □

Comments. The main drawback of Algorithm 1 lies in the fact that it is not a marching technique which updates $\rho(x,t)$ to obtain $\rho(x,t+\Delta t)$. In [1] we have discussed a procedure which advances the characteristics both forward and backward in time without dissipation but it does not seem to apply to the transport diffusion equation, which is the goal of this paper.

Therefore, given H_h, an approximation of finite dimension of $L^2(\Omega)$, consider the following algorithm:

Algorithm 2 (transport+projection). Starting with $m=0$, compute $\rho_h^{m+1}\in \mathrm{H}_h$ from ρ_h^m by solving the linear system:

$$\int_\Omega \rho_h^{m+1}(x)\,\eta_h(x)\,dx=\int_\Omega \rho_h^m(X_h^m(x))\,\eta_n(x)\,dx + \int_\Omega \int_{m\Delta t}^{(m+1)\Delta t} f_h(X_h(x,(m+1)\Delta t;\tau),\tau)\,d\tau\,\eta_h(x)\,dx, \qquad \forall \eta_h\in H_h \tag{1.13}$$

where

$$X_h^m(x)=X_h(x,(m+1)\Delta t; m\Delta t)$$

where X_h is the solution of (1.8) with u_h *a piecewise constant approximation of* u, *satisfying* (1.12).

Comments. 1. Theoretically the second integral in (1.13) can be computed exactly, by computing the polyhedral shape of $X_h^m(T_j)$ for all j. In practice one will have to use a quadrature formula, (see Fig. 3).

2. If u_h satisfies (1.12) then with $\eta_h=1$ it is easy to see that (1.13) implies conservativity:

$$\int_\Omega \rho_h^{m+1}(x)\,dx=\int_\Omega \rho_h^m(x)\,dx \tag{1.14}$$

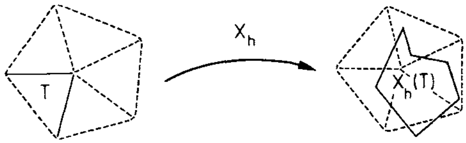

Fig. 3. A possible shape for the deformation of T under the transport field u_h

however conservativity may be lost once a quadrature formula is introduced in the right member of (1.13); it will be satisfied with the precision of the quadrature formula.

3. Algorithm 2 amounts to Algorithm 1 on the time interval $]m\Delta t\ (m+1)\Delta t[$ plus a H_h projection of the result $\rho_h^m(X_h(.,(m+1)\Delta t; m\Delta t))$, which of course is not in H_h.

Choices for H_h. Given a triangulation $\mathcal{T}_h = UT_i$ of Ω, made of triangles tetrahedra in R^3, reasonable choices for H_h include:

$$H_h^0 = \{\eta_h\colon \eta_h|_{T_i} \text{ is constant } \forall T_i\}. \tag{1.15}$$

$$H_h^1 = \{\eta_h \text{ continuous: } \eta_h|_{T_i} \text{ is affine } \forall T_i\}, \tag{1.16}$$

$$H_h^{1'} = \{\eta_h \text{ continuous at the middle of each side (resp. face) of } T_i \text{ only; } \eta_h|_{T_i} \text{ affine } \forall T_i\}, \tag{1.17}$$

$$H_h^2 = \{\eta_h \text{ continuous: } \eta_h|_{T_i} \text{ quadratic } \forall T_i\}. \tag{1.18}$$

Error Analysis. Before studying the problem of numerical quadrature let us find an error bound for algorithm 2.

Theorem 1. Assume that H_h has the following property:

$$\inf_{\eta_h \in H_h} \left(\int_\Omega |\eta - \eta_h|^2 \, ds\right)^{\frac{1}{2}} \leq \|\eta\| \, h^k \tag{1.19}$$

where $\|.\|$ is some (Sobolev) semi-norm.
Then Algorithm 2 gives:

$$\left(\int_\Omega |\rho_h^m(x) - \rho(x, m\Delta t)|^2 \, dx\right)^{\frac{1}{2}} \leq T|\nabla\rho|_{\infty,\Omega} |u_h - u|_{\infty,\Omega} + |\rho_h^0 - \rho^0|_0$$

$$+ \frac{h^k}{\Delta t} T \| \, \|\rho\| \, |_{\infty,]0,T[} + T|f_h - f|_{\infty,Q}, \ \forall m. \tag{1.20}$$

Corollary 1. With H_h^1 or $H_h^{1'}$ and u_h, f_h piecewise constant, Algorithm 2 is $O(h) + O(\Delta t) + O(h^2/\Delta t)$.

With H_h^2 and u_h, f_h piecewise bilinear in (x,t) it is $O(h^2) + O(\Delta t^2) + O(h^3/\Delta t)$.

With H_h^0 and u_h piecewise constant it is $O(h) + O(\Delta t) + O(h/\Delta t)$

Proof of Theorem 1. To make things simple assume first that $f = 0$. Then Π_H denotes the projection operator of $L^2(\Omega)$ on H_h, i.e.

$$\Pi_H \eta = \arg\min_{\substack{\eta_h \in H_h \\ |\eta_h|_0 = 1}} |\eta - \eta_h|_0^2 \tag{1.21}$$

or

$$\int_\Omega (\Pi_H \eta - \eta)\varphi_h \, dx = 0 \qquad \forall \varphi_h \in H_h$$

(recall our notations:

$$|g|_0 = (\int_\Omega g^2 \, dx)^{\frac{1}{2}}). \tag{1.22}$$

With $f=0$, Algorithm 2 is simply

$$\rho_h^{m+1}(.) = \Pi_H \rho_h^m(X_h^m(.)). \tag{1.23}$$

So if $\rho^m(.)$ denotes $\rho(., m\Delta t)$, knowing that $\rho^{m+1}(x) = \rho^m(X^m(x))$ where X is defined by (1.6), we have:

$$|\rho_h^{m+1} - \rho^{m+1}|_0 \leqq |\Pi_H[\rho_h^m(X_h^m) - \rho^m(X^m)]|_0 + |\Pi_H \rho^m(X^m) - \rho^m(X^m)|_0. \tag{1.24}$$

The last term is bounded by using (1.19). For the first term, since Π_H is a contraction we have:

$$\begin{aligned} |\Pi_H[\rho_h^m(X_h^m) - \rho^m(X^m)]|_0 &\leqq |\rho_h^m(X_h^m) - \rho(X^m)|_0 \\ &\leqq |\rho_h^m(X_h^m) - \rho^m(X_h^m)|_0 + |\rho^m(X_h^m) - \rho^m(X^m)|_0. \end{aligned}$$

The last term above is bounded as usual by $|\nabla \rho|_{\infty,Q} |X_h - X|_{\infty,Q}$ hence also by

$$|\nabla \rho|_{\infty,Q} |u_h - u|_{\infty,Q} \Delta t$$

while the first one is equal to $|\rho_h^m - \rho^m|_0$ because of the fundamental formula:

$$\int_\Omega g(X_h(x,t;\tau))\, dx = \int_\Omega g(X_h)\, dX_h. \tag{1.25}$$

Indeed X_h satisfies (1.8) and u_h satisfies (1.12) therefore the Jacobian of the transformation $x \to X_h(x)$ is equal to 1. For the same reason we also have

$$\int_\Omega g(X_h(x,t;\tau) f(x)\, dx = \int_\Omega g(y) f(X_h(y,\tau;t)\, dy. \tag{1.26}$$

All the pieces gathered, (1.20) is obtained after summing for all m.

If $f \neq 0$ then by using (1.26) we rewrite (1.13) as

$$\int_\Omega \rho_h^{m+1} \eta_h \, dx = \int_\Omega \rho_h^m \eta_h(\bar{X}_h^m(x))\, dx + \int_\Omega \int_{m\Delta t}^{(m+1)\Delta t} f_h(X_h(x,(m+1)\Delta t); t)\eta_h \, dx\, dt \tag{1.27}$$

where

$$\bar{X}_h^m(x) = X_h(x, m\Delta t; (m+1)\Delta t) \tag{1.28}$$

then we define (duality argument)

$$\eta_h^{i+1} = \Pi_h \eta_h^i(X_h(x,(m-i)\Delta t; (m+1-i)\Delta t))$$

so that (1.27) now yields

$$\int_\Omega \rho_h^m \eta_h\,dx = \int_\Omega \rho_h^0 \eta_h^m\,dx + \sum_{n=0}^{m} \int_\Omega \int_{n\Delta t}^{(n+1)\Delta t} f_h(X_h(x,(m+1)\Delta t);t)\,\eta_h\,dt\,dx.$$

From there it is easy to get an error estimate by the above method applied on η_h (see [9]).

Remark. The behavior in $\Delta t + h^2/\Delta t$ was numerically observed in [14]. It implies the existence of an optimal Δt of the order of h. □

An interesting phenomenon occurs for the choice (1.17): the linear system (1.13) is diagonal. Indeed if q^k denotes the k^{th} middle node and η_h^k the corresponding basis function, i.e.:

$$\eta_h^k(q^l) = \delta_{kl} \quad \forall k, l; \quad \eta_h^k \in H_h^{1'}, \tag{1.29}$$

then

$$\rho_h^{m+1}(q^k) = \frac{3}{s_k}\left[\int_\Omega \rho_h^m(X_h^m(x))\,\eta_h^k(x)\,dx + \int_\Omega \int_{m\Delta t}^{(m+1)\Delta t} \eta_h^k f_h(X_h)\,d\tau\,dx\right] \tag{1.30}$$

where s_k is the area (resp. volume) of the elements of $\mathcal{T}_h$ which contains q^k (see Fig. 4).

Mass lumping can also be used on (1.13) to obtain an explicit scheme with H_h^1. Then:

$$\rho_h^{m+1}(q^i) = \frac{3}{\sigma^i} \int_\Omega \rho_h^m(X_h^m(x))\,\eta_h^i(x)\,dx + \frac{3}{\sigma^i} \int_\Omega \int_{m\Delta t}^{(m+1)\Delta t} f_h(X_h)\,\eta_h^i(x)\,dx\,d\tau \tag{1.31}$$

for all vertex q^i. There η_h^i is the basis function corresponding to q^i and σ^i is the support of η_h^i (the set of x where $\eta_h^i(x) \neq 0$). However with (1.31) the scheme is no longer conservative. An error estimate can be obtained for (1.31). It can be shown that the scheme is of order $h + \Delta t + h^2/\Delta t$ as long as the grid is regular in the sense that

$$\sigma_i q^i = \sum_{\substack{q^k \text{ mid-node} \\ \text{neighbor to } q^i}} q^k s_k/2. \tag{1.32}$$

However it is not clear whether this condition is purely technical or really necessary (see [14]).

If mass lumping is used in both sides of (1.13) then again convergence and similar error estimates can be shown (see [14]).

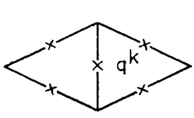

Fig. 4. Degree of freedom of $\rho_h \in H_h^{1'}$ according to (1.17). S_k is the area of the two triangles

Numerical Quadrature

The effect of numerical quadrature on the second integral in (1.13) is to replace the operator Π_H by an approximate projection operator $\tilde{\Pi}_H$.

From the error analysis point of view (see (1.24)) we need that

$$|(\Pi_H - \tilde{\Pi}_H)\,\rho|_0 \leqq C(\rho)\,h^2. \tag{1.33}$$

If a Gauss quadrature formula is used with coefficients ω_j and nodes x^j this means that:

$$\sup_{\eta_h \in H_h} |\int_\Omega \rho\eta_h\,dx - \Sigma\omega_j\,\rho(x^j)\,\eta_h(x^j)|/|\eta_h|_0 \leqq C(\rho)\,h^2. \tag{1.34}$$

This is achieved with a quadrature formula which is exact for polynomials of degree 4. However simpler quadrature formulas, for example with the vertices only, may not to destroy the rate of convergence. To retain conservativity it is better to use (1.26) first and then use the quadrature. The effect of numerical quadrature is studied in [14].

Numerical Tests: a Monodimensional Example

We shall now consider the case

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} = 0 \ \text{ in }]-2, +2[\times]0,1[\\ &\rho(t=0) = \rho^0(x) \quad \text{if } x<0 \\ &\qquad\qquad\quad 0 \qquad\ \ \text{if } x>0;\, u=0.75. \end{aligned} \tag{1.35}$$

The exact solution is

$$\begin{aligned} \rho(x,t) &= \rho^0(x+ut) \quad \text{if } -1<x+ut<0 \\ &= \rho^0(0) \qquad\quad \text{if } x+ut>0 \ \text{ and } \ \rho^0(-1) \ \text{ if } \ x+ut<-1. \end{aligned} \tag{1.36}$$

Since the velocity is constant there is no error in the computations of the characteristics and Algorithm 1 simply gives the interpolate of the exact solution (see Fig. 5); (we maintain $\rho(1,t)=1$ to simulate artificially a boundary layer). Algorithm 2 with H_h^1 takes m times the least square fit of the previous solution over a grid which is translated by $u\Delta t$. If $u\Delta t = h$ of course there is no error but otherwise the curve gradually broadens (see Fig. 6). Naturally a similar behavior is to be expected when H_h^2 is chosen. Notice that this scheme does not like shocks. When $H_h^{1'}$ is taken then the broadening of the hump is not so great because of (1.30); outside $\{x: -1<x+ut<0\}$ ρ_h is exactly zero (see Fig. 7). Shocks are also handled better (Fig. 8). It seems from these calculations that $H_h^{1'}$ is better than H_h^1 when ρ is not regular.

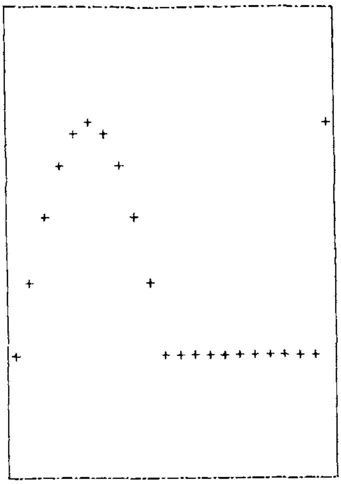
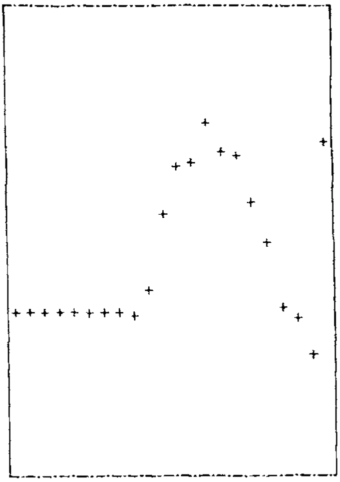

Fig. 5. Non conforming linear discontinuous least square interpolate of the initial condition $\rho^0(x) = \sin(\pi(x+1))$, $x<0$, $\rho^0(x)=0$, $x>0$ $\rho^0(1)=1$

Fig. 6. Solution at $T=1$ of (1.35) with the initial condition of Fig. 5 $\Delta t=0.1$ (10 time steps), with algorithm 2 and conforming linear elements. There is no L^∞ damping but the sin-wave is deformated the $\rho(1,t)=1$ creates also an oscillation in the projected curve. This element does not handle shocks very well. The integrals are computed exactly

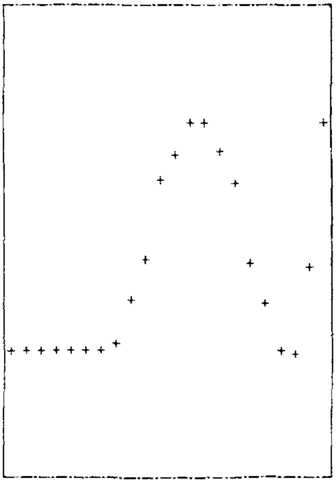
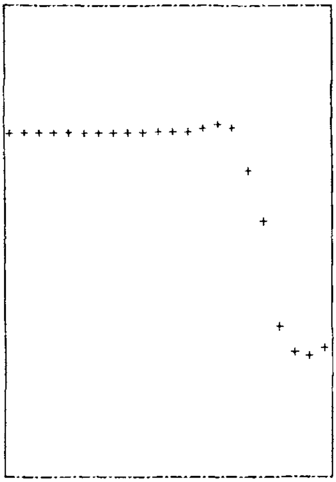

Fig. 7. Same calculation than for Fig. 6 but with linear non conforming elements with 2 nodes per element at distance $h/3$ and $2h/3$. Notice that again there is no L^∞ damping and that the simulated boundary layer does not creates any oscillation

Fig. 8. Solution at $T=1$ of (1.35) with $\rho^0(x)=1$, $x<0$ (i.e. the shock curve in the middle of the figure) with $\Delta t=0.1$. The shock is broadened but with almost no oscillations

2. The Linear Transport-Diffusion Equation

We shall now study the generalization of the previous methods to the linear transport-diffusion equations:

$$\frac{\partial \rho}{\partial t}+u\nabla\rho-\nu\Delta\rho=f \text{ in } \Omega\times]0,T[$$
$$\rho(t=0)=\rho^0, \quad \rho|_\Gamma=\rho_\Gamma. \tag{2.1}$$

As before we assume that

$$\nabla\cdot u=0 \text{ in } \Omega\times]0,T[;\ u\cdot n|_\Gamma=0. \tag{2.2}$$

In general u is the velocity field of an incompressible fluid and ρ can be its temperature or the concentration of a pollutant... Assumptions (2.2) are not essential so long as $u\nabla\rho$ is replaced by $\nabla\cdot(\rho u)$ but they make the analysis simpler.

Equation (2.1) is also:

$$\frac{D\rho}{Dt}-\nu\Delta\rho=f. \tag{2.3}$$

We have seen from (1.5) that $\frac{D\rho}{Dt}$ really means $\frac{\partial}{\partial\tau}\rho(X(x,t;\tau),\tau)|_{\tau=t}$. Therefore if we discretize (2.2) in time with a purely implicit scheme, we obtain a formula to compute ρ^{m+1} from ρ^m:

$$\frac{\rho^{m+1}(x)-\rho^m(X^m(x))}{\Delta t}-\nu\Delta\rho^{m+1}(x)=f^{m+1}(x) \tag{2.4}$$

where $X(x,t;\tau)$ is the solution of

$$\frac{dX}{d\tau}=u(X,\tau);\ X(x,t;t)=x;$$

and where:

$$X^m(x)=X(x,(m+1)\Delta t;m\Delta t). \tag{2.5}$$

Written in variational form (2.4) becomes:

$$\frac{1}{\Delta t}(\rho^{m+1},\eta)+\nu(\nabla\rho^{m+1},\nabla\eta)=\frac{1}{\Delta t}(\rho^m(X^m(.),\eta)+(f^{m+1},\eta), \quad \forall\eta\in H_0^1(\Omega), \tag{2.6}$$

$$\rho^{m+1}-\rho_\Gamma\in H_0^1(\Omega);\ \rho^0 \text{ given}, \tag{2.7}$$

where $(.,.)$ denotes the $L^2(\Omega)$-scalar product:

$$(f,\eta)=\int_\Omega f(x)\eta(x)\,dx \tag{2.8}$$

and $H_0^1(\Omega)$ is the usual Sobolev subspace of functions which are zero on Γ.

Finite Element Approximation of (2.6), (2.7)

If we denote again by H_h one of the spaces defined in (1.16), (1.18) then

$$H_{0h}=\{\eta_h \in H_h:\ \eta_h(q^k)=0 \quad \forall q^k \text{ node corresponding to a degree of freedom on } \Gamma\}. \tag{2.9}$$

Now (2.6) is approximated by

$$\frac{1}{\Delta t}(\rho_h^{m+1}, \eta_h)+\nu(\nabla\rho_h^{m+1}, \nabla\eta_h)=\frac{1}{\Delta t}(\rho_h^m(X_h{}^m(.)), \eta_h)+(f_h^{m+1}, \eta_h) \quad \forall \eta_h \in H_{0h} \tag{2.10}$$

$$\rho_h^{m+1}-\rho_{\Gamma h}\in H_{0h};\ \rho_h^0, \rho_{\Gamma_h} \text{ given approximations of } \rho^0, \rho_\Gamma \tag{2.11}$$

where X_h is the solution of (2.5) with u replaced by an approximation u_h (in general piecewise constant in x and t). If (1.17) is used (non conforming element), $(\nabla\rho_h^{m+1}\, \nabla\eta_h)$ is taken to be $\sum_i \int_{T_i} \nabla\rho_h^{m+1}\, \nabla\eta_h\, dx$.

Comments. Naturally the total derivative in (2.3) could have been discretized either by an explicit scheme or by a Crank-Nicholson scheme. From the results of the last paragraph we can already see that the following combinations are suitable:

- with (2.10) H_{0h} should be constructed from (1.16) or (1.17),
- with an explicit scheme in time (1.17) makes everything explicit,
- with a Crank-Nicholson scheme (1.18) should be chosen. However with (1.17) and a u_h piecewise linear in t the error might be of order $h^2+\Delta t^2+h^2/\Delta t$ thereby allowing bigger time steps ($\Delta t \cong h_{0.6}$).

Error Analysis and Stability

By letting $\eta_h=\rho_h^{m+1}$ in (2.10) we find that

$$(|\rho_h^{m+1}|_0^2+\nu\Delta t|\nabla\rho_h^{m+1}|_0^2)^{\frac{1}{2}} \leqq |\rho_h^m|_0+|f_h^{m+1}|_0\,\Delta t. \tag{2.12}$$

Therefore the method is *unconditionally stable* in the L^2 sense. As ν tends to zero it is also *conservative* because it becomes the algorithm of paragraph 1. Let us show that it has the same error estimates:

Theorem 2. The method (2.9)–(2.11) is of order $h+\Delta t+h^2/\Delta t$ when H_h^1 or $H_h^{1'}$ are selected. More precisely if $\rho_\Gamma=0$, ρ_h^m is computed by (2.10) and ρ^m is computed by (2.4), then:

$$\begin{aligned}(|\rho_h^m-\rho^m|_0^2+\nu\Delta t|\nabla(\rho_h^m-\rho^m)|_0^2)^{\frac{1}{2}} \leqq &\sum_{n=1}^{m}\{\inf_{\eta_h\in H_h}(|\rho^n-\eta_h|_0^2+\nu\Delta t|\nabla(\rho^n-\eta_h)|_0^2)^{\frac{1}{2}}\}\\ &+(|\rho_h^0-\rho^0|_0^2+\nu\Delta t|\nabla(\rho_h^0-\rho^0)|_0^2)^{\frac{1}{2}}+T|f_h^n-f^n|_0\\ &+\sup_n\{|\nabla_e^n|_\infty|u_h-u|_\infty T\}+c\frac{h^2}{\Delta t}\|\rho^n\|_{2,\infty}\end{aligned} \tag{2.13}$$

($c=0$ with H_h^1).

Proof. Let $\delta^m = \rho_h^m - \rho^m$, and assume H_h^1 is used.

By subtraction of (2.4), multiplied by η_h and integrated on Ω, from (2.10), we find that (see (1.28) for the definition of $\bar{X}_h(.)$):

$$\left|\frac{1}{\Delta t}(\delta^{m+1}, \eta_h) - \frac{1}{\Delta t}(\delta^m, \eta_h(\bar{X}_h^m(.)) + \nu(\nabla\delta^{m+1}, \nabla\eta_h)\right| \leqq [|f_h^{m+1} - f^{m+1}|_0 + |\nabla\rho|_{\infty,Q}|u_h - u|_{\infty,Q}]\,|\eta_h|_0; \tag{2.14}$$

where $|\nabla\rho|_{\infty,Q} = \sup|\nabla\rho^m|_{\infty,\Omega}$ can be shown to be bounded by differentiation of (2.4) and the maximum principle.

The second term has been transformed by (1.26); $\frac{1}{\Delta t}(\rho^m, \eta_h(\bar{X}_h^m(.))$ has been added and subtracted. Define η_h^{m+1} by

$$\frac{1}{\Delta t}(\eta_h^{m+1}, \xi_h) + \nu(\nabla\eta_h^{m+1}, \nabla\xi_h) = \frac{1}{\Delta t}(\delta^{m+1}, \xi_h) + \nu(\nabla\delta^{m+1}, \nabla\xi_h) \quad \forall\xi_h \in H_{0h}.$$

Then from (2.14) and (1.25)

$$\frac{1}{\Delta t}|\eta_h^{m+1}|_0^2 + \nu|\nabla\eta_h^{m+1}|_0^2 \leqq \left[\frac{1}{\Delta t}|\delta^m|_0 + |f_h^{m+1} - f^{m+1}|_0 + |\nabla\rho|_{\infty,Q}|u_h - u|_\infty\right]|\eta_h^{m+1}|_0.$$

Therefore

$$(|\eta_h^{m+1}|_0^2 + \Delta t\,\nu|\nabla\eta_h^{m+1}|_0^2)^{\frac{1}{2}} \leqq (|\delta^m|_0^2 + \Delta t\,\nu|\nabla\delta^m|_0^2)^{\frac{1}{2}} + \Delta t(|f_h^{m+1} - f^{m+1}|_0 + |\nabla\rho|_\infty|u_h - u|_\infty). \tag{2.16}$$

Now from (2.15) and the definition of δ^m we find that

$$\eta_h^{m+1} = \rho_h^{m+1} - \tilde{\Pi}_H\,\rho^{m+1}(.) = \delta^{m+1} + \rho^{m+1}(.) - \tilde{\Pi}_H\,\rho^{m+1}(.)$$

where $\tilde{\Pi}_H$ is the projector on H_{0h} with respect to the norm

$$(|.|_0^2 + \nu\Delta t|\nabla.|_0^2)^{\frac{1}{2}} = \|.\|_1. \tag{2.17}$$

So we find that (2.16) yields

$$(|\delta^{m+1}|_0^2 + \nu\Delta t|\nabla\delta^{m+1}|_0^2)^{\frac{1}{2}} \leqq \|\rho^m - \tilde{\pi}_H\,\rho^m\|_1 + (|\delta^m|_0^2 + \nu\Delta t|\nabla\delta^m|_0^2)^{\frac{1}{2}} + \Delta t(|f_h^{m+1} - f^{m+1}|_0 + |\nabla\rho|_\infty|u_h - u|_\infty).$$

Now it remains to sum for all m to complete the proof. The second assertion in the theorem comes from the identity

$$\|\rho^{m+1} - \tilde{\pi}_H\,\rho^{m+1}\|_1 = \inf_{\eta_h \in H_{0h}} \|\rho^{m+1} - \eta_h\|_1$$

when $H_h^{1'}$ is used we have an additional term in the left hand side of (2.14) because

$$(-\Delta\rho^{m+1},\eta_h)=(\nabla_h\rho^{m+1},\nabla_h\eta_h)+\sum_i\int_{\partial T_i}\frac{\partial\rho^{m+1}}{\partial n}\eta_h\,d\Gamma$$

and the last term is not zero if η_h is discontinuous; $\{\partial T_i\}_i$ are the boundaries of the elements of the triangulation.

Define a new interpolation π_h' by

$$(\nabla_h\pi_h'\rho,\nabla_h\eta_h)=(\nabla_h\rho,\nabla_h\eta_h)-\sum_i\int_{\partial T_i}\frac{\partial\rho}{\partial h}\eta_h\,d\Gamma\quad\forall\eta_h\in H_{0h}^{1'}\quad\pi_h'\in H_{0h}^{1'}.$$

Then if $\delta^{m+1}=\rho_h^{m+1}-\pi_h'\rho^{m+1}$, we now have

$$\frac{1}{\Delta t}(\delta^{m+1},\eta_h)+\nu(\nabla_h\delta^{m+1},\nabla_h\eta_h)=\frac{1}{\Delta t}(\rho_h^m(X_h^m(.))-\rho^m(X^m(.)),\eta_h)$$

$$+(f_h^{m+1}-f^{m+1},\eta_h)+\frac{1}{\Delta t}(\rho^{m+1}-\pi_h'\rho^{m+1},\eta_h)$$

which, in turn, yields

$$\|\delta^{m+1}\|_1\leqq\|\delta^m\|_1+\Delta t|f_h^{m+1}-f^{m+1}|_0+|\rho^{m+1}-\pi_h'\rho^{m+1}|_0$$

$$+|\rho^m-\pi_h'\rho^m|_0+|\rho^m(X_h^m(.))-\rho^m(X^m(.))|.$$

Now it can be shown (see [17]) that $|\rho-\pi_h'\rho|_0\leqq c\|\rho^0\|_2h^2$.

Thus the rest of the proof is the same except for this additional term which produces at the end an extra term in the right hand side of (2.13): $\|\rho\|_{2,\infty}ch^2/\Delta t$.

Remark. 1. Theorem 2 does not give a good bound on $|\nabla(\rho_h^m-\rho^m)|_0$ when $\nu\neq0$. From (2.15) one can in fact obtain also

$$(\Sigma\Delta t|\nabla(\rho_h^m-\rho^m)|_0^2)^{\frac{1}{2}}\leqq\frac{1}{\sqrt{\nu}}c(h+\Delta t+h^2/\Delta t).$$

2. The error estimates is produced from (2.13) provided we show that $|\rho^m-\rho|_0$ is $O(\Delta t)$ and $|\nabla\rho|_\infty$ is bounded for all m. Such results are easy to obtain from (2.4).

Practical Implementations. Therefore the same type of quadrature formula must be used: a Gauss formula of precision h^3 for example.

The error analysis also tells us that the time step Δt which gives the least error is

$$\Delta t\cong2h|\nabla(\nabla\rho)|_\infty/\left(\left|\frac{\partial f}{\partial t}\right|_\infty+|\nabla\rho|_\infty\left|\frac{\partial u}{\partial t}\right|_\infty\right).\qquad(2.18)$$

However for very small ν since every time step has a smoothing effect one may prefer to use large time step with perhaps a higher order approximation for H_h.

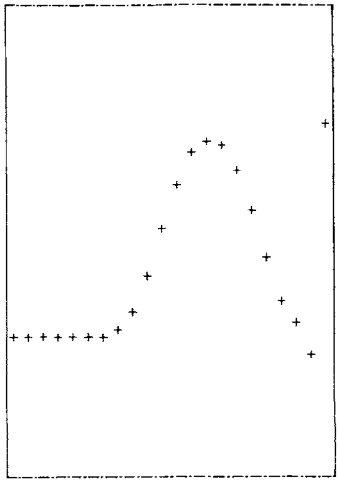

Fig. 9. Solution of (2.22) at $T=1$, after 10 iterations ($\Delta t=0.1$) for $\nu=0.01$. Conforming linear elements are used ($h=0.1$)

Upwinding effect. Let us apply the mean value theorem to the right hand side of (2.10): there exists $\xi(x)$ such that

$$\rho_h^m(X_h^m(x)=\rho_h^m(x)+\tilde{\nabla}\rho_h^m(\xi(x))(X_h^m(x)-x) \tag{2.19}$$

the ˜indicates a subgradient because ρ_h is not in $C^1(\Omega)$ (see Ekland [8]); and from (2.5) there exists θ such that

$$X_h^m-x=u_h(X_h(x,(m+1)\Delta t;\theta),\theta)\Delta t;\ m\Delta t<\theta<(m+1)\Delta t. \tag{2.20}$$

Therefore (2.10) can be rewritten as

$$\frac{1}{\Delta t}(\rho_h^{m+1},\eta_h)+\nu(\nabla\rho_h^{m+1},\nabla\eta_h)=(f_h^{m+1},\eta_h)+\frac{1}{\Delta t}(\rho_h^m,\eta_h)$$
$$+(\tilde{\nabla}\rho_h^m(\xi(x))\,u_h(X_h(\theta(x)),\eta_h)\qquad\forall\eta_h\in H_{0h}. \tag{2.21}$$

Since $\xi(x)$ lies between $X_h(x,(m+1)\Delta t;m\Delta t)$ and x it means that (2.21) is an upwinded finite element semi-implicit approximation.

Compared with the usual upwinded schemes we have the following advantage:
- stability with *symmetric* linear systems,
- optimal error estimates.

Numerical Tests: a Monodimensional Example

As in Paragraph 1 we have tested the method on

$$\frac{\partial\rho}{\partial t}+u\nabla\rho-\nu\Delta\rho=0\ \text{ in }\ ]-1,+1[\times]0,1[;\ u=0.75.$$

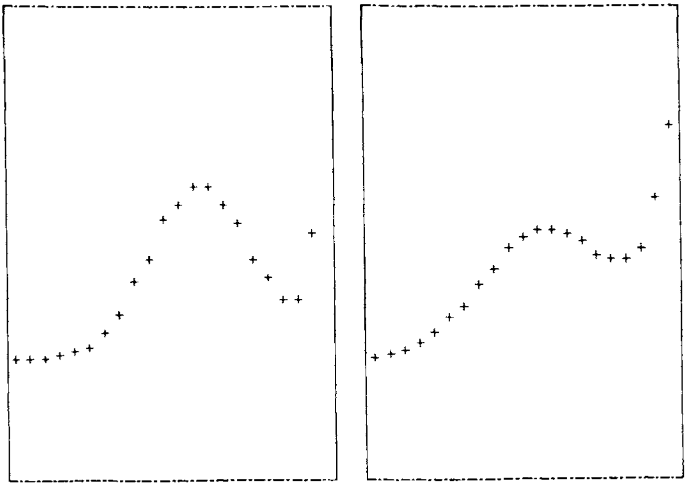

Fig. 10. Same on Fig. 9 but with the non conforming linear discontinuous element of Fig. 7; $\nu = 0.02$ and 0.05. Since the element is discontinuous, $-\nu\Delta\rho$ has been approximated by a finite difference formula

$$\begin{aligned} \rho(x,0) &= \sin \Pi(x+1) \quad \text{if } x \leqq 0 \\ &= 0 \text{ otherwise;} \\ \rho(-1,t) &= 0, \ \rho(1,t) = 1 \end{aligned} \tag{2.22}$$

For small values of ν this problem has a boundary layer at $x = 1$.

Both approximations (linear continuous element and linear discontinuous element) give satisfactory results: no oscillation, reasonable numerical dissipation (see paragraph 1) and a proper treatment of the boundary layer.

The discontinuous element seems to be more dissipative at equal number of degrees of freedom (which implies h (discontinuous) $= 2h$ (continuous)).

3. Application to the Navier-Stokes Equations

Although much more complex and non linear, the Navier-Stokes equations are of the transport-diffusion type:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u \nabla u - \nu \Delta u = f - \nabla p \\ &\nabla \cdot u = 0 \end{aligned}\right\} \quad \text{in } \Omega \times]0, T[$$
$$u(t=0) = u^0; \ u|_\Gamma = 0. \tag{3.1}$$

Indeed in variational form (3.1) can also be rewritten as

$$\begin{aligned} &\left(\frac{Du}{Dt}, v\right) + \nu(\nabla u, \nabla v) = (f, v) \qquad \forall v \in J_0(\Omega) \\ &u(\cdot, t) \in J_0(\Omega) = \{v \in H^1(\Omega)^n; \ \nabla \cdot v = 0 \quad \text{in} \quad \Omega; \ v|_\Gamma = 0\}. \end{aligned} \tag{3.2}$$

As before D/Dt is the total derivative in the direction u, $(.,.)$ is the scalar product of $L^2(\Omega)^m$ ($m=1, n$ or n^2 and n is the dimensionality of the flow ($n=2$ or 3).

We shall discretize in time by a purely implicit scheme, as before, even though in practice one may prefer a Crank-Nicholson scheme; for theoretical considerations it is simpler and it has the same difficulties as the second scheme.

So (3.2) is approximated in time by

$$\frac{1}{\Delta t}(u^{m+1}, v) + \nu(\nabla u^{m+1}, \nabla v) = (f^{m+1}, v) + \frac{1}{\Delta t}(u^m(X^m(.), v) \qquad \forall v \in J_0(\Omega)$$
$$u^{m+1} \in J_0(\Omega) \qquad (3.3)$$

where $X^m(x) = X(x, (m+1)\Delta t; m\Delta t)$ is the solution of

$$\frac{dX}{d\tau} = u^m(X), \qquad X(x, t; t) = x. \qquad (3.4)$$

It is important to stress the good behavior of this algorithm:

1. each time iteration amounts to a *Stokes problem* plus a transport of the previous solution on the characteristics. Therefore the *linear systems* will be *symmetric.*

2. the method is *unconditionally stable.* Indeed letting $v = u^{m+1}$ in (3.3) gives

$$(|u^{m+1}|_0^2 + \nu\Delta t|\nabla u^{m+1}|_0^2)^{\frac{1}{2}} \leqq |f^{m+1}|_0 \Delta t + |u^m|_0$$
$$\leqq |f^{m+1}|_0 \Delta t + (|u^m|_0^2 + \nu\Delta t|\nabla u^m|_0^2)^{\frac{1}{2}}. \qquad (3.5)$$

Spatial Discretization with a Divergent Free Element

To discretize in space we shall begin with the method of Crouzeix-Raviart [6], Thomasset [16], extended to the $3-D$ case by Hecht [11].

Let

$$J_{0h} = \{v_h : v_h|_T \text{ linear; } v_h \text{ continuous at the mid-nodes;}$$
$$\nabla \cdot v_h|_T = 0, \forall T;\ v_h(q^k) = 0 \text{ if } q^k \text{ is a mid-node of } \Gamma\}. \qquad (3.6)$$

The mid-nodes are the middles of each side of triangles in R^2 and of each face of tetrahedras in R^3. J_{0h} is a nonconformal approximation of $J_0(\Omega)$ of order 1. Basis functions of J_{0h} are rather difficult to construct, but it is more a theoretical difficulty than a practical one: the Fortran codes are not complex (see [11]).

Then (3.3) is approximated by

$$\frac{1}{\Delta t}(u_h^{m+1}, v_h) + \nu(\nabla_h u_h^{m+1}, \nabla_h v_h)$$
$$= (f^{m+1}, v_h) + \frac{1}{\Delta t}(u_h^m(X_h^m(.), v_h) \qquad \forall v_h \in J_{0h} \qquad (3.7)$$
$$u_h^{m+1} \in J_{0h};\ u_h^0 \text{ given in } J_{0h}.$$

where $\nabla_h u_h(x)$ is $\nabla u_h(x)$ if $x \in T_i$ and 0 at the discontinuities of u_h.

To make X_h^m computable we define $\tilde{u}_h^m$, piecewise constant in x, by

$$
\begin{aligned}
&(\nabla \times \Psi_h^m, \nabla \times \xi_h) = (u_h^m, \nabla \times \xi_h) \qquad \forall \xi_h \in H_h^1,\ \xi_h \times n|_\Gamma = 0;\\
&\qquad\qquad \Psi_h^m \in H_h^1 \text{ (see (1.16))},\ \Psi_h \times n|_\Gamma = 0;\\
&\tilde{u}_h^m = \nabla \times \psi_h^m;\\
&\frac{dX_h}{d\tau} = \tilde{u}_h^m(X_h(\tau)),\ X_h(x,t;t) = x; \text{ and} \qquad (3.8)\\
&X_h^m(x) = X_h(x,(m+1)\Delta t; m\Delta t).
\end{aligned}
$$

Naturally in practice one would have to use a quadrature formula for the last terms in (3.7) (say a Gauss formula of precision h^3 after having used (1.26)).

Remark 1. Although (3.8) defines $\tilde{u}_h^m$ uniquely, Ψ_h^m is not unique. Numerically one must either add a penalty term or take the (unique) determination that satisfies (see [4]).

$$
\begin{aligned}
&(\nabla \Psi_h^m, \nabla \xi_h) = (u_h^m, \nabla \times \xi_h) \qquad \forall \xi_h \in H_h'; \xi_h \times n = 0 \text{ on } \Gamma\\
&\Psi_h^m \times n = 0 \text{ on } \Gamma.
\end{aligned}
$$

This problem is easier, Ψ_h^m is unique (and $\tilde{u}_h^m \cdot n = 0$ on Γ also).

Stability and Error Analysis

Proposition 3. Scheme (3.7) is unconditionnaly stable:

$$
(|u_h^{m+1}|_0^2 + \Delta t \nu |\nabla_h u_h^{m+1}|_0^2)^{\frac{1}{2}} \leq \Delta t |f^{m+1}|_0 + (|u_h^m|_0^2 + \nu \Delta t |\nabla_h u_h^m|_0^2)^{\frac{1}{2}} \qquad (3.9)
$$

Proof. As above. □

Note that even when ν is very small, (3.7) has a unique solution. Convergence and error analysis for all values of ν is somewhat hopeless since it would constitute also an existence (and perhaps uniqueness) proof of solution of Euler's equations in R^3, (a hard problem in itself).

Therefore let us only give arguments about error estimates which assume existence and uniqueness and regularity of the Euler and Navier-Stokes equations.

If we follow exactly the proof of Theorem 2 of previous paragraph we come to

$$
\begin{aligned}
&(|u_h^{m+1} - u^{m+1}|_0^2 + \nu \Delta t |\nabla_h (u_h^{m+1} - u^{m+1})|_0^2)^{\frac{1}{2}} \leq (h^2 + \nu h \Delta t) |\nabla(\nabla u^m)|_\infty\\
&+ |u^m(X_h^m) - u^m(X^m)|_0 + |u_h^m - u^m|_0 + |f_h^{m+1} - f^{m+1}|_0 \Delta t + c \|u^m\|_{2,\infty} h^2
\end{aligned} \qquad (3.10)
$$

where the third term in the right was bounded by $|\nabla u|_\infty |u_h - u|_\infty \Delta t$ in the previous analysis.

This estimate is not sufficient in our case we must proceed more carefully.

By using the mean value theorem we get as before:

$$|u^m(X_h(.,(m+1)\Delta t); m\Delta t)) - u^m(X(.,(m+1)\Delta t); m\Delta t))|_0 \leq |\nabla u|_{\infty,Q}|X_h^m - X^m|_0 \tag{3.11}$$

where u^m is the solution of (3.3), and $|\nabla u|_{\infty,Q} = \sup_m |\nabla u^m|_\infty$. Now by definition of X_h and X (see (3.8) and (3.4)):

$$\frac{d}{d\tau}|X_h - X|(\tau) \leq |\tilde{u}_h^m(X_h(\tau)) - u^m(X_h(\tau))| + |\nabla u|_{\infty,Q}|X_h - X|(\tau); \tag{3.12}$$

therefore

$$|X_h^m - X^m|(x) \leq e^{|\nabla u|_\infty \Delta t} \int_{m\Delta t}^{(m+1)\Delta t} |\tilde{u}_h^m(X_h(\tau)) - u^m(X_h(\tau))|\, d\tau. \tag{3.13}$$

Hence

$$\int_\Omega |X_h^m - X^m|^2\, dx \leq e^{2|\nabla u|_\infty \Delta t} \int_\Omega \left(\int_{m\nabla t}^{(m+1)\Delta t} |\tilde{u}_h^m(X_h(\tau)) - u^m(X_h(\tau))|\, d\tau \right)^2 dx. \tag{3.14}$$

Recall that the Schwartz inequality implies that

$$\int_a^b g\, dx \leq \sqrt{b-a} \left(\int_a^b g^2\, dx \right)^{\frac{1}{2}}; \text{ therefore}$$

$$|X_h^m - X^m|_0^2 \leq \Delta t\, e^{2|\nabla u|_\infty \Delta t} \int_{m\Delta t}^{(m+1)\Delta t} [|\tilde{u}_h^m(X_h(\tau)) - u^m(X_h(\tau))|_0^2]\, d\tau.$$

Fortunately $X_h(\tau)$ can be replaced by x as in (1.25) because $\nabla \cdot \tilde{u}_h^m = 0$ so we find that

$$|X_h^m - X^m|_0 \leq \Delta t\, e^{|\nabla u|_\infty \Delta t} |\tilde{u}_h^m - u^m|_0. \tag{3.15}$$

If we show that

$$|\tilde{u}_h^m - u^m|_0 \leq |u_h^m - u^m|_0 + c|\nabla u|_\infty h \tag{3.16}$$

then (3.10), (3.11), (3.15) imply that (see (2.17)):

$$\|u_h^{m+1} - u^{m+1}\|_1 \leq [\|u_h^m - u^m\|_1 [1 + 2\Delta t |\nabla u|_\infty e^{|\nabla u|_\infty \Delta t}] + c h \Delta t |\nabla u|_\infty^2 e^{|\nabla u|_\infty \Delta t} + |f^{m+1} - f_h^{m+1}|_0 \Delta t + c(h^2 + \nu h \Delta t) |\nabla\nabla u|_\infty.$$

Therefore we can state the following result:

Theorem 3. Assume that (3.3)–(3.4) has a unique solution in $L^\infty(0, T; W^{2,\infty}(\Omega))$. Let $u_h^m(.)$ be the solution of (3.7)–(3.8) then for h, Δt small enough:

$$(|u_h^m-u^m|_0^2+\nu\Delta t|\nabla_h(u_h^m-u^m)|_0^2)^{\frac{1}{2}}\leqq\Bigg[2\left(T\frac{h^2}{\Delta t}\|u\|_{2,\infty,Q}+T|f_h-f|_{\infty,Q}\right)$$
$$+(|u_h^0-u^0|_0^2+\nu\Delta t|\nabla_h(u_h^0-u^0)|_0^2)^{\frac{1}{2}}$$
$$+c((\nu+h/\Delta t)|\nabla(\nabla u)|_{\infty,Q}|\nabla u|_{\infty,Q}^2)h\Bigg]\exp(|\nabla u|_{\infty,Q}T) \tag{3.17}$$

i.e. the scheme (3.7)-(3.8) is of order $h+\Delta t+h^2/\Delta t$.

Proof. According to the previous discussion we only have to prove (3.16).
Let $v^*=\tilde{u}_h^m-u^m$ and $\tilde{v}_h=\nabla\times\xi_h$ then from the definition of $\tilde{u}_h$ we get

$$|\tilde{u}_h^m-u^m|_0=\frac{(\tilde{u}_h^m-u^m,v^*)}{|v^*|_0}=\frac{1}{|v^*|_0}[(u_h^m-u^m,v^*)+(\tilde{u}_h^m-u_h^m,v^*-\tilde{v}_h)]$$
$$\leqq|u_h^m-u^m|_0+|u_h^m-\tilde{u}_h^m|_0\frac{|v^*-\tilde{v}_h|_0}{|v^*|_0}$$

or equivalently with $\tilde{\omega}_h=\tilde{u}_h^m-\tilde{v}_h$:

$$|\tilde{u}_h^m-u^m|_0^2\leqq|u_h^m-u^m|_0|\tilde{u}_h^m-u^m|_0+|u_h^m-\tilde{u}_h^m|_0|\tilde{\omega}_h-u^m|_0.$$

From this we show easily that (use $a^2\leqq a(b+c)+bc\Rightarrow a\leqq(b+c)$)

$$|\tilde{u}_h^m-u^m|_0\leqq(|u_h^m-u^m|_0+|u^m-\tilde{\omega}_h|_0)$$

and since it is true for all $\xi_h,\tilde{\omega}_h=\nabla\times\xi_h$ it demonstrates (3.16), because $\nabla\cdot u^m=0$.

Spatial Discretization by the Hood-Taylor Element

In [2] the Stokes problem is approximated with conforming elements of degree 2 for the velocity u_h and degree 1 for the pressure p_h on the same triangulation (triangles or tetrahedra). This element is of order h^3 for $|u_h-u|_0$ and h^2 for $|\nabla(u_h-u)|_0$ (see [3]).
More precisely let

$$V_{0h}=\{v_h \text{ continuous: } v_h|_T \text{ quadratic in } x; \tag{3.18}$$
$$v_h|_\Gamma=0\}\subset(H_0^1(\Omega))^n$$
$$Q_h=\{q_h \text{ continuous: } q_h|_T \text{ linear in } x\}\subset H^1(\Omega). \tag{3.19}$$

Then (3.1) is approximated by (Crank-Nicholson's scheme):

$$\frac{1}{\Delta t}(u_h^{m+1},v_h)+\frac{\nu}{2}(\nabla u_h^{m+1},\nabla v_h)+(\nabla p_h^{m+1},v_h)=\frac{1}{\Delta t}(u_h^m(X_h^m(.)),v_h)$$
$$-\frac{\nu}{2}(\nabla u_h^m,\nabla v_h)+(f_h^{m+1},v_h)\quad\forall v_h\in V_{0h}$$
$$u_h^{m+1}\in V_{0h}$$
$$(\nabla q_h,u_h^{m+1})=0\quad\forall q_h\in Q_h;\quad p_h^{m+1}\in Q_h. \tag{3.20}$$

where $X_h^m(x)$ is the solution at time $\tau = m\Delta t$ of

$$\frac{dX_h}{d\tau} = u_h(X_h, \tau); \qquad X_h((m+1)\Delta t) = x. \tag{3.21}$$

As the first term in the right hand side of (3.20) must be computed by a Gauss quadrature formula of precision h^4 the solution of (3.21) is needed for 4 (5 in R^3) points x per triangle (tetrahedron).

In (3.21) u_h is piecewise quadratic in X and ideally piecewise linear in t. Therefore (3.21) is approximated by a method of Runge-Kutta of degree2:

$$X_h^m(x) = x - \frac{\Delta t}{2}[u_h^{m+1}(x) + u_h^m(x - \Delta t\, u_h^{m+1}(x))]. \tag{3.22}$$

This formula makes (3.20) non linear in u_h^{m+1}. Some improvement over (3.22) can be obtained if we change the variable of integration: (see (1.26)).

$$(u_h^m(X_h^m(.)), v_h) = (u_h^m, v_h(\bar{X}_h^{m+1}(.))) + O(|\nabla \cdot u_h|) \tag{3.23}$$

then

$$\bar{X}_h^{m+1} = x + \frac{\Delta t}{2}[u_h^m(x) + u_h^{m+1}(x + \Delta t\, u_h^m(x))]. \tag{3.24}$$

Adams method is even better:

$$X_h^{m+1} = x + \Delta t[u_h^m(x) + u_h^{m-1}(x)]/2 \tag{3.25}$$

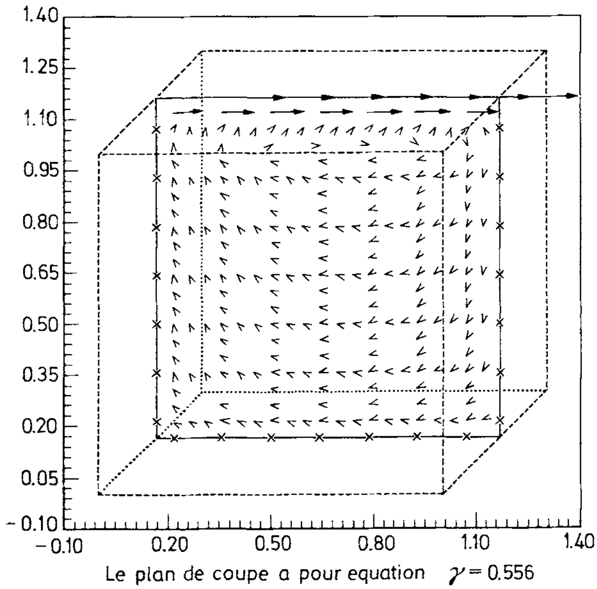

Fig. 11. Visuali∟ation of a 3-*D* flow for the cavity problem at Reynolds number 100. Computation is done by (3.7), (3.8)

because (3.20) is again linear in u_h^{m+1}. In all likelyhood however one can expect that Adams method will give good results only if u_h is a smooth function of t.

Therefore (3.20), (3.25) should be a well balanced scheme of order $h^3/\Delta t+h^2+\Delta t^2$ according to the considerations of the previous paragraph. The interested reader will find in [2] several numerical implementation and tests of this algorithm. A secure error bound is however not as easy as in the previous case because $\nabla \cdot u_h$ is not exactly zero and because (3.21) can no longer be integrated exactly. However if a projection like (3.8) is used then a precision of order $h+\Delta t+h^2/\Delta t$ can be shown as before but it is not optimal for this scheme. We include the visualization of a 3-*D* flow for the cavity problem with $\nu=1/100$ computed with method (3.7), (3.8): see Fig. 11. The result is borrowed from Hecht [11].

Conclusion

For the convection-diffusion equation we have shown that by mixing the method of characteristics and the finite element method we are able to derive first and second order accurate conservative schemes, of the upwinding type, which do not blow up when the diffusion coefficient tends to zero. Moreover these schemes are numerically better than the usual upwinding schemes because they require numerical solution of *symmetric* systems only.

However their numerical implementations require a further step, a quadrature formula for the right hand sides, which is difficult to devise so as to keep the conservativity and the error estimates. Thus in that sense this paper is not complete and will be followed later on by another one dealing only with this difficult problem.

The analysis extends to some non linear problems like the Navier-Stokes equations. We exhibit a stable conservative $0(h+\Delta t+h^2/\Delta t)$ scheme for the Navier-Stokes equations and we show that the scheme of [2] is likely to be $0(h^2+\Delta t^2+h^3/\Delta t)$ provided that the characteristics are properly computed and a suitable quadrature formula is used.

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