Research Article

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On the type 2 poly-Bernoulli polynomials associated with umbral calculus

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Abstract: Type 2 poly-Bernoulli polynomials were introduced recently with the help of modified polyexponential functions. In this paper, we investigate several properties and identities associated with those polynomials arising from umbral calculus techniques. In particular, we express the type 2 poly-Bernoulli polynomials in terms of several special polynomials, like higher-order Cauchy polynomials, higher-order Euler polynomials, and higher-order Frobenius-Euler polynomials.

Keywords: type 2 poly-Bernoulli polynomials, umbral calculus, modified poly-exponential functions

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1 Introduction

The poly-Bernoulli polynomials, which are defined with the help of polylogarithm functions, were studied by Kaneko in [1], while the type 2 poly-Bernoulli polynomials, which are defined with the help of modified polyexponential functions, were investigated very recently in [2]. We note that the modified polyexponential functions are inverse to the polylogarithm functions. Thus, it is very natural to replace the polylogarithms by the modified polyexponential functions in the definition of generating function of poly-Bernoulli polynomials. Indeed, the generating function of type 2 poly-Bernoulli polynomials is obtained in this way (see (1), (3)), and hence we may say that it arises in a natural manner.

Let $k \geq 1$ be an integer, $\mathrm{Ei}_k(x)$ the modified polyexponential function (see (1)), and let $B_n^{(k)}$ be the type 2 poly-Bernoulli numbers (see (3)). In [2], the function $\eta_k(s)$, for $\mathrm{Re}(s) > 0$, is defined as

$$\eta_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \mathrm{Ei}_k(\log(1+t))\mathrm{d}t.$$

It was shown that this function can be continued to an entire function on $\mathbb{C}$ and its values at non-positive integers are given by $\eta_k(-m) = (-1)^m B_m^{(k)}$, $(m \geq 0)$. In addition, for any integer $k \geq 2$, the generating function of the type 2 poly-Bernoulli numbers is given by

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{1}{e^x - 1} \underbrace{\int_0^x \frac{1}{(1+t)\log(1+t)} \int_0^t \frac{1}{(1+t)\log(1+t)} \cdots \int_0^t}_{(k-2)\text{-times}} \frac{t}{(1+t)\log(1+t)} \mathrm{d}t \cdots \mathrm{d}t.$$

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The aim or motivation of this paper is to further derive some properties, recurrence relations, and identities related to the type 2 poly-Bernoulli polynomials by using umbral calculus techniques. Especially, those polynomials are represented in terms of some well-known special polynomials. In general, special polynomials and numbers can be studied by employing various different methods including combinatorial methods, generating functions, differential equations, umbral calculus techniques, p-adic analysis, and probability theory.

The outline of this paper is as follows. In Section 1, we give some necessary definitions and some basic facts about umbral calculus. As to definitions, we recall the definitions of polyexponential functions, type 2 poly-Bernoulli polynomials, higher-order Bernoulli polynomials, higher-order Cauchy polynomials, higher-order Euler polynomials, and Stirling numbers of the first and second kinds. As to umbral calculus, we give very basic facts such as Sheffer sequence, generating functions of Sheffer polynomials, and the formula for representing one Sheffer polynomial by another. For further details on umbral calculus, we let the reader refer to [3–5]. In Section 2, we find an explicit expression for the type 2 poly-Bernoulli polynomials involving Bernoulli numbers and Stirling numbers of the first kind, a recurrence relation for them, and an identity involving the type 2 poly-Bernoulli numbers and Stirling numbers of the first kind. In addition, we express the type 2 poly-Bernoulli polynomials as linear combinations of higher-order Cauchy polynomials, higher-order Euler polynomials, and of higher-order Frobenius-Euler polynomials.

It is one of our future projects to continue to work on various special polynomials and numbers by using umbral calculus, just as we did in the present paper.

Hardy introduced the polyexponential functions [6,7], while Kim-Kim considered the modified polyexponential functions which are given by

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \quad (k \in \mathbb{Z}) \quad (\text{see } [1,8,9]). \tag{1}$$

From (1), we note that

$$\mathrm{Ei}_1(x) = e^x - 1. \tag{2}$$

The type 2 poly-Bernoulli polynomials, which are defined by using the modified polyexponential functions, are given by

$$\frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \quad (\text{see } [1,10,11]). \tag{3}$$

For $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called the type 2 poly-Bernoulli numbers.

For $r \in \mathbb{N}$, the Bernoulli polynomials $\mathbf{B}_n^{(r)}(x)$ of order r are given by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} \mathbf{B}_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{see } [1,3–5,8–16]). \tag{4}$$

When $x = 0$, $\mathbf{B}_n^{(r)} = \mathbf{B}_n^{(r)}(0)$, $(n \geq 0)$, are called the Bernoulli numbers of order r.

Note that $B_n^{(1)}(x) = \mathbf{B}_n^{(1)}(x)$, which will be denoted by $\mathbf{B}_n(x)$, are the Bernoulli polynomials.

It is well known that the Cauchy polynomials of order r are defined by

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} C_n^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{N}) \quad (\text{see } [5]). \tag{5}$$

For $r \in \mathbb{N}$, the Euler polynomials of order r are defined by

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{see } [5]). \tag{6}$$

For $n \geq 0$, the falling factorial sequence is defined by

$$(x)_0 = 1, \quad (x)_n = x(x-1)(x-2)\cdots(x-n+1), \quad (n \geq 1).$$

Here we note that the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad \frac{1}{m!}(\log(1+t))^m = \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!}, \quad (n \geq 0) \quad (\text{see } [5,11,17]). \tag{7}$$

As an inversion formula of (7), the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad \frac{1}{m!}(e^t-1)^m = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}, \quad (n \geq 0) \quad (\text{see } [5,11,13,15]). \tag{8}$$

Let $\mathbb{C}$ be the field of complex numbers and let

$$\mathscr{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\} \tag{9}$$

be the algebra of formal power series. For $\mathbb{P} = \mathbb{C}[x]$, let $\mathbb{P}^*$ denote the vector space of all linear functional on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, and it is well known that the vector space operations on $\mathbb{P}^*$ are defined by

$$\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \quad \langle cL|p(x)\rangle = c\langle L|p(x)\rangle,$$

where c is a complex constant (see [3–5]).

For $f(t) \in \mathscr{F}$, let $\langle f(t)|x^n\rangle = a_n$, $(n \geq 0)$. From (9), we note that

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \quad (n,k \geq 0) \quad (\text{see } [4,12,13]),$$

where $\delta_{n,k}$ is the Kronecker symbol.

The order $o(f(t))$ of the power series $f(t)(\neq 0)$ is the smallest integer k for which a_k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is said to be an invertible series; if $o(f(t)) = 1$, then $f(t)$ is called a delta series.

For $f(t), g(t) \in \mathscr{F}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle. \tag{10}$$

From (10), we note that

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x)\rangle \frac{x^k}{k!} \quad (\text{see } [4]), \tag{11}$$

where $f(t) \in \mathscr{F}$ and $p(x) \in \mathbb{P}$.

Thus, by (11), we get

$$p^{(k)}(0) = \langle t^k|p(x)\rangle = \langle 1|p^{(k)}(x)\rangle \quad (\text{see } [4,16]), \tag{12}$$

where $p^{(k)}(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}p(x)$.

From (12), we note that $t^k p(x) = p^{(k)}(x)$, $(k \geq 0)$. It is not difficult to show that

$$e^{yt}p(x) = p(x+y), \quad \langle e^{yt}|p(x)\rangle = p(y) \quad (\text{see } [15]), \tag{13}$$

where $p(x) \in \mathbb{P}$.

Suppose that $f(t)$ is a delta series and $g(t)$ is an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)(f(t))^k|s_n(x)\rangle = n!\delta_{n,k}$, $(n,k \geq 0)$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $s_n(x) \sim (g(t), f(t))$, we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|s_k(x)\rangle}{k!} g(t)f(t)^k \tag{14}$$

and

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t)f(t)^k|p(x)\rangle}{k!} s_k(x). \tag{15}$$

Thus, by (14), we easily get

$$\frac{1}{g(\overline{f}(t))}e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \tag{16}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = t$, and

$$f(t)s_n(x) = ns_{n-1}(x), \quad (n \in \mathbb{N}). \tag{17}$$

For $s_n(x) \sim (g(t), t)$, we have

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)s_n(x), \tag{18}$$

$$s_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}s_k(x)y^{n-k} = \sum_{k=0}^{n}\binom{n}{k}s_{n-k}(x)y^k, \quad (n \geq 0), \tag{19}$$

and

$$s_n(x) = \frac{1}{g(t)}x^n, \quad (n \geq 0) \quad (\text{see } [15]). \tag{20}$$

We recall here that $s_n(x)$ is called the Appell sequence for $g(t)$ if $s_n(x) \sim (g(t), t)$. For example, the sequence $B_n^{(k)}(x)$ of the type 2 poly-Bernoulli polynomials is an Appell sequence and hence has the properties stated in (18), (19), and (20). In particular, we have

$$B_n^{(k)}(x) = \sum_{j=0}^{n} B_{n-j}^{(k)}x^j. \tag{21}$$

For $s_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$, it is known that

$$s_n(x) = \sum_{m=0}^{n} A_{n,m}r_m(x), \quad (n \geq 0), \tag{22}$$

where

$$A_{n,m} = \frac{1}{m!}\left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))}(l(\overline{f}(t)))^m | x^n \right\rangle \quad (\text{see } [4]). \tag{23}$$

2 Some identities of type 2 poly-Bernoulli polynomials arising from umbral calculus

From (3), (4), and (16), we note that

$$B_n^{(k)}(x) \sim \left(\frac{e^t - 1}{\mathrm{Ei}_k(\log(1+t))}, t\right), \quad \mathbf{B}_n^{(r)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r, t\right). \tag{24}$$

By (17), we get

$$tB_n^{(k)}(x) = nB_{n-1}^{(k)}, \quad t\mathbf{B}_n^{(r)}(x) = n\mathbf{B}_{n-1}^{(r)}, \ (n \geq 1).$$

From (20) and (24), we have the next lemma.

Lemma 1. *For $n \geq 0$, we have*

$$\frac{\mathrm{Ei}_k(\log(1+t))}{e^t - 1}x^n = B_n^{(k)}(x).$$

Now, we observe that

$$\begin{aligned}\mathrm{Ei}_k(\log(1+t)) &= \sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{(m-1)!m^k} = \sum_{m=1}^{\infty}\frac{1}{m^{k-1}}\frac{1}{m!}(\log(1+t))^m \\ &= \sum_{m=1}^{\infty}\frac{1}{m^{k-1}}\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!} = \sum_{l=1}^{\infty}\sum_{m=1}^{l}\frac{S_1(l,m)}{m^{k-1}}\frac{t^l}{l!}.\end{aligned} \tag{25}$$

From Lemma 1 and (25), we have

$$\begin{aligned}B_n^{(k)}(x) &= \frac{1}{n+1}\frac{t}{e^t-1}\mathrm{Ei}_k(\log(1+t))x^{n+1} \\ &= \frac{1}{n+1}\frac{t}{e^t-1}\sum_{l=1}^{\infty}\sum_{m=1}^{l}\frac{S_1(l,m)}{m^{k-1}}\frac{t^l}{l!}x^{n+1} \\ &= \frac{1}{n+1}\sum_{j=0}^{\infty}\mathbf{B}_j\frac{t^j}{j!}\sum_{l=1}^{n+1}\sum_{m=1}^{l}\frac{S_1(l,m)}{m^{k-1}}\binom{n+1}{l}x^{n+1-l} \\ &= \frac{1}{n+1}\sum_{l=1}^{n+1}\sum_{m=1}^{l}\sum_{j=0}^{n+1-l}\frac{S_1(l,m)}{m^{k-1}}\binom{n+1}{l}\binom{n+1-l}{j}\mathbf{B}_j x^{n+1-l-j} \\ &= \frac{1}{n+1}\sum_{l=1}^{n+1}\sum_{m=1}^{l}\sum_{j=0}^{n+1-l}\frac{S_1(l,m)}{m^{k-1}}\binom{n+1}{l,j,n+1-l-j}\mathbf{B}_j x^{n+1-l-j} \\ &= \frac{1}{n+1}\sum_{j=0}^{n}\sum_{l=1}^{n+1-j}\sum_{m=1}^{l}\frac{S_1(l,m)}{m^{k-1}}\binom{n+1}{l,j,n+1-l-j}\mathbf{B}_{n+1-l-j}x^j,\end{aligned} \tag{26}$$

where we used the trinomial coefficients $\binom{n}{a,b,c} = \frac{n!}{a!b!c!}$, with $n = a+b+c$.

Therefore, by (26), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$B_n^{(k)}(x) = \frac{1}{n+1}\sum_{j=0}^{n}\sum_{l=1}^{n+1-j}\sum_{m=1}^{l}\frac{S_1(l,m)}{m^{k-1}}\binom{n+1}{l,j,n+1-l-j}\mathbf{B}_{n+1-l-j}x^j.$$

By (18) and (24), we get

$$B_{n+1}^{(k)}(x) = \left(x - \frac{g'(t)}{g(t)}\right)B_n^{(k)}(x), \quad (n \geq 0), \tag{27}$$

where

$$g(t) = \frac{e^t-1}{\mathrm{Ei}_k(\log(1+t))} \quad \text{and} \quad g'(t) = \frac{\mathrm{d}}{\mathrm{d}t}g(t).$$

We observe that

$$\begin{aligned}\frac{g'(t)}{g(t)} &= \frac{\mathrm{d}}{\mathrm{d}t}(\log g(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(\log(e^t-1) - \log\mathrm{Ei}_k(\log(1+t))) \\ &= \frac{e^t}{e^t-1} - \frac{1}{(1+t)\log(1+t)}\frac{\mathrm{Ei}_{k-1}(\log(1+t))}{\mathrm{Ei}_k(\log(1+t))}.\end{aligned} \tag{28}$$

As is known, the Cauchy numbers of the second kind are defined by

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty}\hat{c}_n\frac{t^n}{n!}. \tag{29}$$

From (27) and (28), we note that

$$\begin{aligned}
B_{n+1}^{(k)}(x) &= \left(x - \frac{g'(t)}{g(t)}\right)B_n^{(k)}(x) \\
&= xB_n^{(k)}(x) - \left(\frac{e^t}{e^t-1} - \frac{1}{(1+t)\log(1+t)}\frac{\mathrm{Ei}_{k-1}(\log(1+t))}{\mathrm{Ei}_k(\log(1+t))}\right)B_n^{(k)}(x) \\
&= xB_n^{(k)}(x) - \frac{e^t}{e^t-1}\frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}x^n + \frac{1}{(1+t)\log(1+t)}\frac{\mathrm{Ei}_{k-1}(\log(1+t))}{\mathrm{Ei}_k(\log(1+t))}\frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}x^n \\
&= xB_n^{(k)}(x) - \frac{te^t}{e^t-1}\frac{\mathrm{Ei}_k(\log(1+t))}{t(e^t-1)}x^n + \frac{t}{(1+t)\log(1+t)}\frac{\mathrm{Ei}_{k-1}(\log(1+t))}{t(e^t-1)}x^n \\
&= xB_n^{(k)}(x) - \frac{1}{n+1}\frac{te^t}{e^t-1}\frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}x^{n+1} + \frac{1}{n+1}\frac{t}{(1+t)\log(1+t)}\frac{\mathrm{Ei}_{k-1}(\log(1+t))}{e^t-1}x^{n+1} \\
&= xB_n^{(k)}(x) - \frac{e^t}{n+1}\sum_{l=0}^{\infty}\sum_{j=0}^{l}\binom{l}{j}\mathbf{B}_{l-j}B_j^{(k)}\frac{t^l}{l!}x^{n+1} + \frac{1}{n+1}\sum_{l=0}^{\infty}\sum_{j=0}^{l}\binom{l}{j}\widehat{c}_{n-j}B_j^{(k-1)}\frac{t^l}{l!}x^{n+1} \qquad (30) \\
&= xB_n^{(k)}(x) - \frac{e^t}{n+1}\sum_{l=0}^{n+1}\sum_{j=0}^{l}\binom{n+1}{j,\,l-j,\,n+1-l}\mathbf{B}_{l-j}B_j^{(k)}x^{n+1-l} \\
&\quad + \frac{1}{n+1}\sum_{l=0}^{n+1}\sum_{j=0}^{l}\widehat{c}_{n-j}B_j^{(k-1)}\binom{n+1}{j,\,l-j,\,n+1-l}x^{n+1-l} \\
&= xB_n^{(k)}(x) - \frac{1}{n+1}\sum_{l=0}^{n+1}\sum_{j=0}^{l}\binom{n+1}{j,\,l-j,\,n+1-l}\mathbf{B}_{l-j}B_j^{(k)}(x+1)^{n+1-l} \\
&\quad + \frac{1}{n+1}\sum_{l=0}^{n+1}\sum_{j=0}^{l}\binom{n+1}{j,\,l-j,\,n+1-l}\widehat{c}_{n-j}B_j^{(k-1)}x^{n+1-l}.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 3. *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$\begin{aligned}
B_{n+1}^{(k)}(x) = xB_n^{(k)}(x) &- \frac{1}{n+1}\sum_{l=0}^{n+1}\sum_{j=0}^{l}\binom{n+1}{j,\,l-j,\,n+1-l}\mathbf{B}_{l-j}B_j^{(k)}(x+1)^{n+1-l} \\
&+ \frac{1}{n+1}\sum_{l=0}^{n+1}\sum_{j=0}^{l}\binom{n+1}{j,\,l-j,\,n+1-l}\widehat{c}_{n-j}B_j^{(k-1)}x^{n+1-l}.
\end{aligned}$$

Now, we compute $\langle \mathrm{Ei}_k(\log(1+t))|x^{n+1}\rangle$, $(n \geq 0)$.

From (1), (3), and (10), we note that

$$\begin{aligned}
\langle \mathrm{Ei}_k(\log(1+t))|x^{n+1}\rangle &= \left\langle \frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1} \,\middle|\, (e^t-1)x^{n+1}\right\rangle = \left\langle \frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1} \,\middle|\, (x+1)^{n+1}-x^{n+1}\right\rangle \\
&= \sum_{l=0}^{n}\binom{n+1}{l}\left\langle \frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1} \,\middle|\, x^l\right\rangle = \sum_{l=0}^{n}\binom{n+1}{l}B_l^{(k)}.
\end{aligned} \qquad (31)$$

On the other hand,

$$\begin{aligned}
\langle \mathrm{Ei}_k(\log(1+t))|x^{n+1}\rangle &= \sum_{m=1}^{\infty}\frac{1}{m^{k-1}}\sum_{l=m}^{\infty}\frac{S_1(l,m)}{l!}\langle t^l|x^{n+1}\rangle = \sum_{l=1}^{\infty}\sum_{m=1}^{l}\frac{1}{m^{k-1}}\frac{S_1(l,m)}{l!}(n+1)!\,\delta_{n+1,l} \\
&= \sum_{m=1}^{n+1}\frac{1}{m^{k-1}}S_1(n+1,m).
\end{aligned} \qquad (32)$$

Therefore, by (31) and (32), we obtain the following theorem.

Theorem 4. *For $n \geq 0$ and $k \in \mathbb{Z}$, we have*

$$\sum_{l=0}^{n}\binom{n+1}{l}B_l^{(k)} = \sum_{m=0}^{n}\frac{1}{(m+1)^{k-1}}S_1(n+1,m+1).$$

Remark 5. Theorem 4 can be deduced also from Theorem 2. From (21) and Theorem 2, we see that

$$\binom{n}{j}B_{n-j}^{(k)} = \sum_{l=1}^{n+1-j}\sum_{m=1}^{l}\frac{1}{n+1}\frac{S_1(l,m)}{m^{k-1}}\binom{n+1}{l}\binom{n+1-l}{j}\mathbf{B}_{n+1-l-j}. \tag{33}$$

Replacing j by $n-j$, noting $\binom{n+1}{j} = \frac{n+1}{n+1-j}\binom{n}{j}$, and summing over j in (33), we obtain

$$\begin{aligned}\sum_{j=0}^{n}\binom{n+1}{j}B_j^{(k)} &= \sum_{j=0}^{n}\sum_{l=1}^{j+1}\sum_{m=1}^{l}\frac{1}{n+1-j}\frac{S_1(l,m)}{m^{k-1}}\binom{n+1}{l}\binom{n+1-l}{n-j}\mathbf{B}_{j+1-l}\\ &= \sum_{l=1}^{n+1}\sum_{m=1}^{l}\binom{n+1}{l}\frac{S_1(l,m)}{m^{k-1}}\sum_{j=l-1}^{n}\frac{1}{n+1-j}\binom{n+1-l}{n-j}\mathbf{B}_{j+1-l}\\ &= \sum_{l=1}^{n+1}\sum_{m=1}^{l}\frac{1}{n+2-l}\binom{n+1}{l}\frac{S_1(l,m)}{m^{k-1}}\sum_{j=0}^{n+1-l}\binom{n+2-l}{j}\mathbf{B}_j.\end{aligned} \tag{34}$$

Now, Theorem 4 follows from (34) by noting that $\sum_{j=0}^{n}\binom{n+1}{j}\mathbf{B}_j = \delta_{n,0}$.

For the next result, we recall that, for any $r \in \mathbb{N}$, the Cauchy polynomials $C_n^{(r)}(x)$ of order r are given by

$$\left(\frac{t}{\log(1+t)}\right)^r(1+t)^x = \sum_{n=0}^{\infty}C_n^{(r)}(x)\frac{t^n}{n!}. \tag{35}$$

We consider the following two Sheffer sequences.

$$B_n^{(k)}(x) \sim \left(\frac{e^t-1}{\mathrm{Ei}_k(\log(1+t))}, t\right), \quad C_n^{(r)}(x) \sim \left(\left(\frac{t}{e^t-1}\right)^r, e^t-1\right). \tag{36}$$

From (22), (23), and (36), we have

$$B_n^{(k)}(x) = \sum_{m=0}^{n}A_{n,m}C_m^{(r)}(x), \tag{37}$$

where, by making use of (8), we show

$$\begin{aligned}A_{n,m} &= \left\langle\left(\frac{t}{e^t-1}\right)^r\frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}\;\middle|\;\frac{1}{m!}(e^t-1)^m x^n\right\rangle\\ &= \sum_{l=m}^{n}\binom{n}{l}S_2(l,m)\left\langle\left(\frac{t}{e^t-1}\right)^r\;\middle|\;\frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}x^{n-l}\right\rangle\\ &= \sum_{l=m}^{n}\binom{n}{l}S_2(l,m)\left\langle\sum_{s=0}^{\infty}\mathbf{B}_s^{(r)}\frac{t^s}{s!}\;\middle|\;B_{n-l}^{(k)}(x)\right\rangle\\ &= \sum_{l=m}^{n}\binom{n}{l}S_2(l,m)\sum_{s=0}^{n-l}\mathbf{B}_s^{(r)}\frac{1}{s!}\langle 1|t^s B_{n-l}^{(k)}(x)\rangle\\ &= \sum_{l=m}^{n}\binom{n}{l}S_2(l,m)\sum_{s=0}^{n-l}\binom{n-l}{s}\mathbf{B}_s^{(r)}B_{n-l-s}^{(k)}.\end{aligned} \tag{38}$$

Therefore, by (37) and (38), we obtain the following theorem.

Theorem 6. *For $n \geq 0$, $r \in \mathbb{N}$, and $k \in \mathbb{Z}$, we have*

$$B_n^{(k)}(x) = \sum_{m=0}^{n} \left\{ \sum_{l=m}^{n} \sum_{s=0}^{n-l} \binom{n}{l, s, n-l-s} S_2(l, m) \mathbf{B}_s^{(r)} B_{n-l-s}^{(k)} \right\} C_m^{(r)}(x),$$

where $C_m^{(r)}(x)$ are the Cauchy polynomials of order r.

For

$$B_n^{(k)}(x) \sim \left(\frac{e^t - 1}{\mathrm{Ei}_k(\log(1+t))}, t \right), \quad E_n^{(r)}(x) \sim \left(\left(\frac{e^t + 1}{2} \right)^r, t \right), \quad (r \in \mathbb{N}),$$

we have

$$B_n^{(k)}(x) = \sum_{m=0}^{n} A_{n,m} E_m^{(r)}(x). \tag{39}$$

Here we note that

$$\begin{aligned} A_{n,m} &= \frac{1}{m!} \left\langle \frac{\mathrm{Ei}_k(\log(1+t))}{e^t - 1} \left(\frac{e^t + 1}{2} \right)^r t^m \,\middle|\, x^n \right\rangle \\ &= \frac{\binom{n}{m}}{2^r} \left\langle \frac{\mathrm{Ei}_k(\log(1+t))}{e^t - 1} \,\middle|\, (e^t + 1)^r x^{n-m} \right\rangle \\ &= \frac{\binom{n}{m}}{2^r} \sum_{j=0}^{r} \binom{r}{j} \left\langle e^{jt} \,\middle|\, \frac{\mathrm{Ei}_k(\log(1+t))}{e^t - 1} x^{n-m} \right\rangle \\ &= \frac{\binom{n}{m}}{2^r} \sum_{j=0}^{r} \binom{r}{j} \langle e^{jt} | B_{n-m}^{(k)}(x) \rangle \\ &= \frac{\binom{n}{m}}{2^r} \sum_{j=0}^{r} \binom{r}{j} B_{n-m}^{(k)}(j). \end{aligned} \tag{40}$$

Therefore, by (39) and (40), we obtain the following theorem.

Theorem 7. *For $n \geq 0$, $r \in \mathbb{N}$, and $k \in \mathbb{Z}$, we have*

$$B_n^{(k)}(x) = \frac{1}{2^r} \sum_{m=0}^{n} \left\{ \binom{n}{m} \sum_{j=0}^{r} \binom{r}{j} B_{n-m}^{(k)}(j) \right\} E_m^{(r)}(x).$$

Let $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. For $r \in \mathbb{N}$, the Frobenius-Euler polynomials of order r are defined by

$$\left(\frac{1-\lambda}{e^t - \lambda} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x; \lambda) \frac{t^n}{n!}. \tag{41}$$

From (16) and (41), we note that

$$H_n^{(r)}(x; \lambda) \sim \left(\left(\frac{e^t - \lambda}{1 - \lambda} \right)^r, t \right). \tag{42}$$

Thus, by (22), (23), and (42), we get

$$B_n^{(k)}(x) = \sum_{m=0}^{n} A_{n,m} H_m^{(r)}(x; \lambda), \tag{43}$$

where

$$\begin{aligned} A_{n,m} &= \frac{1}{m!}\left\langle \frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}\left(\frac{e^t-\lambda}{1-\lambda}\right)^r t^m \,\middle|\, x^n \right\rangle \\ &= \frac{\binom{n}{m}}{(1-\lambda)^r}\left\langle \frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1} \,\middle|\, (e^t-\lambda)^r x^{n-m} \right\rangle \\ &= \frac{\binom{n}{m}}{(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}\left\langle \frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1} \,\middle|\, e^{jt}x^{n-m} \right\rangle \\ &= \frac{\binom{n}{m}}{(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}\left\langle t1 \,\middle|\, \frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}(x+j)^{n-m} \right\rangle \\ &= \frac{\binom{n}{m}}{(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}B_{n-m}^{(k)}(j). \end{aligned} \tag{44}$$

Thus, by (43) and (44), we obtain the following theorem.

Theorem 8. *For $k \in \mathbb{Z}$, $r \in \mathbb{N}$, and $n \geq 0$, we have*

$$B_m^{(k)}(x) = \frac{1}{(1-\lambda)^r}\sum_{m=0}^{n}\left\{\binom{n}{m}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}B_{n-m}^{(k)}(j)\right\}H_m^{(r)}(x;\lambda).$$

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