where F_K is the functional

$$F_K(M)=F(KM) \quad \text{for} \quad M \epsilon \mathfrak{R}.$$

Since Leżański's hypotheses about Ξ and X were not symmetrical, his results are more complicated than those in § 4. He examined only the equations (58) and (60). Instead of the equations (57) and (59) he examined the equations conjugate to (58) and (60) in the space Ξ^*. Besides the equation (58), he examined, more generally, the equation[10])

$$\xi(I+\lambda TK)=\xi_0.$$

However, this generalization is not essential since Leżański's [2] (p. 252) determinant of this equation coincides with the determinant $D(E+\lambda F_K)$ of the operation $I+\lambda T_{F_K}=I+\lambda T_F K$.

Notice that Theorem 2 remains true if we admit the original hypothesis of Leżański[11]).

The connexion between Leżański's [2,3] theory and Ruston's [5,6] theory should be discussed separately. We notice here only that Leżański's formalism is more general than that of Ruston (the question whether they are equivalent remains open). Therefore the theorem on multiplication of determinants holds also in Ruston's theory.

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[10]) $(I+\lambda KT)\xi=\xi_0$ in the original notation of Leżański.

[11]) See Sikorski [7].

On the two-norm convergence

by

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G. Fichtenholz [4] has introduced in some concrete Banach spaces a kind of convergence weaker than that generated by norm. In a previous paper [2] I introduced a general convergence in linear spaces which I called *two-norm convergence,* containing as particular cases the convergences of Fichtenholz. In this paper[1]) I shall complete the results obtained in [2].

1. Let X be an F-space (Banach [3], p. 35) and denote by $\|x\|$ the norm[2]) in X. Suppose that in X a second norm $\|x\|^*$ is defined, not stronger than $\|x\|$, i. e. such that

(i) $$\|x_n\|\to 0 \quad \text{implies} \quad \|x_n\|^*\to 0.$$

A sequence $\{x_n\}$ of elements of X will be called *γ-convergent to x_0* if it is bounded with respect to the norm $\|x\|$[3]) and if $\|x_n-x_0\|^*\to 0$; we shall then write

$$\gamma\text{-}\lim_n x_n=x_0 \quad \text{or} \quad x_n\xrightarrow{\gamma}x_0.$$

Convergence γ will be termed the *two-norm convergence.* The space X supplied with this convergence will be denoted by X_γ — it is evidently an L^*-space (Kuratowski [5], p. 84), moreover, addition of elements and multiplication by scalars are continuous.

A convergence generated by norm will be termed the *norm-convergence.* The convergence γ is in general not equivalent[4]) to a norm-con-

[1]) The results of which were presented on May 23th 1947 to the Polish Mathematical Society, Section of Poznań. Since that time Orlicz [7] has developed a theory of Saks spaces which are closely related to the notion of the two-norm convergence.

[2]) Here by a *norm* is meant an *F-norm;* it is a non-negative functional $\|x\|$, satisfying the postulates: (a) $\|x\|=0$ if and only if $x=0$; (b) $\|x+y\|\leqslant\|x\|+\|y\|$; (c) $a_n\to a_0$, $\|x_n-x_0\|\to 0$ implies $\|a_nx_n-a_0x_0\|\to 0$.

[3]) The sequence is *bounded with respect to* (or *under*) *the norm* $\|x\|$ if $t_n\to 0$ implies $\|t_nx_n\|\to 0$. This notion goes back to Banach.

[4]) Two convergences α and β in L^*-space are said to be *equivalent* if the classes of convergent sequences in both convergencies coincide and the limits under both are equal.

vergence, unless the norms $\|x\|$ and $\|x\|^*$ are essentially the same. This results from the propositions to follow.

1.1. *If the convergence γ is equivalent to a norm-convergence, it is equivalent to the convergence generated by the norm $\|x\|$.*

For the proof see [2], 2.2.2, p. 187.

1.2. *Let X be a Banach space under the norm $\|x\|$. Each of the following conditions is necessary and sufficient that the convergence γ be equivalent to a norm-convergence:*

(a) $\|x_n\|^*\to 0$ *implies boundedness of the sequence* $\{x_n\}$ *under the norm* $\|x\|$;

(b) *the norms* $\|x\|$ *and* $\|x\|^*$ *are equivalent*[5]);

(c) *the space X is complete under the norm* $\|x\|^*$.

Proof. (a) *implies* (b). Let $\|x_n\|^*\to 0$, then as in the proof of 1.1 we can show that $\|x_n\|\to 0$; conversely $\|x_n\|\to 0$ implies $\|x_n\|^*\to 0$ by (i).

(b) *implies* (c). Indeed, X is a Banach space under the norm $\|x\|$.

(c) *implies* (a). Since (i) is satisfied, by a theorem of Banach ([3], p. 41) the norms $\|x\|$ and $\|x\|^*$ are equivalent.

The sufficiency of the condition (b) is obvious. The condition (a) is necessary. Indeed, let $\|x_n\|^*\to 0$, suppose that $\|x_n\|\to\infty$, and set $\bar{x}_n = x_n/\|x_n\|$. Then $\bar{x}_n \xrightarrow{\gamma} 0$, whence by 1.1 $\|\bar{x}_n\|\to 0$. This, however, is impossible, for $\|\bar{x}_n\|=1$.

The proposition 1.2 is true also if X is a B_0-space (Mazur and Orlicz [6]).

A convergence α in X will be said to be *metrical* if it is possible to introduce in X a distance $\varrho(x,y)$ so that $x_n\xrightarrow{\alpha}x_0$ be equivalent to $\varrho(x_n,x_0)\to 0$.

1.3. *If the convergence γ is metrical, then γ is equivalent to the convergence generated by the norm $\|x\|$.*

Proof. Suppose the contrary. Then there exists a sequence $x_n\xrightarrow{\gamma}0$ such that $\inf_n\|x_n\|>\varepsilon>0$. Write $x_{kn}=kx_n$; clearly $\gamma\text{-}\lim_n x_{kn}=0$ for $k=1,2,\ldots$; there does not exist, however, any sequence $n_k\to\infty$ of indices such that $\gamma\text{-}\lim_k x_{kn_k}=0$. Indeed, let $\vartheta_k=k^{-1}$, then $\|\vartheta_k x_{kn_k}\|=\|x_{n_k}\|$ does not tend to 0. Thus the convergence γ is deprived of the following property which belongs to all metrical convergencies: if $\lim_n x_{kn}=x_k$ for $k=1,2,\ldots$ and $\lim_k x_k=x_0$, then there exists a sequence $n_k\to\infty$ of indices such that $\lim_k x_{kn_k}=x_0$.

[5]) *i. e.* the convergences generated by these norms are equivalent.

2. Now we add two postulates. The space X is not complete under the norm $\|x\|^*$ unless the convergence γ is equivalent to a norm-convergence. By X^* we shall denote the completion of the space X under the norm $\|x\|^*$. We now suppose the following postulates:

(ii) If the sequence $\{x_n\}$ is bounded under the norm $\|x\|$, $x_0\in X^*$, and $\|x_n-x_0\|^*\to 0$, then $x_0\in X$.

(iii) If $x_n, x_0\in X$ and $\|x_n-x_0\|^*\to 0$, then $\|x_0\|\leqslant\varliminf_n\|x_n\|$.

The postulate (ii) implies that the convergence γ is *sequentially complete*[6]) ([2], p. 189), and it is obvious that (ii) is also necessary for the sequential completeness of the convergence γ.

Now we present some examples of spaces with the γ-convergence satisfying the conditions (i)-(iii)[7]).

A. The space $\boldsymbol{M}_\gamma$. For $X=\boldsymbol{M}$ set

$$\|x\|^*=\int_0^1|x(t)|\,dt,$$

then $X^*=\boldsymbol{L}$. The convergence γ may be characterized as follows:

$\gamma\text{-}\lim_n x_n=x_0$ means that $\operatorname{ess\,sup}|x_n(t)|<K$ and $\lim_n \operatorname{as} x_n(t)=x_0(t)$.

This follows from the fact that for essentially bounded sequences the convergence in mean is equivalent to asymptotical convergence.

B. The space $\boldsymbol{L}_\gamma$. For $X=\boldsymbol{L}$ write

$$\|x\|^*=\int_0^1|x(t)|\,[1+|x(t)|]^{-1}\,dt,$$

then $X^*=\boldsymbol{S}$. Conditions (ii) and (iii) are satisfied by Fatou's lemma. The convergence γ may be characterized as follows:

$\gamma\text{-}\lim_n x_n=x_0$ means that $\int_0^1|x_n(t)|\,dt<K$ and $\lim_n \operatorname{as} x_n(t)=x_0(t)$.

C. The space $\boldsymbol{L}^2_\gamma$. For $X=\boldsymbol{L}^2$ write

$$\|x\|^*=\int_0^1|x(t)|\,dt,$$

[6]) *i. e.* has the following property: if $p_n\to\infty$, $q_n\to\infty$ implies $\gamma\text{-}\lim_n(x_{p_n}-x_{q_n})=0$, then the sequence $\{x_n\}$ is γ-convergent.

[7]) For the definitions of the spaces $\boldsymbol{M}, \boldsymbol{L}, \boldsymbol{L}^2, \boldsymbol{S}, \boldsymbol{m}, \boldsymbol{s}$ see Banach [3], p. 9-12; these spaces are denoted there by (M), (L), (L^2), (S), (m), (s).

then $X^*=L$. We easily see that conditions (i)-(iii) are satisfied: (i) — by the inequality of Schwarz, (ii) and (iii) — by Fatou's lemma. The convergence γ is characterized as follows:

$$\gamma\text{-}\lim_n x_n = x_0 \text{ means that } \int_0^1 |x_n(t)|^2 dt < K \text{ and } \int_0^1 |x_n(t)-x_0(t)|\,dt \to 0.$$

D. The space V_γ. Let V denote the space of functions $x=x(t)$ of bounded variation in $\langle 0,1\rangle$; the norm is defined as $\|x\|=|x(0)|+ +\operatorname{var} x(t)$. For $X=V$ write $\|x\|^*=\sup|x(t)|$, then $X^*=$ the space of bounded functions which are uniform limits of functions belonging to V. Conditions (i)-(iii) are satisfied, the convergence γ is characterized as follows:

$\gamma\text{-}\lim x_n = x_0$ means that $\operatorname{var} x_n(t)\leqslant K$ and $x_n(t)\to x(t)$ uniformly in $\langle 0,1\rangle$.

E. The space m_γ. For $X=m$ denote by $x=\{u_n\}$ the generic element of m and write

$$\|x\|^* = \sum_{n=1}^{\infty} 2^{-n}|u_n|[1+|u_n|]^{-1}.$$

Then $X^*=s$. Conditions (i)-(iii) are evidently satisfied. The convergence γ is characterized as follows:

$\gamma\text{-}\lim_p x_p = x_0$ means that $|u_{pn}|<K$ and $\lim_p u_{pn}=u_{0n}$ for $n=1,2,\ldots$ (here $x_p=\{u_{pn}\}$, $x_0=\{u_{0n}\}$).

3. A functional $\xi(x)$ defined in X_γ is said to be *γ-linear* if it is additive and $\gamma\text{-}\lim_n x_n = x_0$ implies $\lim_n \xi(x_n)=\xi(x_0)$.

Every additive functional continuous under the norm $\|x\|^*$ is γ-linear; hence if the space X^* is of B_0-type, then there exist non trivial γ-linear functionals. Every γ-linear functional is obviously linear under the norm $\|x\|$. So it may happen that in X three kinds of linear functionals are defined. We shall see below that in particular cases those types of linear functionals may coincide.

Fichtenholz has shown ([4], p. 199) that the general form of γ-linear functionals in M_γ is

$$\xi(x)=\int_0^1 x(t)g(t)\,dt \tag{1}$$

where $g(t)$ is an arbitrary Lebesgue integrable function. The same author has shown that the general form of γ-linear functionals in V_γ is

$$\xi(x)=Cx(0)+\int_0^1 g(t)\,dx(t)$$

with arbitrary continuous $g(t)$.

3.1. *The general form of γ-linear functionals in L^2_γ is* (1) *with $g(t)$ essentially bounded.*

Proof. The sufficiency being obvious, we prove only the necessity. If $\xi(x)$ is a γ-linear functional, it must be linear under the norm $\|x\|$, hence of form (1) with $g(t)\in L^2$ (Banach [3], p. 64). Suppose, if possible, that $\operatorname{ess\,sup}|g(t)|=\infty$, then there exists a sequence $\{a_n\}$, such that $a_n\to\infty$, and such that the set

$$E_n = \underset{t}{E}\{a_n<|g(t)|<a_n+1\}$$

is of positive measure. Write

$$x_n(t)=\begin{cases} g(t)\{\int_{E_n}|g(t)|^2dt\}^{-1} & \text{for } t\in E_n,\\ 0 & \text{elsewhere.}\end{cases}$$

Obviously

$$\int_0^1 |x_n(t)|^2 dt = 1$$

and

$$\int_0^1 |x_n(t)|\,dt=\int_{E_n}|x_n(t)|\,dt=\int_{E_n}|g(t)|dt\{\int_{E_n}|g(t)|^2dt\}^{-1}\leqslant\frac{a_n+1}{a_n^2}$$

hence $x_n\xrightarrow{\gamma}0$. Now

$$\xi(x_n)=\int_0^1 x_n(t)g(t)\,dt=1;$$

this, however, is impossible, for ξ is γ-linear.

Proposition 3.1 shows that the γ-linear functionals may be identical with the functionals linear under the norm $\|x\|^*$. The following proposition shows that those functionals may also coincide with the functionals linear under the norm $\|x\|$.

3.2. *The general form of the γ-linear functionals in m_γ is*

$$\xi(x)=\sum_{n=1}^{\infty} a_n u_n \tag{2}$$

with $\sum_{n=1}^{\infty}|a_n|<\infty$.

Proof. Let the functional ξ be γ-linear. Denoting by e_n the n-th unit vector in m, write

$$z_n=\sum_{k=1}^{n} u_k e_k,$$

then $z_n\xrightarrow{\gamma}x$, whence $\xi(z_n)\to\xi(x)$. Now set $a_n=\xi(e_n)$, thus

$$\xi(x)=\lim_n \xi\Big(\sum_{k=1}^{n}u_k e_k\Big)=\lim_k\sum_{k=1}^{n}a_k u_k=\sum_{k=1}^{\infty}a_k u_k.$$

The series $\sum_{k=1}^{\infty} a_k u_k$ must converge for every $\{u_k\} \epsilon \boldsymbol{m}$, hence $\sum_{k=1}^{\infty} |a_k| < \infty$. It follows that every linear γ-functional is of the form (2). It is obvious that the converse proposition is also true.

If the space X^* is not of B_0-type, all the linear functionals may be trivial. This is shown by

3.3. *Any γ-linear functional in $\boldsymbol{L}_\gamma$ is identically equal to 0.*

Proof. Every γ-linear functional in $\boldsymbol{L}_\gamma$ is linear under the norm

$$\|x\| = \int_0^1 |x(t)|\, dt,$$

hence of form (1) with essentially bounded $g(t)$. Suppose that $g \neq 0$, then there must exist an $\varepsilon > 0$ such that

$$H = \mathop{E}_t \{|g(t)| > \varepsilon\}$$

is a set of positive measure. Let $H_n \subset H$, $|H_n| \to 0$, $|H_n| > 0$ and write

$$x_n(t) = \begin{cases} |H_n|^{-1} \operatorname{sign} g(t) & \text{for } t \epsilon H_n \\ 0 & \text{elsewhere.} \end{cases}$$

Since

$$\lim_n \text{as } x_n(t) = 0 \quad \text{and} \quad \int_0^1 |x_n(t)|\, dt = 1,$$

we obtain $\gamma\text{-}\lim_n x_n = 0$. On the other hand

$$\xi(x_n) = \int_0^1 x_n(t) g(t)\, dt = |H_n|^{-1} \int_{H_n} |g(t)|\, dt \geqslant \varepsilon,$$

hence $\xi(x_n)$ non$\to 0$, which contradicts the γ-linearity of ξ.

4. Let X and Y be two Banach spaces, each provided with a γ-convergence. An operation $U(x)$ from X to Y is called *γ-γ-linear* if it is additive and $x_n \xrightarrow{\gamma} x_0$ implies $U(x_n) \xrightarrow{\gamma} U(x_0)$. If the convergence γ in Y is metrical (hence equivalent to that generated by the norm $\|y\|$), any γ-γ-linear operation will be termed *γ-linear*.

An important tool in the functional analysis is the theorem of Banach on the linearity of the limit of a sequence of linear operations in F-spaces. That theorem does not hold for γ-linear operations, as may easily be shown on the example of linear functionals in $\boldsymbol{V}_\gamma$. Hence it is desirable to find sufficient conditions bearing on the space to ensure the validity of that theorem. One such condition[8]) is

[8]) This condition is a generalization of a condition due to Saks [8].

(iv) Given any $\varepsilon > 0$ and $x_0 \epsilon X$ such that $\|x_0\| \leqslant 1$ there is a $\delta > 0$ such that every element x satisfying the inequalities $\|x\| \leqslant 1$, $\|x\|^* < \delta$ is of the form $x = x_1 - x_2$ where $\|x_i\| \leqslant 1$, $\|x_i - x_0\|^* < \varepsilon$.

We suppose in the sequel conditions (i), (ii), and (iv) to be satisfied, and the spaces X, Y^*, Y to be Banach spaces.

4.1. *If the operations $U_n(x)$ are γ-linear and converge in $\overline{X}_\gamma$ to $U(x)$, then $U(x)$ is also γ-linear.*

Proof. Denote by S the solid sphere $\|x\| \leqslant 1$ in X. If we define the distance in S as $\|x_1 - x_2\|^*$, S becomes a complete metric space. For $x \epsilon S$ write

$$V_n(x) = U_n(x|S), \qquad V(x) = U(x|S).$$

We observe that $x_1, x_2, x_1 + x_2 \epsilon S$ implies $V_n(x_1 \pm x_2) = V_n(x_1) \pm V_n(x_2)$, $V(x_1 \pm x_2) = V(x_1) \pm V(x_2)$. The operations V_n are continuous in S and converge in S to V; hence they are equicontinuous at one point x_0 (see for instance [1], p. 5). Thus, given $\eta > 0$, there exists an $\varepsilon > 0$ such that $\|x - x_0\| < \varepsilon$, $x \epsilon S$ implies $\|V_n(x) - V_n(x_0)\| < \eta$ for $n = 1, 2, \ldots$ We choose now by (iv) a $\delta > 0$ corresponding to ε and x_0. If $\|x\|^* < \delta$, $x \epsilon S$, then there are $x_1, x_2 \epsilon S$ such that $x = x_1 - x_2$, $\|x_1 - x_0\|^* < \varepsilon$, $\|x_2 - x_0\|^* < \varepsilon$; hence

$$\|V_n(x)\| = \|V_n(x_1) - V_n(x_2)\| \leqslant \|V_n(x_1) - V_n(x_0)\| + \|V_n(x_2) - V_n(x_0)\| < 2\eta.$$

Now the continuity of V in the sphere S follows easily and this implies the γ-linearity of U in the space X_γ.

4.2. *If the operations $U_n(x)$ are γ-γ-linear and converge in X_γ to $U(x)$, then $U(x)$ is γ-γ-linear.*

Proof. To prove the proposition it suffices to show that

$$x_n \xrightarrow{\gamma} 0 \quad \text{implies} \quad U_n(x_n) \xrightarrow{\gamma} 0.$$

Let $x_n \xrightarrow{\gamma} 0$, $\vartheta_n \to 0$; then $\vartheta_n = \varepsilon_n \tau_n^2$ with $\varepsilon_n = \pm 1$. Write $V_n(x) = \tau_n U_n(x)$; these operations fulfil the condition that

$$\|z_n\| \to 0 \quad \text{implies} \quad \|V_p(z_n)\| \to 0 \quad \text{for} \quad p = 1, 2, \ldots$$

Indeed, choose $t_n \to \infty$ so that $\|t_n z_n\| \to 0$, then $V_p(t_n z_n) \xrightarrow{\gamma} 0$, whence $\|t_n^{-1} V_p(t_n z_n)\| = \|V_p(z_n)\| \to 0$. Now, $\|V_n(x)\| \to 0$ for every x, since $\tau_n \to 0$. By the Banach Theorem ([3], p. 80)

$$\sup_{\|z\| \leqslant 1} \|V_n(z)\| < K,$$

which implies $\|\vartheta_n U_n(x_n)\| = \|V_n(\varepsilon_n \tau_n x_n)\| \to 0$.

Thus the sequence $\{U_n(x_n)\}$ is bounded with respect to the norm $\|y\|$. The operations are γ-linear as operations from X to Y^*; hence as in the proof of 4.1 there is a $\delta>0$ such that $\|U_n(x)\|^*<\varepsilon$ if $\|x\|^*<\delta$, $\|x\|\leqslant 1$.

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Analytic operations in real Banach spaces

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In the treatise Functional Analysis and Semigroups Hille [5] has developped a complete theory of analytic operations in complex Banach spaces. The fact that the spaces under consideration are complex is essential for the methods used by Hille, for they are grounded on the use of the theorem of Hartoggs. On the other hand in many applications of Functional Analysis analytic operations in real Banach spaces play an essential role. Therefore it seems worth while to transfer the theory of Hille to the case of real Banach spaces. That is the purpose of this paper. We show that *mutatis mutandis* the main theorems of the theory can be restored in real Banach spaces. In the development of the theory we follow closely the ideas of Hille: we consider firstly the series of discontinuous powers (called the *p-powers*) and then pass to the continuous powers. The paper is divided into two parts. In the first, after introductory considerations, we establish some properties of analytic operations, in the second, sections 5 and 6, we formulate the extension principle and by its use we extend locally every convergent power series to a power series in an appropriate complex Banach space, converging also locally. Although the results of the first part of the paper might be obtained from the extension principle, we preferred to exhibit them independently, and so to use the extension principle only in the cases where its use seems to be indispensable. The principal tool in tackling the real case is the use of Leja's theorems on sequences of polynomials.

1. Preliminary theorems. In this section we present the auxiliary theorems dealing with the polynomials of real and complex variables taking on values from Banach spaces. Those theorems are well known for numerically valued polynomials.

Let

$$P(u)=\sum_{k=0}^{n} a_k u^k$$