ON THE TREATMENT OF SINGULAR BETHE-SALPETER EQUATIONS

A. Bastai

Istituto di Fisica dell'Università, Torino
Istituto Nazionale di Fisica Nucleare, Sez. di Torino

L. Bertocchi

Istituto di Fisica dell'Università, Bologna
Istituto Nazionale di Fisica Nucleare, Sez. di Bologna

S. Fubini *) and G. Furlan **)

CERN - Geneva

M. Tonin

Istituto di Fisica dell'Università, Padova
Istituto Nazionale di Fisica Nucleare, Sez. di Padova

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*) and Istituto di Fisica dell'Università, Torino.

**) On leave from Istituto di Fisica dell'Università di Trieste, Trieste, Italy.

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A B S T R A C T

The properties of the bound-state solutions of the Bethe-Salpeter equation in the ladder approximation are investigated in the case of potentials strongly singular at the origin. The analogy with potential scattering and the connection to field theory are discussed. A method is given which allows the reduction of the singular problem to a regular one and explicit solutions for eigenvalue and eigenfunctions are presented in a particular case.

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I. INTRODUCTION

During the last years the relativistic Bethe-Salpeter [1] equation has been the object of several theoretical investigations. The reasons for this renewed interest in the Bethe-Salpeter equation are the following. First, because it gives a covariant treatment of the two-body problem in which elastic unitarity and reasonable analytic properties are properly taken into account. Secondly, the Bethe-Salpeter equation is connected, through the crossing relation, to the problem of diffraction scattering and multiple production at high energy [2]. However, in many cases, one is faced with the problem that the usual treatment of integral equations, as applied to the relativistic two-body problem, leads to meaningless results.

More specifically if one studies the interaction of two scalar bosons either through a vector particle or through the exchange of two scalar ones, the resulting integral equation is not of the Fredholm type, that is all Fredholm traces are divergent at high momenta. An analogous situation appears for the equation of the electron-positron system bound by a photon.

In this paper we wish to discuss the origin of this theoretical difficulty and we shall show by some examples how to overcome it. The interest of this investigation is twofold. First of all it is an attempt to obtain a finite solution of many bound state problems in elementary particle physics; secondly it will be suggested that the infinities appearing in the Bethe-Salpeter equation have a close relation with the infinities appearing in charge renormalization. Therefore it is hoped that our result will be of use for a better understanding and treatment of the infinity problem in quantum field theory. In Sections II and III we shall discuss the behaviour at small distances both in non-relativistic potential scattering and in the Bethe-Salpeter equation and show that the reason of the divergent traces lies in an improper treatment of the wave function at small distances. In Sections IV and V we will show that a correct treatment of the small distance behaviour of the wave function allows to recast the Bethe-Salpeter equation into a form in which no more infinities appear.

Finally, in Section VI we shall discuss the relation between our result and the full field theoretical problem.

II. NON-RELATIVISTIC THEORY

Let us consider the non-relativistic Schroedinger equation

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + k^2 - V(r) \right] \varphi(r) = 0 \qquad \text{(II.1)}$$

when the potential V(r) has a power behaviour at the origin

$$V(r) \sim \frac{f}{r^\beta} \qquad \text{(II.2)}$$

The differential equations (II.1) can be classified into three classes, according to the value of β, i.e., according to the relative strength of the centrifugal and potential terms near the origin:

I) $\beta < 2$ (regular potential). The behaviour at the origin

$$\varphi(r) \sim r^{\ell+1} \qquad \text{(II.3)}$$

is completely determined by the kinetic part of the Hamiltonian.

II) $\beta = 2$, i.e., $V(r) = -\frac{a}{r^2} + \bar{V}(r)$, where $\bar{V}(r)$ behaves regularly at the origin. The wave function at the origin still behaves like a power of the co-ordinate r

$$\varphi(r) \sim r^{\bar{\ell}+1} \qquad \text{(II.4)}$$

but now the exponent

$$\bar{\ell} = \frac{1}{2} + \sqrt{(\ell + \tfrac{1}{2})^2 - a} \qquad \text{(II.5)}$$

depends in an essential manner on the strength of the potential near the origin.

III) $\beta > 2$. In this case the potential term dominates on the centrifugal one at small distances and the wave function no more shows a power behaviour at small r.

The preceding classification of potentials is a well-known one. It is, however, very interesting that the same classification can be carried through in the relativistic Bethe-Salpeter equation and in field theory.

A very important point in Schroedinger theory is the following. Let us transform the differential equation (II.1) into an integral equation by expanding

$$\varphi(r) = \int \varphi(q)\, j_l(qr)\, dq \tag{II.6}$$

where $j_l(qr)$ are the eigenfunctions of the free Hamiltonian

$$H_0 = \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \tag{II.7}$$

Then the resulting integral equation

$$(q^2 - k^2)\,\varphi(q) = \int K(q, q')\,\varphi(q')\,dq' \tag{II.8}$$

satisfies the Fredholm conditions only for potentials of class I. In other words: for classes II and III if one tries to solve Eq. (II.8) using the Fredholm method all traces diverge at large momenta (in case II the divergence is logarithmic).

The reason for those infinities is quite clear: by means of Eq. (II.6) one is expanding the wave function in a series of eigenfunctions which do not have the right behaviour near the origin. Therefore the infinite integrals appearing in this model are the consequence of an improper treatment of the equation near the origin. Those infinities can be easily avoided if one splits the Hamiltonian in the following way

$$H = \bar{H}_0 + \bar{V}(r) \tag{II.9}$$

where

$$\bar{H}_0 = \frac{d^2}{dr^2} - \frac{\ell(\ell+1) - a}{r^2} = \frac{d^2}{dr^2} - \frac{\bar{\ell}(\bar{\ell}+1)}{r^2} \tag{II.10}$$

If we now expand our wave function in terms of the eigenfunctions of $\bar{H}_0$: $j_{\bar{\ell}}(qr)$ <u>which have the right behaviour at the origin</u> the new integral equation will be of the Fredholm kind so that all traces will now be finite.

We want now to remark the very peculiar problems which are present when we are considering potentials of class II. In this case we have a direct comparison, at small r, between the strengths of the attractive and centrifugal potential, which appears explicitly in the expression for $\bar{\ell}$

$$\bar{\ell} = \frac{1}{2} \pm \sqrt{(\ell + \frac{1}{2})^2 - a}$$

As long as we limit ourselves to values of the coupling constant "a", $a < a_c$, $a_c = (\ell + \frac{1}{2})^2$, the redefined $\bar{\ell}$ is real and it is easy to discriminate between the two roots choosing the expression (II.5) for $\bar{\ell}$ [3]. In the particular case in which $\bar{V}(r) \equiv 0$ the Schroedinger equation becomes a free one (aside the change $\ell \to \bar{\ell}$) and no bound states are present. If we consider values $a > a_c$ the problem is more difficult ($\bar{\ell}$ becomes complex) and, formally, the bound states problem can be solved [4]: however, the solutions are not of high physical interest because the ground state corresponds to $E = -\infty$. A concrete case where this distinction between values of the coupling constant, higher or smaller than a critical one is really of importance, is given by the Coulomb scattering of a Klein-Gordon particle: here

$$V(r) = -\frac{(Z\alpha)^2}{r^2} - \frac{2EZ\alpha}{r} = -\frac{a}{r^2} - \frac{b}{r} \qquad \text{(II.11)}$$

and, in S wave, the critical coupling constant is $\alpha Z_c = \frac{1}{2}$, $Z_c = \frac{137}{2}$. The equation to be solved becomes a Schroedinger equation with a Coulomb potential b/r and with a modified centrifugal barrier with "angular momentum" $\bar{\ell}$. So the eigenvalue condition will be simply given by

$$\frac{b}{2\sqrt{E}} = (\bar{\ell} + n + 1) \qquad \text{(II.12)}$$

and the only effect of the singular part of the potential is to change $\ell \to \bar{\ell}$.

Another characteristic feature of the class II potentials regards the analyticity properties both in "$\bar{\ell}$" and in "a". The $1/r^2$ singularity at the origin produces a branch cut in the "ℓ" complex plane, with branch point $\ell = -\frac{1}{2} \pm \sqrt{a}$. Conversely, the solutions show, through Eq. (II.5), a non-analytic dependence on the coupling constant near $a = 0$ and this can be another way of understanding the failure of the Fredholm method where one chooses as Green function the free one; this implicitly rests on the analyticity properties of the solution as a function of a.

All the qualitative characteristics of the potentials of class II will be still present in the analogous relativistic field theories (at least when we study particular models that is particular sets of Feynman graphs). As we will show, critical values of the coupling constant and non-analyticity in ℓ and a will be encountered.

Another way of discussing the infinities appearing for $1/r^2$ potential which is very useful for further comparison with field theory is based on the direct study of the singular solution around the origin.

Let us consider for simplicity an S wave bound state calculated from the Schroedinger equation with a central potential of class I: in this case the bound state condition is given by the vanishing of the coefficient of the singular part of the wave function which at the origin behaves as $1/r$. Correspondingly in momentum space the bound state condition is given by setting to zero the coefficient of that part of the wave function $\psi(u)$ which for $u = p^2 \to \infty$ behaves as $1/u+s$ ($-s$ is the binding energy).

This condition can be put in a more elegant form by defining the analogous of the non-renormalized and renormalized coupling constants

$$g_0 = \lim_{u \to \infty} (u+s)\psi(u)$$

$$g = \lim_{u \to -s} (u+s)\psi(u) \qquad \text{(II.13)}$$

and since g (which is the coefficient of the asymptotic part of the wave function in co-ordinate space) is finite, the bound state condition can be written

$$Z = \frac{g_0}{g} = 0 \tag{II.14}$$

The situation is completely different when the potential is of class II: in this case the behaviour of the wave function for $r \to 0$ is not only determined by the centrifugal part, but depends also on the potential, which enters in the indicial equation at the origin. The bound state condition requires now that the coefficient of the less regular part of the wave function vanishes: in momentum space this corresponds to impose the vanishing of the dominant term for $u \to \infty$. Now, since the behaviour of the irregular part of the wave function for $r \to 0$ is different from $1/r$, this means that the dominant part of the wave function for $u \to \infty$ is different from $1/u+s$ and therefore the function which must be equated to zero is different from the unrenormalized coupling constant which, being still defined as the coefficient of the $1/u+s$ term, becomes meaningless.

Study of singular potential scattering from this point of view is now in progress.

III. CLASSIFICATION OF RELATIVISTIC EQUATIONS

We now turn to the relativistic Bethe-Salpeter [5] equation in order to see whether the ideas developed in the last section can be generalized to this case.

The Bethe-Salpeter equation for two interacting scalar particles has the form

$$(\Box_1-\mu^2)(\Box_2-\mu^2)\phi(x_1,x_2)= V[(x_1-x_2)]\phi(x_1,x_2) \tag{III.1}$$

where $V(x_1-x_2)$ is the "relativistic potential" due to the exchange of one or more particles. We shall now be interested in the behaviour of the "wave function" $\phi\ (x_1x_2)$ when the four-dimensional distance $r^2 = (x_1-x_2)^2 = 0$, i.e., when $\ddot{x} = x_1-x_2$ is on the light cone. The relative influence of the kinetic and the potential term can be easily tested by trying at small r a behaviour like $\phi \sim r^{\delta}$. The kinetic term contains now fourth order derivatives in r so that the leading term is now $r^{\delta-4}$. As in the last section we shall classify the potentials in three classes on the basis of their behaviour at small r:

I) $\beta < 4$. The kinetic term dominates at small distances so that a representation of the wave function in plane waves reproduces the correct behaviour of the true solution on the light cone. In this "regular case" the integral equation one obtains by passing to momentum representation is of the Fredholm kind and the problem of obtaining eigenvalues can be handled by techniques similar to those used in potential scattering.

II) $\beta = 4; V(r) = \frac{a}{r^4} + \bar{V}(r)$. Kinetic and potential terms give contributions of the same order. We still have power behaviour r^{δ} at the origin, but now the exponent δ depends in an essential manner on the parameter a of the potential. The integral equation in momentum representation is no more of the Fredholm type since the plane wave expansion does not reproduce the correct behaviour of the wave function at small r. We shall see in the next section that, by recasting Eq. (III.1) in an appropriate manner, one can deal with this difficulty and obtain a completely finite answer.

III) $\beta > 4$. The potential term gives the dominating contribution and the behaviour at the origin is no more a power law. The same classification can be carried through for the case of two interacting Dirac particles

$$\left(\gamma_\mu^{(1)} \frac{\partial}{\partial x_\mu^{(1)}} - m\right)\left(\gamma_\mu^{(1)} \frac{\partial}{\partial x_\mu^{(1)}} - m\right) \psi(x_1, x_2) =$$

$$= V(x_1 - x_2)\, \psi(x_1, x_2) \qquad \text{(III.2)}$$

Now only second order derivatives appear in the kinetic term so that the potentials $V(r) \sim \frac{f}{r^\beta}$ have to be classified according to $\beta \lesseqgtr 2$. Let us now discuss the different field theoretical Hamiltonians on the basis of the properties of their ladder Bethe-Salpeter equation. In the case of scalar theories we see that for φ^3, φ^4, φ^5 interactions the potential is $D_F(x)$, $D_F^2(x)$, $D_F^3(x)$. Recalling that the D_F function behaves like $1/r^2$ near the light cone we see that those three Hamiltonians give rise to equations of class I, II, and III respectively. Looking now to spin $\frac{1}{2}$ theories we see that interactions of fermions through a vector particle belongs to class II whereas a four-fermion theory belongs to class III.

A simple way of summarizing our classification is obtained on the basis of the dimensionality $(\text{length})^{\varrho}$ of the coupling constant. The Hamiltonians belong to classes I, II, or III depending whether $\varrho \lesseqgtr 0$ [7].

It is amusing to compare our discussion with the usual classification in renormalizable (R) and non-renormalizable (NR) field theories. It is clear that (NR) theories correspond to our class III. We have, however, a further subdivision of renormalizable theories in classes I and II which is based on the form of the vertex function. In class I the vertex integral converges whereas in class II it diverges logarithmically. Let us discuss this point in more detail. The Bethe-Salpeter wave function $\phi\,(q_1 q_2)$ can be considered as the vertex function between our quantized, spin zero, field B and an external field Γ (see Fig. 1). We define, as normally, the unrenormalized and renormalized coupling constants as the following limits of the vertex function

$$g_0 = \lim_{u_1, u_2 \to \infty} (u_1 + m^2)(u_2 + m^2)\,\phi(u_1, u_2) \tag{III.3}$$

$$g = \lim_{u_1, u_2 \to -m^2} (u_1 + m^2)(u_2 + m^2)\,\phi(u_1, u_2) \qquad u_i = q_i^2 \tag{III.4}$$

In a theory of class I both g_0 and g are simultaneously finite. In analogy with potential scattering we can impose the bound state condition by requiring $g_0 = 0$. This can be visualized [8] by saying that we introduce a fictitious "elementary particle" Γ, we compute by standard techniques the ratio $Z = \frac{g_0}{g}$ and then we require the particle to have no elementary coupling by setting finally $g_0 = 0$.

From the point of view of Bethe-Salpeter equation in ladder approximation the vertex function is given in Fig. 2. The behaviour of $\phi(u_1u_2)$ for large u_1u_2 is $g_0/(u_1+m^2)(u_2+m^2)$ and setting $g_0 \to 0$ means requiring the wave function to behave regularly for large u (i.e., small r).

Whenever we have to deal with interactions of class II the Bethe-Salpeter equation shows the important feature that the binding potential has, at small r, the same behaviour as the centrifugal part, and therefore it determines the behaviour of the vertex function when $u_1, u_2 \to \infty$. This behaviour turns out to be different from the bare form of the vertex function $g_0/(u_1+m^2)(u_2+m^2)$, therefore the limit in Eq. (III.3) does not exist.

The situation is now much the same as the Schroedinger equation with a $1/r^2$ potential and the infinite result one gets when trying to calculate the unrenormalized coupling in terms of the renormalized one is a sign that we have wrongly questioned the theory. We see, therefore, that the study of singular Bethe-Salpeter equations can be useful to understand the meaning of the infinities appearing in field theory.

IV. THE BEHAVIOUR AT THE ORIGIN

In this paper we will be interested in the scalar case of class II. It will be seen in Section IV that this allows the treatment of the ladder approximation for the λA^2B^2 and the $gB\overleftrightarrow{\partial}_\mu BA^\mu$ theories.

In this case it is useful to separate the potential V into a singular and a regular part

$$V(R) = \frac{a}{R^4} + \bar{V}(R) \qquad R = \sqrt{x^2} \tag{IV.1}$$

After translational invariance we can separate the C.M. and the relative motion

$$\phi(x_1, x_2) = \psi(x)\, e^{iP\cdot\frac{x_1+x_2}{2}} \tag{IV.2}$$

where $x = x_1 - x_2$ and P_μ is the total four momentum. In the centre-of-mass system Eq. (IV.1) becomes

$$\left[\left(\Box_x - 1 + \frac{E^2}{4}\right)^2 + E^2\frac{\partial^2}{\partial x_4^2}\right]\psi(x) = \left[\frac{a}{R^4} + \bar{V}(R)\right]\psi(x) \tag{IV.3}$$

(Euclidean metric is understood)

We wish to discuss in more detail the behaviour of our equation at small distances. In order to do so, we write the reduced equation which is obtained from Eq. (IV.3) by selecting only those terms which are dominant at $R \to 0$.

$$\Box\Box\,\psi(x) = \frac{a}{R^4}\,\psi(x) \tag{IV.4}$$

We see the very important fact that, although the full Eq. (IV.3) is invariant only for three-dimensional rotations in the space orthogonal to P_μ, the behaviour at small distances is obtained on the basis of Eq. (IV.4) which shows full four-dimensional invariance. So we can at once separate the angular dependence introducing four dimensional spherical harmonics [9)]

$$\psi(x) = \sum_{\alpha l m} \frac{f_\alpha(R)}{R} Y_{\alpha l m}(\theta_1, \theta_2, \varphi) \tag{IV.5}$$

and we arrive to the radial equation

$$O\,O\, f_\alpha(R) = \frac{a}{R^4} f_\alpha(R) \tag{IV.6}$$

where

$$O = \frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR} - \frac{\nu^2}{R^2} \tag{IV.7}$$

$$\nu = \alpha + 1$$

and α is the four dimensional angular momentum.

We insert now the power solution

$$f_\alpha(R) = \text{const.} \cdot R^{\gamma+1} \tag{IV.8}$$

in Eq. (IV.6) and we get for γ the indicial equation

$$\left[(\gamma-1)^2 - \nu^2\right]\left[(\gamma+1)^2 - \nu^2\right] - a = 0 \tag{IV.9}$$

or

$$\gamma^4 - 2\gamma^2(\nu^2+1) + (\nu^2-1)^2 - a = 0 \qquad \text{(IV.9')}$$

which leads at once to the four solutions $\gamma_1, \gamma_2, \gamma_3 = -\gamma_1, \gamma_4 = -\gamma_2,$ where

$$\begin{cases} \gamma_1 = \left\{1+\nu^2+\left[4\nu^2+a\right]^{1/2}\right\}^{1/2} \\ \gamma_2 = \left\{1+\nu^2-\left[4\nu^2+a\right]^{1/2}\right\}^{1/2} \end{cases} \qquad \text{(IV.10)}$$

In Fig. 3 we plot Eq. (IV.9') with the real solutions for γ, as a function of a; (note the symmetry with respect to the zero) (see Fig. 3).

As in potential theory we now have also the appearance of critical values [10] of the coupling constant for which pairs of roots of γ become complex. They are $a_2 = \alpha^2(\alpha+2)^2$ and $a_1 = -4(\alpha+1)^2$ respectively. For $a > a_2$ we get two real solutions, for $a_2 > a > a_1$ four real solutions, for $a_1 > a$ no real solutions are present.

We will limit ourselves, as in potential scattering, to the range of a, $a_2 > a \geq 0$.

Obviously not all the solutions of Eq. (IV.9) have to be retained, but in order $\varphi_{cl}(R)$ to be an acceptable wave function we have to impose boundary conditions. Wick has shown that the differential equation (IV.3) is equivalent to the original integral form of the

Bethe-Salpeter equation if we require a regular behaviour of the wave function at the boundaries. Analogously we will assume, also in our case, regularity conditions at $R = 0$, that is

$$\lim_{R \to 0} R^{\varepsilon - 1} \varphi_{\alpha}(R) = 0 \tag{IV.11}$$

Condition (IV.11) enables us to rule out the solutions which at the origin behave as $R^{1-\gamma_1}$, $R^{1-\gamma_2}$, so that in Fig. 3 we have to consider only the two solutions corresponding to $\gamma > 0$.

It is important to observe at this point that the presence in the potential of (IV.3) of a contact term $\delta^4(x)$ has no effect if the constant "a" is below the critical value. In fact, in this case, the regularity conditions imposed on the wave function cause the product $\psi(x)\,\delta^{(4)}(x)$ to vanish.

We conclude this section by collecting some properties of the solutions of the indicial equation

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 0 \tag{IV.12}$$

(The sum of the solutions is zero)

$$\gamma_1^2 + \gamma_2^2 = 2(\nu^2 + 1) \tag{IV.13}$$

$$\gamma_1 + \gamma_2 = \left\{ 2(\nu^2+1) + 2\left[(\nu^2-1)^2 - a\right]^{1/2} \right\}^{1/2} \tag{IV.14}$$

Further we get in the limit $a \to 0$

$$\gamma_1 \to \alpha + 2 \qquad \gamma_2 \to \alpha \tag{IV.15}$$

In addition we want to point out that in the free case the indicial equation (IV.9) is factorized in the product of two terms and every of them gives two roots: one corresponding to a regular solution, another to an irregular one. More precisely, the first term contains γ_1, γ_4, the second one γ_2, γ_3 and these pairs of roots show a symmetry, in the sense that they differ only by a change of sign, that is $\gamma_i' = -\gamma_i$. As we will show, this property can be maintained in the $a \neq 0$ case.

V. THE METHOD OF SOLUTION

Let us now discuss the solution of Eq. (III.5).

We observe first that the mass E of the bound state does not enter into the indicial equation at $R = 0$. We can therefore get rid of this term and consider only the simplified problem $E = 0$. This case already contains all the difficulties of the more realistic problems, in which $E \neq 0$, and could therefore be a basis for their solution. Moreover, this choice introducing a complete four-dimensional symmetry allows the separation of the total equation in four-dimensional polar coordinates.

So we can use the decomposition (IV.5) and we get for the radial wave function the following equation

$$(O-1)^2 \varphi_\alpha(R) = \left(\frac{a}{R^4} + \bar{V}(R)\right) \varphi_\alpha(R) \tag{V.1}$$

We have now to supplement it with the boundary conditions, which, as we have referred in the previous section, are regularity conditions, namely $\varphi_\alpha(R)$ has to be zero both for $R \to 0$, $R \to \infty$.

The condition at the infinity will enable us to rule out the solutions which increase exponentially as $R \to \infty$. Therefore, remembering the discussion of the previous section, the wanted solution will be a linear combination of the two solutions regular at the origin (whose behaviour is R^{γ_1+1}, R^{γ_2+1}) which goes at the infinity as a decreasing exponential.

To solve the problem we develop a method which is the direct generalization to the relativistic case of the procedure used in Section II for non-relativistic potentials of class II. Thus the central idea will be to include in the "free" or kinetic part that term of the potential which enters into the indicial equation at $R = 0$. Then we can develop $\varphi_\alpha(R)$ in eigenfunctions of this operator which certainly have the required behaviour at the origin. The other advantage of our procedure is that, once redefined the free part, it is possible to reduce the problem to a case with a regular potential $\bar{V}(R)$, whose treatment is well known.

To this end we introduce two operators O_1, O_2, such that

$$O_1 O_2 = O O - \frac{a}{R^4} \tag{V.2}$$

(Remember that $OO - \frac{a}{R^4}$ determines the behaviour at $R = 0$.) We put

$$O_1 = \frac{d^2}{dR^2} + \frac{1+\beta_1}{R}\frac{d}{dR} - \frac{\nu^2 - A_1}{R^2}$$

$$O_2 = \frac{d^2}{dR^2} + \frac{1+\beta_2}{R}\frac{d}{dR} - \frac{\nu^2 - A_2}{R^2} \tag{V.3}$$

To determine the parameters β_i, A_i we have only to write the indicial equation in function of them. More precisely

$$O_1 O_2 R^{\gamma+1} = 0$$

gives

$$\left[(\gamma-1)^2 + \beta_1(\gamma-1) - (\nu^2 - A_1)\right]\left[(\gamma+1)^2 + \beta_2(\gamma+1) - (\nu^2 - A_2)\right] = 0 \tag{V.4}$$

So, as in the free case, the indicial equation is factorized in the product of two terms

$$[(\gamma-1)^2+\beta_1(\gamma-1)-(\nu^2-A_1)]=0$$

$$[(\gamma+1)^2+\beta_2(\gamma+1)-(\nu^2-A_2)]=0 \qquad \text{(V.4')}$$

If we call γ_{I}, γ_{II} the solutions of the first equation and γ_{III}, γ_{IV} those of the second one, we get

$$\beta_1 = 2 - (\gamma_{I}+\gamma_{II})$$

$$\beta_2 = -2 - (\gamma_{III}+\gamma_{IV}) \qquad \text{(V.5)}$$

and remembering property (IV.12) it follows

$$\beta_1+\beta_2 = 0 \qquad \text{(V.6)}$$

Similarly

$$A_1 = \nu^2 + (\gamma_{I}-1)(\gamma_{II}-1)$$

$$A_2 = \nu^2 + (\gamma_{III}+1)(\gamma_{IV}+1) \qquad \text{(V.7)}$$

It is evident from Eq. (V.5) that we have six possible choices for β_1 corresponding to the six different pairings with γ_1, γ_2, γ_3, γ_4. More explicitly there are the possibilities $(\beta_1 \equiv \beta)$:

a) $\beta = 2-(\gamma_1+\gamma_2)$ b) $\beta = 2-(\gamma_3+\gamma_4)$

c) $\beta = 2-(\gamma_1+\gamma_4)$ d) $\beta = 2-(\gamma_2+\gamma_3)$

e) $\beta = 2-(\gamma_1+\gamma_3)$ f) $\beta = 2-(\gamma_2+\gamma_4)$

We can make a first distinction between the six solutions for β_1 by dividing them into two groups:

A) to this group belong cases a), b), c), d) which are characterized by the property which is already present in the free $(a = 0)$ case that the γ's coming from O_2 can be obtained by changing the sign of those deriving from O_1 (remember $\gamma_3 = -\gamma_1$, $\gamma_4 = -\gamma_2$).

B) this property does not hold for cases e), f). For group B) we have at once

$$\beta = 2 \tag{V.8}$$

and

$$\begin{aligned} A_1 &= \nu^2 - (\gamma_1^2 - 1) \\ A_2 &= \nu^2 - (\gamma_2^2 - 1) \end{aligned} \tag{V.9}$$

and remembering Eq. (IV.13) we have also $A_1+A_2 = 0$.

For group A) on the other hand it is easily seen that

$$A_1 = A_2 = \beta^2/2 - \beta \tag{V.10}$$

To maintain completely the symmetry properties of the free case, from now on we will limit ourselves to the solutions for β_1 which belong to group A)[11] so that we can finally write

$$O_1 = \frac{d^2}{dR^2} + \frac{1+\beta}{R}\frac{d}{dR} - \frac{\nu^2 - A}{R^2}$$

$$O_2 = \frac{d^2}{dR^2} + \frac{1-\beta}{R}\frac{d}{dR} - \frac{\nu^2 - A}{R^2} \qquad \text{(V.11)}$$

Moreover, property (V.10) enables us to introduce a unique operator $\bar{O}$

$$\bar{O} = R^{\beta/2}\, O_1\, R^{-\beta/2} = R^{-\beta/2}\, O_2\, R^{\beta/2} \qquad \text{(V.12)}$$

and

$$\bar{O} = \frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR} - \frac{\bar{\nu}^2}{R^2} \qquad \text{(V.13)}$$

$$\bar{\nu}^2 = \nu^2 - A + \beta^2/4 = \nu^2 + \beta - \beta^2/4 \qquad \text{(V.14)}$$

which can also be written

$$\bar{\nu}^2 = \frac{\nu^2 + 1 - \gamma_I \gamma_{II}}{2} = \frac{(\gamma_I - \gamma_{II})^2}{4} \qquad \text{(V.14')}$$

To recast Eq. (V.1) in a more significative form we observe that from (V.11)

$$2\,O = O_1 + O_2 - \frac{2A}{R^2} \tag{V.15}$$

so that we can write

$$(O-1)^2 - \frac{a}{R^4} = O^2 - \frac{a}{R^4} - 2\,O + 1 = O_1 O_2 - (O_1 + O_2) + 1 + \frac{2A}{R^2} =$$

$$= (O_1 - 1)(O_2 - 1) + \frac{2A}{R^2}$$

and Eq. (V.1) becomes

$$(O_1 - 1)(O_2 - 1)\,\varphi_\alpha(R) = \left[\bar{V}(R) - \frac{2A}{R^2}\right]\varphi_\alpha(R) \tag{V.16}$$

Moreover, defining a new wave function

$$\bar{\varphi}_\alpha(R) = R^{-\beta/2}\,\varphi_\alpha(R) \tag{V.17}$$

Eq. (V.16) can be written as

$$\begin{cases} (\bar{O}-1)\,\bar{\Psi}_\alpha(R) = \left[\bar{V}(R) - \dfrac{2A}{R^2}\right]\varphi_\alpha(R) \\ (\bar{O}-1)\,\bar{\varphi}_\alpha(R) = \bar{\Psi}_\alpha(R) \end{cases} \tag{V.18}$$

or

$$R^{-\beta}\,\bar{O}\,R^{\beta}\,\bar{O}\,\bar{\varphi}_\alpha(R) = \left[\bar{V}(R) - \frac{2A}{R^2}\right]\varphi_\alpha(R) \tag{V.18'}$$

These algebraic results are independent of one of the possible choices of β. We find it convenient to specify the choice by selecting that solution for β, which in the limit $a \to 0$ gives back the regular case, namely

$$\beta \to 0, \quad \bar{\nu} \to \nu, \quad \bar{D} \to D \tag{V.19}$$

This requires that, as $a \to 0$,

$$\gamma_{I} + \gamma_{II} \to 2$$

so that from (IV.14)

$$\gamma_{I} = \gamma_1 \qquad \gamma_{II} = \gamma_4 = -\gamma_2$$

$$\beta = 2 - (\gamma_1 - \gamma_2) \qquad \text{case c)} \tag{V.20}$$

and using expressions (IV.10) we get

$$\beta = 2 - \left\{ 2(\nu^2+1) - 2\left[(\nu^2-1)^2 - a\right]^{1/2} \right\}^{1/2} \tag{V.21}$$

In the same way

$$\bar{\nu} = \left\{ \frac{\nu^2 + 1 + \gamma_1\gamma_2}{2} \right\}^{1/2} = \left\{ \frac{\nu^2 + 1 + \left[(\nu^2-1)^2 - a\right]^{1/2}}{2} \right\}^{1/2} \tag{V.22}$$

or

$$\bar{\nu} = \frac{\gamma_1 + \gamma_2}{2}$$

and conversely with this choice

$$\gamma_1 = \bar{\nu} + 1 - \beta/2$$

$$\gamma_2 = \bar{\nu} - 1 + \beta/2 \qquad (V.23)$$

The system of Eqs. (V.18) is now in such a form that the new kinetic operator $\bar{O}$ determines the correct behaviour of the wave function at small R. Therefore, if one expands $\varphi_\alpha(R)$ and $\Psi_\alpha(R)$ in terms of eigenfunctions of $\bar{O}$ (i.e., Bessel functions $J_{\bar{\nu}}(qR)$) the system (V.18) is now transformed into a Fredholm form in which no more infinities appear.

We have been able to solve exactly the problem in the case $\bar{V}(R) = \frac{b}{R^2}$. We do not discuss here the details of the mathematical treatment which are described in the following paper [12], but we limit ourselves to the results. The eigenvalue condition is simply given by

$$b = 4(n+\bar{\nu})(n+\bar{\nu}+1) \qquad (V.24)$$

where $\bar{\nu}$ is given by (V.22) and $n = 0,1,2,\ldots$. Moreover, the first $(n = 0)$ eigenfunction is

$$\varphi_\alpha^{(0)}(R) = \frac{R^{\bar{\nu}+1} K_{1-\beta/2}(R)}{2^{\bar{\nu}-\beta/2}\,\Gamma(\bar{\nu}+2-\beta/2)} \qquad (V.25)$$

and $K_{1-\beta/2}(R)$ is the Bessel function of imaginary argument.

It is amusing to see that, as in the analogous non-relativistic case $\frac{a}{r^2} + \frac{b}{r}$, the role of the most singular part of the potential is only to redefine the value of the effective angular momentum from ν to $\bar{\nu}$.

VI. COMPARISON WITH FIELD THEORY

Until now we have discussed the Bethe-Salpeter equation as a phenomenological two body relativistic model without inquiring about the field theoretical origin of the "potential".

We wish now to discuss in some examples the relativistic equations obtained in the ladder approximation from field theoretical Hamiltonians.

1) Exchange of a scalar boson

The interaction Hamiltonian is

$$H_I = f\, B^2(x)\, A(x) \tag{VI.1}$$

where $A(x)$ is the exchanged field of mass μ and the ladder graphs are shown in Fig. 4. The potential is simply given by

$$V(R) = D_F(x,\mu) = \frac{\mu}{4\pi^2}\,\frac{K_1(\mu R)}{R} \tag{VI.2}$$

Since at small distances

$$V(R) \sim \frac{1}{R^2} \tag{VI.3}$$

the resulting Bethe-Salpeter equation is of class I and no special treatment is needed.

2) Exchange of a boson pair

The interaction is now

$$H_I = g\, B^2(x)\, A^2(x) \tag{VI.4}$$

and the ladder graphs are shown in Fig. 5. The potential is now

$$V(R) = D_F^2(x,\mu) \underset{R \to 0}{\sim} \frac{1}{R^4} \tag{VI.5}$$

so that the Bethe-Salpeter equation is now of class II and the treatment of Sections III, IV, and V can be directly applied to it. If one wants to take into account a counter-term

$$\delta H = \lambda B^4(x) \tag{VI.6}$$

one gets the extra contact potential

$$\delta V = -\lambda \delta^{(4)}(x) \tag{VI.7}$$

We have seen at the end of Section IV that, if the correct boundary conditions at the origin are taken into account, this extra term has no effect on the eigenvalue problem.

3) Exchange of a vector boson

The interaction Hamiltonian is

$$H_I = i\lambda B \overleftrightarrow{\partial}_\mu B A^\mu(x) \tag{VI.8}$$

and the ladder Bethe-Salpeter equation (for $P_\mu = 0$) becomes

$$(\Box - 1)^2 \psi(x) = -\lambda^2 \left[2 D_F (\Box \psi) + 2 \Box (D_F \psi) - \psi \Box D_F \right] \tag{VI.9}$$

Equation (VI.9) can be recasted in the following form

$$(\Box - 1 + 2h^2 D_F)^2 \Psi(x) = -4h^2 D_F \Psi(x) + 4h^4 D_F^2 \Psi(x) + + h^2 \Psi(\Box D_F) \tag{VI.10}$$

where the choice of the sign is such that it gives an attractive potential. Now, in the simple case of zero mass of the A_μ field the D_F term in the left-hand side of Eq. (VI.10) acts as an extra centrifugal potential so that going to angular momentum representation our problem can be reduced to a combination of the potentials appearing in Eqs. (VI.2), (VI.5), (VI.7) with the redefinition of the angular momentum

$$(\alpha+1)^2 \to (\alpha+1)^2 - \frac{h^2}{2\pi^2} \tag{VI.11}$$

As in the case 2) the counter-term has no effect on the eigenvalue problem. We see therefore that our method allows to study the sum of the vertex ladder graphs which, in the usual treatment, will turn out to be infinite. Moreover, in the simple case of exchanged particles of zero masses, one indeed obtains an exact eigenvalue solution. Thus, if we consider the general interaction Hamiltonian

$$H_I = f B^2(x) A(x) + g B^2(x) A^2(x) + i h B \overleftrightarrow{\partial}_\mu B A_\mu(x) \tag{VI.12}$$

we get the eigenvalue condition

$$b_n = 4\left(\bar{\nu}+n\right)\left(\bar{\nu}+n+1\right) \tag{VI.13}$$

where

$$b = \frac{f^2}{4\pi^2} - \frac{h^2}{\pi^2} \tag{VI.14}$$

$$\bar{\nu} = \left\{ \frac{1+(\alpha+1)^2 - \frac{h^2}{2\pi^2} + \left[\left(\alpha(\alpha+2) - \frac{h^2}{2\pi^2}\right)^2 - a\right]^{1/2}}{2} \right\}^{1/2} \tag{VI.15}$$

$$a = \frac{g^2}{16\pi^4} + \frac{h^2}{4\pi^4} \tag{VI.16}$$

R E F E R E N C E S

1) H.A. Bethe and E.E. Salpeter, Phys. Rev. 84, 1232 (1951).

M. Gell-Mann and F.E. Low, Phys. Rev. 84, 350 (1951).

2) See for instance L. Bertocchi, S. Fubini and M. Tonin, Nuovo Cimento 25, 626 (1962), where other references are given.

3) See for instance L.D. Landau and E.M. Lifshitz, Quantum Mechanics, Pergamon Press, (1958), Chap. V. A more general criterion has been suggested to us by Professor G. Dell'Antonio: if we require that the Hamiltonian is a self-adjoint and hypermaximal operator (in the von Neumann sense), this imposes to the wave function $\varphi(r)$ the behaviour $\lim_{r\to 0} r^{-1/2}\varphi(r) = 0$.

4) K.M. Case, Phys. Rev. 80, 797 (1958).

5) In this connection we recall the very important result obtained by Wick who has shown that the causality condition allows to transform Eq. (III.1) into an equivalent equation in which the space time metric is now Eucledian. In this new representation $r^2 = 0$ will just mean the point at the origin [6].

Our treatment of the Bethe-Salpeter equation will make in the following explicit use of the Euclidean metric.

6) G.C. Wick, Phys. Rev. 94, 1124 (1954).

7) A well-known exception is the vector theory with no current conservation which, due to the form of the vector boson propagator, is non-renormalizable even if the coupling constant is dimensionless.

8) See for instance A. Salam, Nuovo Cimento 25, 224 (1962), where other references are given.

9) See for instance A. Erdelyi et al., Bateman Manuscript Project, McGraw-Hill (1953), Vol. II.

10) This feature is connected to the non-analytic behaviour of the high-energy limit of the scattering amplitude as a function of the coupling constant, as found in similar problems by R.F. Sawyer: Complex Angular Momenta in Perturbation Theory, University of Wisconsin preprint (1963).

11) With the particular choice $\tilde{V}(R) = 0$ the case $\beta = 2$ has been solved by M. Banerjee, L. Kisslinger and A. Levinson (private communication) with results qualitatively in agreement with ours.

12) A. Bastai, L. Bertocchi, G. Furlan and M. Tonin, following paper.

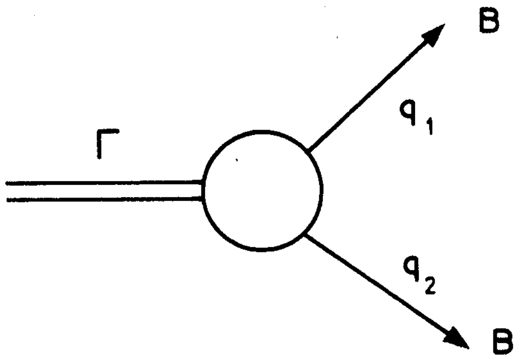


FIG. 1

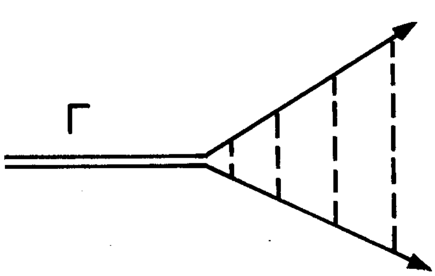
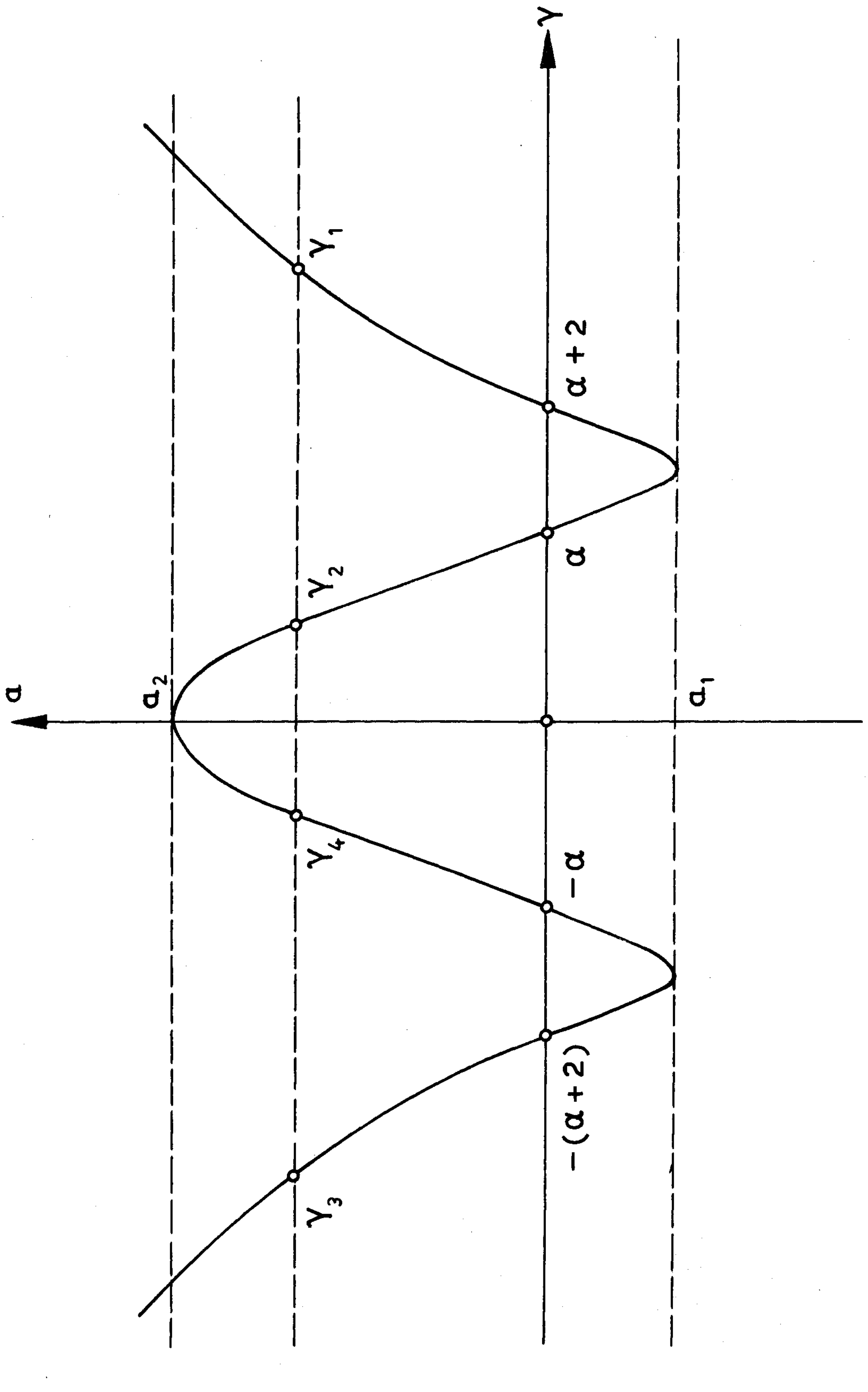


FIG. 2



γ
α
a_2
a_1
γ_1
γ_2
γ_3
γ_4
$\alpha + 2$
α
$-\alpha$
$-(\alpha+2)$

FIG. 3

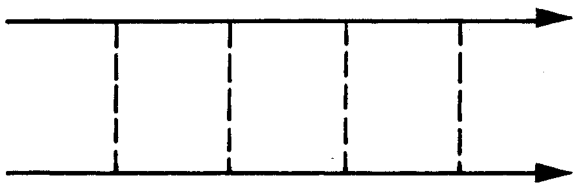


FIG. 4

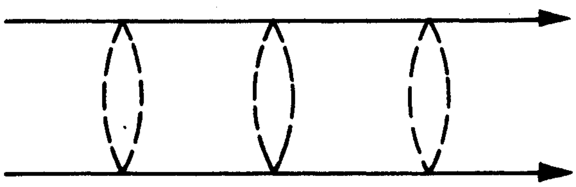


FIG. 5