

On the truth of nanoscale for nanobeams based on nonlocal elastic stress field theory: equilibrium, governing equation and static deflection *

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(Communicated by Xing-ming GUO)

Abstract This paper has successfully addressed three critical but overlooked issues in nonlocal elastic stress field theory for nanobeams: (i) why does the presence of increasing nonlocal effects induce reduced nanostructural stiffness in many, but not consistently for all, cases of study, i.e., increasing static deflection, decreasing natural frequency and decreasing buckling load, although physical intuition according to the nonlocal elasticity field theory first established by Eringen tells otherwise? (ii) the intriguing conclusion that nanoscale effects are missing in the solutions in many exemplary cases of study, e.g., bending deflection of a cantilever nanobeam with a point load at its tip; and (iii) the non-existence of additional higher-order boundary conditions for a higher-order governing differential equation. Applying the nonlocal elasticity field theory in nanomechanics and an exact variational principal approach, we derive the new equilibrium conditions, domain governing differential equation and boundary conditions for bending of nanobeams. These equations and conditions involve essential higher-order differential terms which are opposite in sign with respect to the previously studies in the statics and dynamics of nonlocal nano-structures. The difference in higher-order terms results in reverse trends of nanoscale effects with respect to the conclusion of this paper. Effectively, this paper reports new equilibrium conditions, governing differential equation and boundary conditions and the true basic static responses for bending of nanobeams. It is also concluded that the widely accepted equilibrium conditions of nonlocal nanostructures are in fact not in equilibrium, but they can be made perfect should the nonlocal bending moment be replaced by an effective nonlocal bending moment. These conclusions are substantiated, in a general sense, by other approaches in nanostructural models such as strain gradient theory, modified couple stress models and experiments.

Key words bending, effective nonlocal bending moment, nanobeam, nanomechanics, nanoscale, nonlocal elastic stress, strain gradient

Chinese Library Classification O343

2000 Mathematics Subject Classification 74A60

* Received Sept. 29, 2009 / Revised Nov. 19, 2009

Project supported by a grant from Research Grants Council of the Hong Kong Special Administrative Region (No. CityU 117406)

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1 Introduction

Since the discovery of carbon nanotubes (CNTs) by Iijima^[1] in the early 1990s, there has been intensive research on CNTs, such as the studies of Treacy et al.^[2] and Ball^[3]. There are two main approaches to the analysis of CNTs: atomic modelling and continuum modelling. The most common atomic modelling method is the molecular dynamics (MD) method^[4-6]. As the MD method considers each individual molecule and its multiple mechanical or chemical web-interactions, its use requires extremely fast computing facilities and thence it is largely confined to relatively restricted systems with a limited number of molecules.

Due to the excessively extensive computation that is needed to run a full MD program, many researchers have resorted to the continuum modelling approach, because it can be practically applied to model, compute and analyze CNTs in large-scale systems. This approach has resulted in the elastic, linear or nonlinear beam and shell models for the analysis of static and dynamic responses and the stability and vibration of CNTs. One of the earliest efforts in this area was that of Yakobson et al.^[7], who used a continuum shell model to study the instability of CNTs that were subjected to compressive loading. The buckling of CNTs was subsequently investigated by Ru^[8-9] using a different continuum shell model. These continuum beam and shell models may be classified as classical or local models, because the stress at a reference point is a function of the local strain field at that point. Although justified experimentally to a certain extent^[10], local continuum models do not exhibit intrinsic size-dependence features and do not allow inclusions and property inhomogeneities that are fundamental and significant in atomic modelling. Size-dependent effects^[11-17] become significantly more prominent at the nanoscale, and are thus a compelling research area in continuum nanomechanical modelling.

One promising continuum mechanics model in nanomechanics which considers size-dependent effects is the nonlocal field theory which was first extensively developed by Eringen^[18-35] since the early seventies in the last century. The nonlocal field theories establish the stress at a point in a domain to depend not only on the classical local stress at that particular point, but also on the spatial integrals that represent the weighted averages of the local stress contribution of all other points in the domain. For over three decades afterwards, many branches of the nonlocal field theories have been established by Eringen and his associates including nonlocal pure elastic continua^[18-19,21-22,32], nonlocal fluid mechanics^[20,23], nonlocal electromagnetism^[24], nonlocal thermoelasticity^[25-26], nonlocal memory-dependent elasticity^[26-27], nonlocal piezoelectricity^[33-34], etc. Very good accounts of the theory in many aspects of physics and engineering have been published in a number of his monographs^[28-31,35], the readers are referred, in particular, to the book on nonlocal continuum field theories by Eringen^[35].

Application of the nonlocal elastic field theory to modelling of CNTs and other nanostructures only drew extensive attention in the beginning of the twenty-first century. The lack of wide attention earlier was mainly due to the requirement of solving a kernel function representing the nonlocal modulus $\alpha|x' - x|$ which satisfies the field equations. Unfortunately, solving such exact kernel function has been very deterrent and it requires the application of correct Green functions which could hardly be identified or determined unless under very specific circumstances. For certain elastic continua of practical engineering interest, e.g., a nanobeam or a nanotube, the nonlocal modulus may obey specific conditions such as one-dimensional moduli, two-dimensional moduli and three-dimensional moduli. Incidentally, for a two-dimensional moduli, Eringen^[32,35] reported that it was possible to further reduce the nonlocal stress field equation to a second-order partial differential equation in a two-dimensional region which, however, required modelling approximation and induced modeling errors depending on certain atomic parameters of as much as 6%. Considering a one-dimensional nanostructure, a second-order ordinary differential equation (ODE) can further be obtained. By using this second-order ODE, it is very possible to analytically solve the static and dynamic responses of CNTs and other one-dimensional nanostructures such as nanorods, nanobeams, etc., for bending, vibra-

tion, buckling and wave propagation.

Based on this second-order ODE, the work on continuum modelling and the analyses of CNTs and the other nanostructures revived and prospered. The early works began with Peddieson et al.^[36] who applied a nonlocal stress model to analyze static bending of a simply supported nanobeam subject to sinusoidal loading, a cantilever nanobeam with a point load at the tip, and a cantilever with variable distributed load. At almost the same time, Sudak^[37] investigated column buckling of multiwalled carbon nanotubes and predicted decreasing critical buckling load with respect to increasing nanoscale effect. Subsequently, there appeared abundant works in nonlocal modelling following this nonlocal stress approach (a total of more than 80 papers including^[36-37] and a text since 2003 have been identified), including bending of nanobeams (5 papers and a text) and nanoplates (1 paper); vibration of nanobeams (12 papers), nanoring/nanoarches (1 paper), nanoplates (1 paper) and nanoshells (1 paper); buckling of nanobeams (8 papers) and nanoshells (9 papers); wave propagation of nanobeams (8 papers) and nanoshells (7 papers); general structural analyses of nanobeams (3 papers) and nanoplates (1 paper), and many others.

Among these published works, some adopt the thin beam, thin plate or thin shell theory which specifies that the normal to mid-surfaces before deformation remain normal and inextensible after deformation, while the others consider shear deformation which allows rotation of normal after deformation with a shear correction factor. The thin-structural theories are employed for nanobeams (28 papers and a text), nanoplates (3 papers), and nanoshells (19 papers) while the thick structural, i.e., Timoshenko and Mindlin theories, are employed for nanobeams (17 papers) and nanoplates (1 paper). It will be clear in this paper that these nonlocal modeling works [36-37 and the rests] can be specifically classified as the “partial nonlocal stress models” because they have been formulated by directly extending the classical models without rigorous verification. Consequently, there are certain very important nonlocal terms that have been incorrectly derived and higher-order terms inadvertently neglected. The models, in fact, do not satisfy the condition of equilibrium.

It will also be established in this paper that the existence of higher-order nonlocal bending moment terms is so much critical that it alters completely the understanding of the role of nanoscale and also, without them, the two intriguing observations as clearly specified in the beginning of the abstract are concluded: (i) the presence of increasing nonlocal effects induce reduced nanostructural stiffness in many, but not consistently in all (CF nanobeam with uniform load as one example), cases of study in virtually all previously published works in this subject although physical intuition according to the nonlocal field theories^[35] tells otherwise; and (ii) that nanoscale effects are missing in the solutions in many exemplary cases of study^[36].

Concurrent to the development of nonlocal stress field theories and their many application studies, there exist intensive development in other branches of nanomechanics such as the strain gradient theory^[38-39], molecular-structural-mechanics method^[40], and a modified couple stress theory based on a variational formulation and the principle of minimum total potential energy^[41-43]. In particular, Nix and Gao^[38] successfully concluded that the hardness of crystalline materials is very much size-dependent by using the concept of geometrically necessary dislocations. They further established a well-known relation that not only relates the hardness of a given depth of indentation compared with the hardness of the bulk, but also concluded that the former can be significantly greater than the latter if the depth of indentation decreases to micro or even nanoscales. This relation was later verified by many microindentation experiments. On the other hand, Lam et al.^[39] derived a new set of higher-order metrics to characterise strain gradient behaviors and subsequently developed a strain gradient elastic bending theory for plane-strain beams. The latter investigated the bending of a Bernoulli-Euler beam which contains an internal material length scale parameter that captures the size effect. Although the strain gradient theory^[39] is similar to the nonlocal local stress field theory reported in this paper at a first glance, they are different in the principle and fundamental

concept. The strain gradient theory is based on the strain energy density which depends on both the symmetric part of the first-order deformation gradient (the classical strain) and on the second-order deformation gradient while the nonlocal stress develops from a nonlocal sense of stress where the stress at a reference point is affected by the stress throughout the region of interest in an integral sense through the influence of a nonlocal modulus. The nonlocal stress concept will become clear in the following sections. Based on the molecular-structural-mechanics method with interlayer van der Waals interactions represented by Lennard-Jones potential and a nonlinear truss rod model, Li and Chou^[40] proved that the fundamental vibration frequencies of single- and double-walled carbon nanotubes based on the classical continuum shell model are 40%–60% lower than those based on the atomistic modeling. Although developed from entirely different concepts of physics, these papers^[39–43] all point to a similar consequence to be concluded in this paper that any size-dependent nanoscale parameter tends to induce increased structural stiffness. The conclusion was further substantiated by experiments that showed (i) much higher tensile strengths of finely structured microlaminate films than the strengths of monolithic films^[44]; (ii) significant increased hardness^[38]; and (iii) significant increased bending stiffness of a nano-cantilever with decreasing thickness^[39,45]. Comparison with molecular dynamic simulations on nanotubes for wave propagation^[46] and buckling^[47] also indicate stiffness strengthening behavior for nanotubes with respect to classical solutions.

One potential application of nanostructures is a cantilever nanobeam which has practical use in microelectromechanical systems (MEMS) and nanoelectromechanical systems (NEMS) as an actuator. Using the partial nonlocal stress model for such nano-cantilever with a point load at its tip, one arrives at a solution identical to the local stress, classical cantilever model without any nanoscale effect at all^[36]. For a similar nano-cantilever with distributed loadings, the small-scale effect does present. This is somewhat puzzling because a concentrated point load may be viewed as a distributed load acting within a small finite region. Hence, the partial nonlocal stress model introduces a surprising, discrete, and discontinuous jump in solution for a finite-region distributed load (practically a point load) and a theoretically perfect point load.

Applying the nonlocal elasticity field theory in nanomechanics first proposed by Eringen^[35] and an exact variational principal approach, this paper derives the exact equilibrium conditions, domain governing differential equation, and boundary conditions for bending of nanobeams. These equations and conditions involve essential higher-order terms which are missing in virtually all models and analyses in previously published works on the statics and dynamics of nonlocal nanostructures. Such negligence of higher-order terms in these works results in contradictory nanoscale effects with respect to the conclusion of this paper. Effectively, based on a new nonlocal stress model this paper discovers the truth of nanoscale on equilibrium conditions, governing differential equation and boundary conditions. The true basic static responses for bending of nanobeams with various boundary conditions are also derived. In this context, a new and asymptotic representation of the one-dimensional nanobeam model with nonlocal stress is derived and presented. An asymptotic governing differential equation of infinite order of the strain gradient model and the corresponding infinite number of boundary conditions are also derived and discussed. For practical applications, it explores a reduced higher-order solution to the asymptotic nonlocal model. A few nanobeam examples with various loading and boundary conditions are solved to illustrate the true effects of nanoscale based on nonlocal modelling.

The presented nanobeam bending solutions based on nonlocal stress model should be useful to engineers who are designing MEMS and NEMS devices. Moreover, the higher-order strain gradient solutions serve as benchmarks for reference, convergence, and accuracy of numerical solutions for bending of nanobeams obtained from other mathematical and computational approaches such as molecular dynamics simulations.

2 Derivation of new field equations

This section begins with a brief description of the fundamental concepts of nonlocal elastic stress field theory and the thin beam or the so-called Euler-Bernoulli beam theory. It then applies the nonlocal elastic stress concept to model a nanobeam. Using an exact variational principle approach, it subsequently derives via a rigorous energy formulation new equilibrium conditions and domain governing equations of motion with higher-order effects which are missing in all relevant previously published works.

2.1 Classical and nonlocal elastic field equations

The theory of nonlocal stress field within a domain was first introduced into solid mechanics and materials by Eringen and his associates. It is based on atomic theory of lattice dynamics and experimental observations on phonon dispersion. In accordance with this theory, the nonlocal stress field at a reference point r within a homogeneous and isotropic solid V , which can be one- or multi-dimensional, depends not only on the strains at r but also on strains at all other points of the body. For such a homogeneous and isotropic solid, the linear nonlocal elastic field theory is governed by [32,35]

$$\sigma_{ij,i} + \rho(f_j - \ddot{u}_j) = 0, \quad \sigma_{ij}(r) = \int_V \alpha(|r' - r|, \tau) \sigma'_{ij}(r') dV(r'), \quad (1a, 1b)$$

$$\sigma'_{ij}(r') = \lambda e_{kk}(r') \delta_{ij} + 2\mu e_{ij}(r'), \quad e_{ij}(r') = \frac{1}{2} \left(\frac{\partial u_j(r')}{\partial r'_i} + \frac{\partial u_i(r')}{\partial r'_j} \right), \quad (1c, 1d)$$

where $\sigma_{ij}(r)$, ρ , f_j , and u_j are, respectively, the nonlocal stress tensor, mass density, body force density, and displacement vector at a reference point r in the body, at time t , while \ddot{u}_j , the second derivative of u_j with respect to time t is the acceleration vector at r , and $i, j = 1$ or $i, j = 1, 2$ or $i, j = 1, 2, 3$ depending on the relevant dimension.

Equation (1c) shows the classical constitutive relation where the classical or local stress tensor at r , denoted as $\sigma'_{ij}(r')$, is related to the linear strain tensor $e_{ij}(r')$ at any point r in the body at time t , with λ and μ being Lamé constants, and δ_{ij} being Kronecker delta. It is clear that for nonlocal elasticity, the classical or local constitutive relation (1c) has to be replaced by the nonlocal constitutive relation (1b). Referring to Eq. (1b), $\sigma_{ij}(r)$ at r depends not only on the classical local stress $\sigma'_{ij}(r')$ at that particular point but also on spatial integrations which represent weighted averages of the contributions of local stress of all points within the body V . The spatial weight is represented by a specific nonlocal modulus $\alpha(|r' - r|, \tau)$ which depends on a dimensionless nanolength scale

$$\tau = \frac{e_0 a}{L} \quad (2)$$

of the material of V in which a is an internal characteristic length (e.g., lattice parameter, C-C bond length, granular distance, etc.), L is an external characteristic length (e.g., crack length, wave length, etc.) and e_0 is a material constant. The magnitude of e_0 is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics. In a macroscopic analysis when the effects of nanoscale become infinitely insignificant in the limit $\tau \rightarrow 0$, the effects of strains at points $r \neq r'$ are negligible, the nonlocal modulus approaches to the Dirac delta function and hence $\sigma_{ij}(r') = \sigma'_{ij}(r')$. Consequently, the classical elasticity for continuum mechanics should be recovered in the limit of vanishing nonlocal nanoscale.

Although it is difficult mathematically to obtain the solution of nonlocal elasticity problems due to the spatial integrals in the nonlocal relations, these integro-partial equations can be approximately transformed to equivalent differential constitutive equations within a two-dimensional region and under certain conditions using Green's function with a certain approximation error^[32,35] as

$$(1 - \tau^2 L^2 \nabla^2) \sigma_{ij} = \sigma'_{ij}, \quad (3)$$

where $\nabla^2 = \frac{\partial}{\partial r_k} \frac{\partial}{\partial r_k}$ (summation on k applies) is the Laplace operator and $i, j, k = 1, 2$. For a two-dimensional domain in a Cartesian coordinate system, the Laplace operator becomes $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

2.2 Nonlocal nanobeam modelling

The nonlocal stress theory above is now applied to model a nanobeam with nonlocal stress effect. Consider an Euler-Bernoulli or thin nanobeam with middle surface A , length L , and thickness h . According to the Euler-Bernoulli beam theory, a straight line normal to the mid-plane before deformation remains straight and perpendicular to the deflected midplane after deformation. Based on this assumption, the in-plane displacement $u(x, z, t)$ and the transverse displacement $w(x, t)$ are related by

$$u(x, z, t) = -z \frac{dw(x, t)}{dx}, \quad (4)$$

and the strain-displacement relation is

$$\varepsilon_{xx} = -z \frac{d^2 w(x, t)}{dx^2}, \quad (5)$$

where x is the longitudinal coordinate measured from the left end on the mid-plane of the nanobeam, z is the normal coordinate measured from the midplane, t is the time, and ε_{xx} is the normal strain.

In accordance with the nonlocal stress theory presented in Section 2.1 and in particular the approximate two-dimensional nonlocal stress model expressed in Eq. (3), the simplified nonlocal constitutive equation for normal stress and normal strain in a one-dimensional Euler-Bernoulli nanobeam with nonlocal effect is

$$\sigma_{xx} - \tau^2 L^2 \frac{d^2 \sigma_{xx}}{dx^2} = E \varepsilon_{xx}, \quad (6)$$

where τ , given by Eq. (2), is a dimensionless scale coefficient that incorporates the nanoscale effect and E is the Young's modulus. The equation above can be non-dimensionalized using the following dimensionless parameters

$$\bar{\sigma}_{xx} = \frac{\sigma_{xx}}{E}, \quad \tau = \frac{e_0 a}{L}, \quad \bar{x} = \frac{x}{L}, \quad \bar{z} = \frac{z}{L}, \quad \bar{w} = \frac{w}{L}, \quad (7)$$

as

$$\bar{\sigma}_{xx} - \tau^2 \bar{\sigma}_{xx}^{(2)} = \varepsilon_{xx}, \quad (8)$$

or, using Eq. (5), as

$$\tau^2 \bar{\sigma}_{xx}^{(2)} - \bar{\sigma}_{xx} = \bar{z} \bar{w}_{xx}^{(2)}, \quad (9)$$

where $(\cdot)^{(n)} = d^n/d\bar{x}^n$, derivatives with respect to the dimensionless coordinate \bar{x} . This is an ordinary second-order differential equation and the general solution can be expressed as

$$\bar{\sigma}_{xx} = B_1(\bar{z}) e^{\bar{x}/\tau} + B_2(\bar{z}) e^{-\bar{x}/\tau} + \sum_{n=1}^{\infty} A_n \bar{w}^{(2n)}, \quad (10)$$

where the constants of integration $B_1(\bar{z})$ and $B_2(\bar{z})$ in the homogeneous solution are functions of \bar{z} in general and the last term in the equation above is the particular solution. By differentiating Eq. (10) twice and then substituting it into Eq. (9), one obtains

$$A_n = -\tau^{2(n-1)} \bar{z}. \quad (11)$$

Because the nonlocal stress σ_{xx} is expected to approach the classical local stress σ'_{xx} in the limit of vanishing nanoscale effect $\tau \rightarrow 0$, i.e., $\lim_{\tau \rightarrow 0} \bar{\sigma}_{xx} = \varepsilon_{xx}$, the constants of integration must be zero, or $B_1(\bar{z}) = B_2(\bar{z}) = 0$. Hence, the general solution for the nonlocal stress is

$$\bar{\sigma}_{xx} = - \sum_{n=1}^{\infty} \tau^{2(n-1)} \bar{z} \frac{d^{2n} \bar{w}}{d\bar{x}^{2n}} = \sum_{n=1}^{\infty} \tau^{2(n-1)} \varepsilon_{xx}^{(2(n-1))}. \quad (12)$$

The nonlinear, nonlocal constitutive relation above is new and it was first derived by Lim and Wang^[48]. By referring to the definition of bending moment, $M_{xx} = \int_A \sigma_{xx} z dA$, the dimensionless nonlocal bending moment

$$\bar{M}_{xx} = \frac{M_{xx} L}{EI_{xx}} \quad (13)$$

can be expressed as

$$\begin{aligned} \bar{M}_{xx} &= - \sum_{n=1}^{\infty} \tau^{2(n-1)} \bar{w}^{(2n)} \\ &= - \left(\bar{w}^{(2)} + \tau^2 \bar{w}^{(4)} + \tau^4 \bar{w}^{(6)} + \tau^6 \bar{w}^{(8)} + \tau^8 \bar{w}^{(10)} + \dots \right), \end{aligned} \quad (14)$$

where I_{xx} is the second moment of area with respect to the cross-section of the nanobeam.

2.3 Exact strain energy via a variational principle approach

The strain energy stored in a deformed structure, whether macro or nano in size, is not affected by the presence or absence of nonlocal effects. We now consider a nanobeam with nonlinear, nonlocal constitutive relation expressed by Eq. (12). According to the basic definition, the gain in strain energy per unit volume, i.e., the strain energy density u , of a deformed body with respect to its undeformed condition is defined as the sum (integration) of stress over the history of strain starting from an unstrained condition, or

$$u = \int_0^{\varepsilon_{xx}} \sigma_{xx} d\varepsilon_{xx}. \quad (15)$$

Substituting Eq. (12) into Eq. (15), we obtain

$$u = E \sum_{n=1}^{\infty} \tau^{2(n-1)} \int_0^{\varepsilon_{xx}} \varepsilon_{xx}^{(2(n-1))} d\varepsilon_{xx}. \quad (16)$$

Performing integration over the history of strain starting from an unstrained condition yields the strain energy density as

$$u = u_1 + u_2 + u_3, \quad (17)$$

where

$$u_1 = \frac{1}{2} E \varepsilon_{xx}^2, \quad u_2 = \frac{1}{2} E \sum_{n=1}^{\infty} (-1)^{n+1} \tau^{2n} \left(\varepsilon_{xx}^{(n)} \right)^2, \quad (18a, 18b)$$

$$u_3 = E \sum_{n=1}^{\infty} \left\{ \tau^{2(n+1)} \sum_{m=1}^n \left[(-1)^{m+1} \varepsilon_{xx}^{(m)} \varepsilon_{xx}^{(2(n+1)-m)} \right] \right\}. \quad (18c)$$

The strain energy of the deformed body with volume V is thus

$$U = \int_V u dV = \int_V (u_1 + u_2 + u_3) dV. \quad (19)$$

The variational principle is applied to determine the equilibrium conditions and boundary conditions. Variation of the strain energy in Eq. (19) yields

$$\begin{aligned}
\delta U &= \int_V (\delta u_1 + \delta u_2 + \delta u_3) dV \\
&= \frac{EI_{xx}}{L} \int_0^1 \left(- \sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2(n+1))} \right) \delta \bar{w} d\bar{x} \\
&\quad + \frac{EI_{xx}}{L} \left[\sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2n+1)} \delta \bar{w} - \sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2n)} \delta \bar{w}^{(1)} \right. \\
&\quad + \sum_{n=1}^{\infty} (2n-1) \tau^{2n} \bar{w}^{(2n+1)} \delta \bar{w}^{(2)} - \sum_{n=1}^{\infty} 2n \tau^{2(n+1)} \bar{w}^{(2(n+1))} \delta \bar{w}^{(3)} \\
&\quad + \sum_{n=1}^{\infty} (2n-1) \tau^{2(n+1)} \bar{w}^{(2n+1)} \delta \bar{w}^{(4)} - \sum_{n=1}^{\infty} 2n \tau^{2(n+2)} \bar{w}^{(2(n+1))} \delta \bar{w}^{(5)} \\
&\quad \left. + \sum_{n=1}^{\infty} (2n-1) \tau^{2(n+2)} \bar{w}^{(2n+1)} \delta \bar{w}^{(6)} + \dots \right]_0^1, \tag{20}
\end{aligned}$$

or expressed in terms of nonlocal bending moments,

$$\begin{aligned}
\delta U &= \frac{EI_{xx}}{L} \int_0^1 \left(- \bar{M}_{xx}^{(2)} + 2 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2(n+1))} \right) \delta \bar{w} d\bar{x} \\
&\quad + \frac{EI_{xx}}{L} \left[\left(\bar{M}_{xx}^{(1)} - 2 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n+1)} \right) \delta \bar{w} + \left(- \bar{M}_{xx} + 2 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n)} \right) \delta \bar{w}^{(1)} \right. \\
&\quad - \left(\tau^2 \bar{M}_{xx}^{(1)} + 2 \sum_{n=1}^{\infty} \tau^{2(n+1)} \bar{M}_{xx}^{(2n+1)} \right) \delta \bar{w}^{(2)} + \left(2\tau^4 \sum_{n=1}^{\infty} \tau^{2(n-1)} \bar{M}_{xx}^{(2n)} \right) \delta \bar{w}^{(3)} \\
&\quad - \left(\tau^4 \bar{M}_{xx}^{(1)} + 2 \sum_{n=1}^{\infty} \tau^{2(n+2)} \bar{M}_{xx}^{(2n+1)} \right) \delta \bar{w}^{(4)} + \left(2\tau^4 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n)} \right) \delta \bar{w}^{(5)} \\
&\quad \left. - \left(\tau^6 \bar{M}_{xx}^{(1)} + 2\tau^6 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n+1)} \right) \delta \bar{w}^{(6)} + \dots \right]_0^1. \tag{21}
\end{aligned}$$

2.4 New higher-order governing differential equation, static bending equilibrium equations, and boundary conditions via variational principle

For static bending of a nanobeam with nonlocal stress, the work W exerted by the external transverse load $p = p(x)$ per unit length, end transverse loads $P|_{x=0,L}$ and end moments $R|_{x=0,L}$ is

$$W = \int_0^L p w dx + [Pw]_{x=0}^{x=L} + \left[R \frac{dw}{dx} \right]_{x=0}^{x=L} = L^2 \int_0^1 p \bar{w} d\bar{x} + L[P\bar{w}]_{x=0}^{x=1} + [R\bar{w}^{(1)}]_{x=0}^{x=1}. \tag{22}$$

Variation of this work for bending is

$$\delta W = L^2 \int_0^1 p \delta \bar{w} d\bar{x} + L[P \delta \bar{w}]_{x=0}^{x=1} + [R \delta \bar{w}^{(1)}]_{x=0}^{x=1}. \tag{23}$$

Defining a bending energy functional as

$$I = U - W, \tag{24}$$

and variation of this energy functional, referring to Eqs. (21) and (23), yield

$$\begin{aligned}
\delta I = \delta(U - W) = & \int_0^1 \left\{ -\frac{EI_{xx}}{L} \left[\sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2(n+1))} \right] - L^2 p \right\} \delta \bar{w} d\bar{x} \\
& + \frac{EI_{xx}}{L} \left[\left(\bar{M}_{xx}^{(1)} - 2 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n+1)} - \frac{PL^2}{EI_{xx}} \right) \delta \bar{w} + \left(-\bar{M}_{xx} + 2 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n)} - \frac{RL}{EI_{xx}} \right) \delta \bar{w}^{(1)} \right. \\
& - \left(\tau^2 \bar{M}_{xx}^{(1)} + 2 \sum_{n=1}^{\infty} \tau^{2(n+1)} \bar{M}_{xx}^{(2n+1)} \right) \delta \bar{w}^{(2)} + \left(2\tau^4 \sum_{n=1}^{\infty} \tau^{2(n-1)} \bar{M}_{xx}^{(2n)} \right) \delta \bar{w}^{(3)} \\
& - \left(\tau^4 \bar{M}_{xx}^{(1)} + 2 \sum_{n=1}^{\infty} \tau^{2(n+2)} \bar{M}_{xx}^{(2n+1)} \right) \delta \bar{w}^{(4)} + \left(2\tau^4 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n)} \right) \delta \bar{w}^{(5)} \\
& \left. - \left(\tau^6 \bar{M}_{xx}^{(1)} + 2\tau^6 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n+1)} \right) \delta \bar{w}^{(6)} + \dots \right]_0^1. \tag{25}
\end{aligned}$$

For stability, the variational principle requires that the energy functional be at local extremum such that variation of the energy functional vanishes, or

$$\delta I = \delta(U - W) = 0. \tag{26}$$

Since $\delta \bar{w}$ does not vanish for nontrivial solution, the higher-order governing differential equation with nonlocal effects is

$$-\left[\sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2(n+1))} \right] = \bar{p}, \tag{27}$$

where $\bar{p} = \frac{pL^3}{EI_{xx}}$. In terms of nonlocal dimensionless bending moment, as

$$-\bar{M}_{xx}^{(2)} + 2 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2(n+1))} = \bar{p}. \tag{28}$$

Defining a dimensionless effective nonlocal bending moment with respect to the dimensionless nonlocal bending moment of nanobeam as

$$\begin{aligned}
\bar{M}_{\text{ef}} &= \bar{M}_{xx} - 2 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n)} = \sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2n)} \\
&= -\bar{w}^{(2)} + \tau^2 \bar{w}^{(4)} + 3\tau^4 \bar{w}^{(6)} + 5\tau^6 \bar{w}^{(8)} + 7\tau^8 \bar{w}^{(10)} + \dots,
\end{aligned} \tag{29}$$

hence the higher-order governing differential equation (28) becomes

$$\bar{M}_{\text{ef}}^{(2)} = -\bar{p}. \tag{30}$$

This equation, derived from an exact variational approach, is related to the moment equilibrium condition of a nonlocal stress element as shown in Fig. 1 below.

The corresponding higher-order boundary conditions in Eq. (25) can be grouped into the natural boundary conditions and the geometric boundary conditions. These boundary condi-

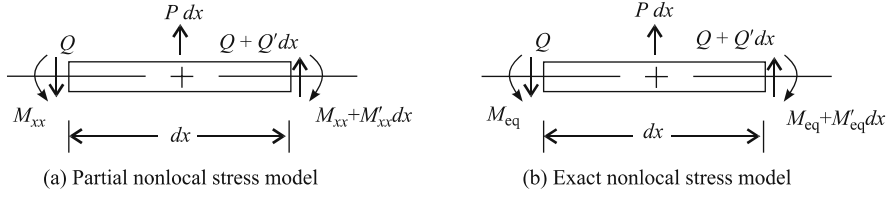


Fig. 1 Static equilibrium for an element of a nanobeam

tions in terms of displacement functions are

$$\left. \begin{aligned}
 & \sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2n+1)} = \bar{P} \quad \text{or} \quad \bar{w} = 0 \\
 & - \sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2n)} = \bar{R} \quad \text{or} \quad \bar{w}^{(1)} = 0 \\
 & \sum_{n=1}^{\infty} (2n-1) \tau^{2n} \bar{w}^{(2n+1)} = 0 \quad \text{or} \quad \bar{w}^{(2)} = 0 \\
 & - \sum_{n=1}^{\infty} 2n \tau^{2(n+1)} \bar{w}^{(2(n+1))} = 0 \quad \text{or} \quad \bar{w}^{(3)} = 0 \\
 & \sum_{n=1}^{\infty} (2n-1) \tau^{2(n+1)} \bar{w}^{(2n+1)} = 0 \quad \text{or} \quad \bar{w}^{(4)} = 0 \\
 & - \sum_{n=1}^{\infty} 2n \tau^{2(n+2)} \bar{w}^{(2(n+1))} = 0 \quad \text{or} \quad \bar{w}^{(5)} = 0 \\
 & \sum_{n=1}^{\infty} (2n-1) \tau^{2(n+2)} \bar{w}^{(2n+1)} = 0 \quad \text{or} \quad \bar{w}^{(6)} = 0 \\
 & \quad \quad \quad \vdots \quad \quad \quad \vdots
 \end{aligned} \right\}_{\bar{x}=0,1}, \quad (31a)$$

where $\bar{P} = \frac{PL^2}{EI_{xx}}$ and $\bar{R} = \frac{RL}{EI_{xx}}$ are the dimensionless end load and end moment, respectively, or

$$\left. \begin{aligned}
 & \bar{M}_{xx}^{(1)} - 2 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n+1)} = \bar{M}_{\text{ef}}^{(1)} = \bar{P} \quad \text{or} \quad \bar{w} = \bar{R} \\
 & - \bar{M}_{xx} + 2 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n)} = -\bar{M}_{\text{ef}} = \bar{R} \quad \text{or} \quad \bar{w}^{(1)} = 0 \\
 & \tau^2 \bar{M}_{xx}^{(1)} + 2 \sum_{n=1}^{\infty} \tau^{2(n+1)} \bar{M}_{xx}^{(2n+1)} = -\tau^2 \left(\bar{M}_{\text{ef}}^{(1)} - 2\bar{M}_{xx}^{(1)} \right) = 0 \quad \text{or} \quad \bar{w}^{(2)} = 0 \\
 & 2\tau^4 \sum_{n=1}^{\infty} \tau^{2(n-1)} \bar{M}_{xx}^{(2n)} = \tau^4 \left(\bar{M}_{\text{ef}}^{(2)} - 3\bar{M}_{xx}^{(2)} \right) = 0 \quad \text{or} \quad \bar{w}^{(3)} = 0 \\
 & \tau^4 \bar{M}_{xx}^{(1)} + 2 \sum_{n=1}^{\infty} \tau^{2(n+2)} \bar{M}_{xx}^{(2n+1)} = -\tau^4 \left(\bar{M}_{\text{ef}}^{(1)} - 2\bar{M}_{xx}^{(1)} \right) = 0 \quad \text{or} \quad \bar{w}^{(4)} = 0 \\
 & 2\tau^4 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n)} = \tau^6 \left(\bar{M}_{\text{ef}}^{(2)} - 3\bar{M}_{xx}^{(2)} \right) = 0 \quad \text{or} \quad \bar{w}^{(5)} = 0 \\
 & \tau^6 \bar{M}_{xx}^{(1)} + 2\tau^6 \sum_{n=1}^{\infty} \tau^{2n} \bar{M}_{xx}^{(2n+1)} = -\tau^6 \left(\bar{M}_{\text{ef}}^{(1)} - 2\bar{M}_{xx}^{(1)} \right) = 0 \quad \text{or} \quad \bar{w}^{(6)} = 0 \\
 & \quad \quad \quad \vdots \quad \quad \quad \vdots
 \end{aligned} \right\}_{\bar{x}=0,1}, \quad (31b)$$

in terms of nonlocal bending moments. In each term, either the natural boundary condition on the left or the geometric boundary condition on the right vanishes but not both at the same time.

To derive the static equilibrium conditions, we first consider the equilibrium of an element of a nanobeam as illustrated in Fig. 1 where Fig. 1(a) shows the classical, local stress equilibrium

condition while the other shows the nonlocal one in which \bar{M}_{ef} is the dimensionless effective nonlocal bending moment defined in Eq. (29).

For both cases, the equilibrium of vertical forces yields an identical condition

$$\bar{Q}^{(1)} + \bar{p} = 0, \quad (32)$$

where

$$\bar{Q} = \frac{QL^2}{EI_{xx}} \quad (33)$$

is the dimensionless shear force in which Q is the dimensional shear force. The moment equilibrium conditions for Figs. 1(a) and 1(b) are similar but one is expressed in \bar{M}_{xx} while the other in \bar{M}_{ef} . Differentiating these two conditions and making use of Eq. (33) show that the moment equilibrium of \bar{M}_{ef} in Fig. 1(b) actually yields the governing differential equation (30) while the other yields an incorrect differential equation. Consequently, it is clear that the true moment equilibrium condition for a nonlocal stress element should be derived via an effective nonlocal bending moment parameter rather than by directly extending the local bending moment of a classical element to the nonlocal bending moment of a nonlocal stress element in Fig. 1(a). From Eqs. (30) and (32), the equations for static equilibrium with higher-order nonlocal effects are

$$\bar{Q}^{(1)} + \bar{p} = 0, \quad \bar{M}_{\text{ef}}^{(1)} - \bar{Q} = 0. \quad (34a, 34b)$$

For a nanobeam with free support conditions, we have from Eqs (31a–b) that

$$\begin{aligned} \bar{M}_{\text{ef}}^{(1)} \Big|_{\bar{x}=0,1} &= \bar{Q} \Big|_{\bar{x}=0,1} = \bar{P}, \\ -\bar{M}_{\text{ef}} \Big|_{\bar{x}=0,1} &= \bar{R}, \\ -\tau^2 \left(\bar{M}_{\text{ef}}^{(1)} - 2\bar{M}_{xx}^{(1)} \right) \Big|_{\bar{x}=0,1} &= 0, \\ \tau^4 \left(\bar{M}_{\text{ef}}^{(2)} - 3\bar{M}_{xx}^{(2)} \right) \Big|_{\bar{x}=0,1} &= 0, \\ -\tau^4 \left(\bar{M}_{\text{ef}}^{(1)} - 2\bar{M}_{xx}^{(1)} \right) \Big|_{\bar{x}=0,1} &= 0, \\ &\dots \end{aligned} \quad (35)$$

where shear forces, effective nonlocal bending moments, and all other natural boundary conditions vanish. For simply supported conditions, we have

$$\bar{w} \Big|_{\bar{x}=0,1} = \bar{w}^{(2)} \Big|_{\bar{x}=0,1} = \bar{w}^{(4)} \Big|_{\bar{x}=0,1} = \dots = 0, \quad (36a)$$

as the geometric boundary conditions, and

$$\begin{aligned} -\bar{M}_{\text{ef}} \Big|_{\bar{x}=0,1} &= \bar{R}, \\ \left[2\tau^4 \sum_{n=1}^{\infty} \tau^{2(n-1)} \bar{M}_{xx}^{(2n)} \right] \Big|_{\bar{x}=0,1} &= \tau^4 \left(\bar{M}_{\text{ef}}^{(2)} - 3\bar{M}_{xx}^{(2)} \right) \Big|_{\bar{x}=0,1} = 0, \\ &\dots \end{aligned} \quad (36b)$$

as the natural boundary conditions. Similarly, for clamped conditions, we have

$$\bar{w} \Big|_{\bar{x}=0,1} = \bar{w}^{(1)} \Big|_{\bar{x}=0,1} = \bar{w}^{(2)} \Big|_{\bar{x}=0,1} = \bar{w}^{(3)} \Big|_{\bar{x}=0,1} = \bar{w}^{(4)} \Big|_{\bar{x}=0,1} = \dots = 0, \quad (37)$$

where the deflection, slope, point of inflexion and all other geometric boundary conditions vanish. The physical interpretation of the higher-order natural boundary conditions is yet to be

ascertained and an in-depth investigation of their nature will be reported.

More generally, the boundary conditions in Eqs. (31a–b) or Eqs. (35–37) may not completely vanish concurrently at $\bar{x} = 0, 1$. For specified point load and point couple, specified displacement, specified displacement gradient and/or specified higher-order load/moment natural effects or displacement gradient at the boundaries, specified values may replace the zero values at the locations.

It is interesting to note that higher-order nonlocal effects only exist in bending moment equilibrium in Eq. (34b), defined as the effective nonlocal bending moment in Eq. (29), but not shear force equilibrium in Eq. (34a). This is because the bending moments are derived from the in-plane nonlocal stress components given by Eq. (1b) which is the key subject of the two-dimensional nonlocal stress theory. While reducing or simplifying the nonlocal stress to Eq. (6), we have implicitly adopted the assumption of a two-dimensional domain as explained by Eringen^[32,35]. Hence, the vertical shear force, which is out-of-plane with respect to the middle surface, should not be governed by Eq. (6) and hence no nonlocal effects exist. Simultaneous consideration of in-plane and out-of-plane nonlocal effects would render Eq. (6) inapplicable but instead a three-dimensional kernel^[32,35] should be adopted in the original nonlocal stress model expressed in Eqs. (1a–d) and solved using the Green function. Hence, many papers (12 papers) which consider coupling of in-plane two-dimensional (2D) nonlocal stress and transverse cross-sectional 2D nonlocal shear stress using the Timoshenko model are not only not rigorous at all, but also yield highly suspicious results with respect to the original nonlocal stress model^[32,35].

The nonlocal higher-order governing differential equations (27) and (28), the associated boundary conditions (31a–b), and the nonlocal higher-order equilibrium conditions have been derived in an exact and rigorous manner via the variational principle. These new equations will be shown in the following sections to not only solve many analytical puzzles previously encountered, but also overturn all existing models and results related to the statics and dynamics of a CNTs, nanobeams, nanoplates, nanoshells, etc., based on the nonlocal stress model.

3 Exact bending solution for a cantilever nanobeam with an end point load

A classical example is considered here to highlight the effects of nonlocal nanoscale effects. The surprising prediction earlier that a cantilevered nonlocal nanobeam with a tip point load is free from any nonlocal influences is discussed^[36]. In the example, some common molecule and nanotube values are adopted: the C-C bond length $a = 0.12$ nm, nanobeam length $L = 10$ nm, and $e_0 = 0 \rightarrow 1.67$, and hence, the nanoscale parameter $\tau = 0 \rightarrow 0.02$. In some of the numerical cases, the choice of τ has been rather uncommon although not impossible for nanobeams. In these examples, $e_0 = 1$ for $5 \leq L/a \leq 55$ has been adopted, and in the extreme if $L/a = 5$ is considered, we have $\tau = 0.2$. These examples are reproduced here only for the purpose of comparison.

For a cantilever (CF) nanobeam clamped at $\bar{x} = 0$, without end moment $\bar{R}|_{\bar{x}=1}=0$, and with a point load $\bar{P}|_{\bar{x}=1} = P_0$ or

$$\bar{P}_0 = \frac{P_0 L^2}{EI_{xx}}, \quad (38)$$

in dimensionless terms, at $\bar{x} = 1$, the nonlocal higher-order governing differential equation (27) yields

$$-\left[\sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2(n+1))} \right] = 0. \quad (39)$$

Retaining the first two terms, the equation above becomes

$$-\tau^2 \bar{w}^{(6)} + \bar{w}^{(4)} = 0. \quad (40)$$

The solution to the sixth-order differential equation above comprising only the homogeneous solution can be expressed as

$$\bar{w} = C_0 + C_1\bar{x} + C_2\bar{x}^2 + C_3\bar{x}^3 + C_4e^{\bar{x}/\tau} + C_5e^{-\bar{x}/\tau}, \quad (41)$$

where C_i ($i = 0, 1, 2, \dots, 5$) are constants of integration. From Eqs. (35) and (37), the clamped and free support conditions, taking only terms up to τ^2 for consistency, are

$$\bar{w}|_{\bar{x}=0} = \bar{w}^{(1)}|_{\bar{x}=0} = \bar{w}^{(2)}|_{\bar{x}=0} = 0, \quad (42a)$$

at the clamped end $\bar{x} = 0$, and

$$\begin{aligned} [\tau^2 \bar{w}^{(5)} - \bar{w}^{(3)}]_{\bar{x}=1} &= \bar{Q}|_{\bar{x}=1} = \bar{P}_0 \\ [-\tau^2 \bar{w}^{(4)} + \bar{w}^{(2)}]_{\bar{x}=1} &= -\bar{M}_{\text{ef}}|_{\bar{x}=1} = 0 \end{aligned} \quad (42b)$$

$$\tau^2 [3\tau^2 \bar{w}^{(5)} + \bar{w}^{(3)}]_{\bar{x}=1} = 0$$

at the free end $\bar{x} = 1$. Substituting Eq. (41) into Eqs. (42a,42b), the constants of integration are

$$\begin{aligned} C_0 &= \tau^2 \bar{P}_0, \quad C_1 = \frac{\tau(2 - \tau e^{1/\tau} - 2e^{2/\tau})}{2(1 + e^{2/\tau})} \bar{P}_0, \quad C_2 = \frac{\bar{P}_0}{2}, \\ C_3 &= -\frac{\bar{P}_0}{6}, \quad C_4 = -\frac{\tau^2(4 - \tau e^{1/\tau})}{4(1 + e^{2/\tau})} \bar{P}_0, \quad C_5 = -\frac{\tau^2(4e^{1/\tau} + \tau)e^{1/\tau}}{4(1 + e^{2/\tau})} \bar{P}_0. \end{aligned} \quad (43)$$

Hence, the general solution using a nonlocal stress model for a cantilever nanobeam with a point load $\bar{P}|_{\bar{x}=1} = P_0$ is

$$\begin{aligned} \bar{w} &= \frac{\bar{P}_0}{12} \left[6\bar{x}^2 - 2\bar{x}^3 + \frac{6\tau(2 - \tau e^{1/\tau} - 2e^{2/\tau})}{1 + e^{2/\tau}} \bar{x} + 12\tau^2 \right. \\ &\quad \left. - \frac{3\tau^2(4 - \tau e^{1/\tau})}{1 + e^{2/\tau}} e^{\bar{x}/\tau} - \frac{3\tau^2(4e^{1/\tau} + \tau)}{1 + e^{2/\tau}} e^{(1-\bar{x})/\tau} \right]. \end{aligned} \quad (44)$$

Because only the first two terms are retained in Eq. (40), the dimensionless moment from Eq. (14) should be consistently approximated by

$$\bar{M}_{xx} = -\bar{P}_0 \left[1 - \bar{x} - \frac{4 - \tau e^{1/\tau}}{2(1 + e^{2/\tau})} e^{\bar{x}/\tau} - \frac{4e^{1/\tau} + \tau}{2(1 + e^{2/\tau})} e^{(1-\bar{x})/\tau} \right]. \quad (45)$$

The dimensionless effective nonlocal bending moment can be derived by substituting Eq. (45) into Eq. (29), with only two terms, as

$$\bar{M}_{\text{ef}} = -\bar{w}^{(2)} + \tau^2 \bar{w}^{(4)} = -\bar{P}_0(1 - \bar{x}), \quad (46)$$

or alternatively, by integrating twice the static equilibrium conditions Eqs. (34a,b) using $p = 0$, and the boundary conditions $\bar{Q}|_{\bar{x}=1} = \bar{M}_{\text{ef}}^{(1)}|_{\bar{x}=1} = \bar{P}_0$ and $-\bar{M}_{\text{ef}}|_{\bar{x}=1} = 0$.

The maximum transverse deflection occurs at the free end $\bar{x} = 1$ while the maxima for bending moment and effective nonlocal bending moment occur at the clamped end $\bar{x} = 0$. They are, respectively, given by

$$\bar{w}_{\text{max}} = \frac{\bar{P}_0}{12} \left[4 + 12\tau^2 + \frac{3\tau(4 - \tau^2)(1 - e^{2/\tau}) - 30\tau^2 e^{1/\tau}}{1 + e^{2/\tau}} \right], \quad (47)$$

$$(\bar{M}_{xx})_{\max} = -\bar{P}_0, \quad (\bar{M}_{ef})_{\max} = -\bar{P}_0. \quad (48, 49)$$

The effect of nanoscale τ on the dimensionless, static deflection \bar{w} as expressed in Eq. (44) is illustrated in Fig. 2. The nanoscale τ ranges from 0, a classical or local beam solution, to 0.02. As observed in the figure, increasing nanoscale τ tends to reduce the static deflection of the nanobeam. Hence, the classical, local stress field theory overestimates the static deflection of a nanobeam.

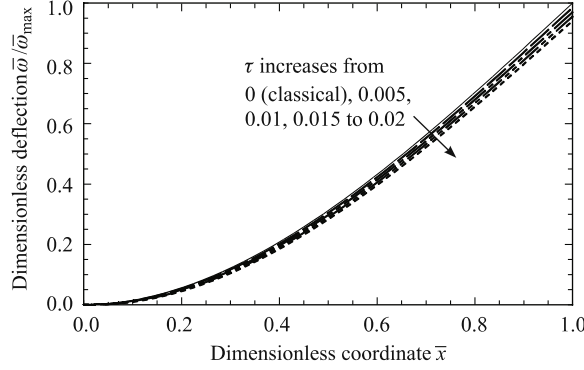


Fig. 2 The effect of τ on \bar{w}/\bar{w}_{\max} for a CF nanobeam with point load $P|_{\bar{x}=1} = P_0$

The identical CF nanobeam with a point load at the tip was analyzed using the partial nonlocal stress model^[36,48]. The dimensionless deflection expressions, respectively after standardizing the parameters, are

$$\bar{w} = \frac{\bar{P}_0}{6} (3\bar{x}^2 - \bar{x}^3), \quad (50)$$

and

$$\bar{w} = \frac{\bar{P}_0}{6} \left[3\bar{x}^2 - \bar{x}^3 + 6\tau \left(\cot \frac{1}{\tau} \right) \bar{x} - 6\tau^2 \left(1 + \cot \frac{1}{\tau} \sin \frac{\bar{x}}{\tau} - \cos \frac{\bar{x}}{\tau} \right) \right]. \quad (51)$$

Equation (50) indicates that the dimensionless deflection derived via the partial nonlocal stress model^[36] results in the very puzzling bending solution that a CF nanobeam modeled by the nonlocal stress elastic field theory^[32,35] yields the identical classical CF beam solution which is not affected at all by the nanoscale effect. Effectively, it concludes that such a CF nanobeam in the presence of a nonlocal stress field behaves as though there were no nonlocal stress effect! It will be easily verified that for a CF nanobeam with distributed loadings using this partial nonlocal stress model, the nanoscale effect, in fact, does present. This is somewhat puzzling because a concentrated point load may be viewed as a distributed load acting within a small finite region. Consequently, the partial nonlocal stress model introduces a surprising, discrete and discontinuous jump in solution for a finite-region distributed load (practically a point load) and a theoretically perfect point load.

Further, comparing Eqs. (50) and (51) with Eq. (44), it is obvious that the partial nonlocal stress model predicts higher \bar{w} with increasing τ while the exact nonlocal stress model otherwise.

4 Further discussion

4.1 Applicability of kernel function for two-dimensional nonlocal moduli

It is noted that the widely used nonlocal stress ordinary differential equation

$$(1 - \tau^2 L^2 \nabla^2) \sigma_{ij} = \sigma'_{ij} \quad (3)$$

is derived from the elastic field equation (1b) by assuming a two-dimensional nonlocal modulus for the kernel function and thus induces a modeling error of as much as 6% [32,35]. Hence this nonlocal stress expression has been specifically meant for a 2D nonlocal stress field within a 2D domain. It is well justified when the 2D nonlocal stress is applied to the mid-plane or mid-surface of a nanobeam or a nanoplate. However, to apply the identical 2D nonlocal stress to the mid-surface and the transverse cross section of a thick beam, i.e., a Timoshenko beam for describing respectively the coupled flexural deformation and shear deformation, requires rigorous justification. Direct coupling of the two nonlocal moduli for two distinct 2D domains is very much in doubt. Even if coupling of two 2D nonlocal moduli is possible, the different and anisotropic orientation of atoms implies that the nonlocal nanoscale parameter $\tau = e_0 a/L$ should be anisotropic for a particular shear correction factor.

4.2 Applicability of an approximate nonlocal strain gradient model

There are a number of papers which adopted an approximate nonlocal model with strain gradients as follows:

$$\sigma_x = E \left[\varepsilon_{xx} + (e_0 a)^2 \frac{\partial^2 \varepsilon_{xx}}{\partial x^2} \right] \quad \text{or} \quad \bar{\sigma}_x = \varepsilon_{xx} + \tau^2 \frac{\partial^2 \varepsilon_{xx}}{\partial \bar{x}^2}. \quad (52a, 52b)$$

Comparing Eq. (52b) with Eq. (8) shows that this approximate nonlocal model assumes the higher-order nonlocal stress, first established by Eringen^[32,35] as expressed in Eqs. (1c) and (3), as the local or classical stress,

$$\frac{\partial^2 \bar{\sigma}_{xx}}{\partial \bar{x}^2} \approx \frac{\partial^2 \bar{\sigma}'_{xx}}{\partial \bar{x}^2} = \frac{\partial^2 \varepsilon_{xx}}{\partial \bar{x}^2}. \quad (53)$$

Further comparing Eq. (52b) with Eq. (12) shows that the approximation above is equivalent to retaining only two terms in the asymptotic solution of the second-order ordinary differential equation for nonlocal stress given in Eq. (12). Such approximate model, although not invalid, must be applied with care that it is only valid to the order of τ^2 .

4.3 On the truth of nonlocal nanoscale effects with respect to the partial and exact nonlocal stress models

Virtually all published works on the statics and dynamics of nonlocal nanostructures are based on the partial nonlocal stress models. These works presented a variety of topics, including bending, vibration, buckling and wave propagation, using different solution methodologies. Although seemingly independent, however, all the works can be traced back, either directly or indirectly, to originate from a common equilibrium model, i.e., the “classical equilibrium model with partial nonlocal effects associated” or the partial nonlocal equilibrium model, as illustrated in Fig. 1(a). Such partial models somehow inadvertently neglect the existence of higher-order nonlocal effects of the “nonlocal bending moment” of a thin, structural nano-element which, consequently, renders a nano-element not in static or dynamic equilibrium. Equivalently, the model in Fig. 1(a) is not in static equilibrium.

As far as this partial nonlocal element “in equilibrium” is concerned, it is an extension of an element in local, classical equilibrium by directly replacing the local bending moment with the nonlocal bending moment without rigorously examining whether such generalization is actually valid. It has been established in this paper that the direct extension of bending moment equilibrium is completely perfect should the nonlocal bending moment be replaced by an effective nonlocal bending moment, derived in Eq. (29), which includes higher-order differential terms, as illustrated in Fig. 1(b).

For a nanobeam subject to a general loading p , the partial nonlocal stress model yields a domain governing differential equation as [48],

$$\sum_{n=1}^{\infty} \tau^{2(n-1)} \bar{w}^{(2(n+1))} = \bar{p}, \quad (54a)$$

or, retaining only two terms, as [36, 48],

$$\tau^2 \bar{w}^{(6)} + \bar{w}^{(4)} = \bar{p} , \quad (54b)$$

while the exact nonlocal stress model yields

$$- \left[\sum_{n=1}^{\infty} (2n-3) \tau^{2(n-1)} \bar{w}^{(2(n+1))} \right] = \bar{p} , \quad (27)$$

or retaining only two terms,

$$-\tau^2 \bar{w}^{(6)} + \bar{w}^{(4)} = \bar{p} . \quad (55)$$

Comparing Eq. (54b) with (55), it is clear that the partial nonlocal stress model predicts completely reverse effects of the nanoscale represented by $\tau = e_0 a / L$. While the former predicts increasing static deflection, or equivalently stiffness softening effect, with the consideration of nonlocal nanoscale irrespective of how small the nanoscale is, the latter predicts lower static deflection, or equivalently stiffness strengthening effect, through an exact nonlocal elasticity model as derived in this paper.

The conclusion of stiffness strengthening effect of nanobeams with increasing nanoscale effect is consistent qualitatively with other published research works via other non-nonlocal elasticity approaches. Some noted instance including the strain gradient theory^[38-39]; molecular-structural-mechanics method for free vibration of a single-walled and doubled-walled carbon nanotubes^[40] which concluded that the fundamental frequencies of the classical solution could be significantly lower than the atomistic simulation solutions by 40% to 60%; and a modified couple stress theory^[41-43]. The conclusion was further substantiated by experiments that showed (i) much higher tensile strengths of finely structured microlaminate films than the strengths of monolithic films^[44]; (ii) significant increased hardness of nanoindentation of crystalline materials^[38]; and (iii) significantly increased bending stiffness of a nano-cantilever with decreasing thickness^[39,45]. Comparison with molecular dynamic simulations on nanotubes for wave propagation^[46] and buckling^[47] also indicates stiffness strengthening behavior for nanotubes with respect to classical solutions.

5 Conclusions

Through an exact nonlocal stress field modeling and a rigorous variational principle formulation, this paper has successfully addressed three critical but overlooked issues in bending of nanobeams. (i) In all cases of study, the presence of increasing nonlocal stress effects in fact induce increased nanostructural stiffness, i.e., decreasing static deflection, which is consistent with physical intuition according to the nonlocal field theories. (ii) There is no intriguing conclusion that nanoscale effects are absent in the bending solutions of nanobeams, i.e., all nanobeam bending solutions are affected by the presence of nonlocal stress in the field. In addition, the classical bending solutions are recovered when in the limit of vanishing nonlocal nanoscale. (iii) Higher-order boundary conditions are derived which are consistent with a higher-order governing differential equation.

On the other hand, the paper also derived new equilibrium equations, which include essential higher-order nonlocal terms missing in the partial nonlocal stress models, via an exact variational principle approach. The domain governing higher-order differential equations and higher-order boundary conditions have also been established. It is also concluded that the widely accepted equilibrium conditions of the partial nonlocal stress models are in fact not in equilibrium, but they can be made perfect should the nonlocal bending moment be replaced by an effective nonlocal bending moment.

The analytical results for vibration and buckling of nanobeams are currently in progress.

Although new results are pending and to be reported, it has been preliminarily verified that increasing nanoscale stress effect yields increasing nanobeam stiffness which, in turn, induces increasing natural frequency and increasing buckling load.

Acknowledgements The work described in this paper was supported by a grant from Research Grants Council of the Hong Kong Special Administrative Region (Project No. CityU 117406). Special thanks also go to Dr J.C. Niu of Shandong University, P. R. China, for some valuable discussions and assistance.

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