ON THE UNIFORM ASYMPTOTIC STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF THE NEUTRAL TYPE

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Abstract

Consider the functional equations of neutral type

(1) $\frac{d}{dt} D(t,x_t) = f(t,x_t)$ and (2) $\frac{d}{dt} [D(t,x_t) - G(t,x_t)] =$

$= f(t,x_t) + F(t,x_t)$ where D,f are bounded linear operators from $C[a,b]$ into R^n or C^n for each fixed t in $[0,\infty)$, $F = F_1 + F_2$, $G = G_1 + G_2$, $|F_1(t,\phi)| \le \nu(t)|\phi|$, $|G_1(t,\phi)| \le r(t)|\phi|$, $r(t)$, bounded and for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $|F_2(t,\phi)| \le \varepsilon|\phi|$, $|G_2(t,\phi)| \le \varepsilon|\phi|$, $t \ge 0$, $|\phi| < \delta(\varepsilon)$. The authors prove that if (1) is uniformly asymptotically stable, then there is a ζ_o, $0 < \zeta_o < 1$ such that for any $p > 0$, $0 < \zeta < \zeta_o$ there are constants $\nu_o > 0$, $M_o > 0$, $s_o > 0$ such that if $\pi(t) < M_o$, $t \ge s_o, \frac{1}{p}\int_t^{t+p} \nu(s)ds < \zeta\nu_o$, $t > 0$ then the solution $x = 0$ of (2) is uniformly asymptotically stable. The result generalizes previous results which consider only terms of the form F_1,G_1 or F_2,G_2 but not both simultaneously, and the stronger hypothesis $\lim_{t \to \infty} \pi(t) = 0$.

<u>List of Symbols</u>

≧	φ	σ	ν	π
∞	Ω	ϵ	α	ε
[ ]	→	∫	ℓ	η
θ	×	μ	ϒ	ζ
β	/	Σ	>	

ON THE UNIFORM ASYMPTOTIC STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF THE NEUTRAL TYPE

J. K. Hale and A. F. Ize

Suppose $r \geq 0$ is a given real number, $R = (-\infty,\infty)$, E is a real or complex n-dimensional linear vector space with norm $|\cdot|$, $C([a,b],E)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into E with the topology of uniform convergence. If $[a,b] = [-r,0]$, we let $C = ([-r,0],E)$ and designate the norm of an element ϕ in C by $|\phi| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$. If Ω is an open subset of $R \times C$ and $f, D: \Omega \to E$ are given continuous functions, we say that the relation

$$\frac{d}{dt} D(t,x_t) = f(t,x_t) \tag{1}$$

is a functional differential equation. A function x is said to be a solution of (1) if there are $\sigma \in R$, $A > 0$ such that $x \in C([\sigma-r,\sigma+A),E)$, $(t,x_t) \in \Omega$, $t \in (\sigma,\sigma+A)$ and x satisfies (1) on $(\sigma,\sigma+A)$. Notice this definition implies that $D(t,x_t)$ and not $x(t)$ is continuously differentiable on $(\sigma,\sigma+A)$. For a given $\sigma \in R$, $\phi \in C$, $(\sigma,\phi) \in \Omega$, we say $x(\sigma,\phi)$ is a solution of (1) with initial value (σ,ϕ) if there is an $A > 0$ such that $x(\sigma,\phi)$ is a solution of (1) on $[\sigma-r,\sigma+A)$ and $x_\sigma(\sigma,\phi) = \phi$.

Our objective is to study the relationship between the uniform asymptotic stability of the linear neutral differential equation

$$\frac{d}{dt} D(t,x_t) = f(t,x_t) \tag{2}$$

and the perturbed equation

$$\frac{d}{dt} [D(t,x_t) - G(t,x_t)] = f(t,x_t) + F(t,x_t), \tag{3}$$

where $D(t,x_t) = \phi(0) - g(t,\phi)$, $g(t,\cdot)$, $f(t,\cdot)$ are bounded linear operators from C into E for each fixed t in $[0,\infty)$, $g(t,\phi)$ is continuous for $(t,\phi) \in [0,\infty) \times C$

$$g(t,\phi) = \int_{-r}^{0} [d_\theta \mu(t,\theta)]\phi(\theta), \qquad f(t,\phi) = \int_{-r}^{0} [d_\theta \eta(t,\theta)]\phi(\theta)$$

$$g(t,\phi) \le K|\phi| \qquad |f(t,\phi)| \le \ell(t)|\phi|, \quad (t,\phi) \in [0,\infty) \times C$$

for some non-negative constant K, continuous non-negative function ℓ and $\mu(t,\cdot)$, $\eta(t,\cdot)$ are $n \times n$ matrix functions of bounded variation on $[-r,0]$. We also assume that g is uniformly non-atomic at zero, that is, there exists a continuous, non-negative, non-decreasing function $\gamma(s)$ for s in $[0,r]$ such that

$$\gamma(0) = 0 \qquad \left|\int_{-s}^{0} [d_\theta \mu(t,\theta)]\phi(\theta)\right| \le \gamma(s)|\phi|.$$

Throughout the paper, we assume that $D - G, F$ satisfy enough smoothness conditions to ensure that a solution of (3) exist through each point $(\sigma,\phi) \in [0,\infty) \times C$, is unique, depends continuously upon (σ,ϕ) and can be continued to the right as long as the trajectory remains in a bounded set in $[0,\infty) \times C$. Sufficient conditions for these properties to be true are contained in [2].

Basic to this investigation is the variation of constants formula given in [1]. If the solution $x_t(\sigma,\phi)$ of the linear system is designated by $T(t,\sigma)\phi$, then there is an $n \times n$ matrix function $B(t,s)$ defined for $0 \leq s \leq t + r$, $t \in [0,\infty)$, continuous in s from the right, of bounded variation in s, $B(t,s) = 0$, $t \leq s \leq t + r$, such that the solution $x(\sigma,\phi)$ of (3) is given by

$$x_t(\sigma,\phi) = T(t,\sigma)\phi + \int_\sigma^t [-\{d_s B_t(\cdot,s)\}G(s,x_s) + B_t(\cdot,s)F(s,x_s)ds], \quad t \geq \sigma. \tag{4}$$

Furthermore, by [1], if the solution $x = 0$ of (6) is uniformly asymptotically stable, there are constants $M \geq 1$, $\alpha > 0$, such that

$$\begin{aligned} &|T(t,\sigma)\phi| \leq Me^{-\alpha(t-\sigma)}|\phi|, \quad t \geq \sigma \geq 0, \ \phi \in C, \\ &|B_t(\cdot,s)| \leq Me^{-\alpha(t-s)}, \quad t \geq s \geq 0 \\ &\int_\sigma^s |d_u B_t(\cdot,u)| \leq Me^{-\alpha(t-s)}, \quad t \geq s \geq \sigma \geq 0. \end{aligned} \tag{5}$$

In the following we will also assume that

$$G = G_1 + G_2, \quad F = F_1 + F_2 \tag{6}$$

where

$$\begin{aligned} &|F_1(t,\phi)| \le \nu(t)|\phi| \\ &|G_1(t,\phi)| \le \pi(t)|\phi|, \quad t \ge 0, \quad \phi \in C \end{aligned} \tag{7}$$

where $\pi(t), \nu(t)$ are continuous, $\pi(t), \int_t^{t+1} \nu(s)ds$ are bounded for $t \ge 0$ and for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that

$$|F_2(t,\phi)| \le \varepsilon|\phi|, \quad |G_2(t,\phi)| \le \varepsilon|\phi|, \quad t \ge 0, \ |\phi| < \delta(\varepsilon). \tag{8}$$

We can now prove the following

<u>Theorem</u>. Suppose F_1, G_1 satisfy (7) and F_2, G_2 satisfy (8). If system (2) is uniformly asymptotically stable, then there is a ζ_o, $0 < \zeta_o < 1$ such that for any $p > 0$, $0 < \zeta < \zeta_o$ there are constants $\nu_o > 0$, $M_o > 0$, $s_o > 0$, such that if

$$\pi(t) < M_o, \quad t \ge s_o \tag{9}$$

$$\frac{1}{p}\int_t^{t+p} \nu(s)ds \le \zeta\nu_o, \quad t \ge 0, \tag{10}$$

then the solution $x = 0$ of (3) is uniformly asymptotically stable.

Proof. Let $R^+ = [0,\infty)$. The boundedness hypotheses on $\pi(t)$, $\int_t^{t+1} \nu(s)ds$ and using an argument very similar to the one in lemma 1, of [3] imply for any $\beta > 0$, there are $\delta_1(\beta) > 0$, $M_1(\beta) > 0$ such that for any $\sigma \in R^+$ the solution $x = x(\sigma,\phi)$ of (1.1) through (σ,ϕ) satisfies $|x_t(\sigma,\phi)| \leq M_1(\beta)|\phi|$ for $\sigma \leq t \leq \sigma + 2\beta$, provided that $|\phi| \leq \delta_1(\beta)$. From the hypothesis of uniform asymptotic stability there are constants $M \geq 1$, $\alpha > 0$, such that B and T in (4) satisfy (5). S[illegible]e $0 < \zeta < 1/(2M-1)$. Then $\zeta < 1$ and $\zeta < (1+\zeta)/2M$. Let $M_2(\beta,\zeta) = \max [M(1+\pi^*(0) + \varepsilon); M_1(\beta)]2/(1-\zeta)$. Choose $M_o > 0$, $\beta > 0$, $\varepsilon > 0$ such that

$$\eta \overset{\text{def}}{=} M\pi^*(0)e^{-\alpha\beta} + M_2(\beta,\zeta)M_o + M_2(\beta,\zeta)\varepsilon(1+\alpha^{-1}) + \zeta < (1+\zeta)/2M, \quad (11)$$

where $\pi^*(s) = \sup_{s\leq t} \pi(t)$. The choice of M_o,β,ε satisfying (11) can be made in the following way. First choose β so that

$$M\pi^*(0)e^{-\alpha\beta} < (1+\zeta)/6M - \zeta/3,$$

then choose M_o so that

$$M_2(\beta,\zeta)M_o < (1+\zeta)/6M - \zeta/3$$

and finally choose ε so that

$$M_2(\beta,\zeta)\varepsilon(1+\alpha^{-1}) < (1+\zeta)/6M - \zeta/3.$$

Let $s_o = \sigma + \beta$ and suppose (9) is satisfied.

From the hypotheses on F_2, G_2, for the above $\varepsilon > 0$, there is a $\delta_2(\varepsilon) > 0$ such that

$$|F_2(t,\phi)| \leq \varepsilon|\phi|, \qquad |G_2(t,\phi)| \leq \varepsilon|\phi|$$

for $|\phi| < \delta_2(\varepsilon)$. Choose $\delta > 0$ such that

$$M_2(\beta,\zeta)\delta < \min\,(\delta_1(\beta),\delta_2(\varepsilon)).$$

For any $p > 0$, choose ν_o so that

$$pM_2(\beta,\zeta)\nu_o = (e^{\alpha p}-1)/(2e^{\alpha p}-1)$$

and suppose (10) is satisfied for this ν_o.

If $k = k(t-\sigma)$ is the integer such that $kp \leq t - \sigma < (k+1)p$ then

$$\int_{\sigma}^{t} e^{-\alpha(t-u)}\nu(u)du = \int_{\sigma+kp}^{t} e^{-\alpha(t-u)}\nu(u)du + \sum_{j=0}^{k-1}\int_{\sigma+jp}^{\sigma+(j+1)p} e^{-\alpha(t-u)}\nu(u)du$$

$$\leq p\zeta\nu_o + \sum_{j=o}^{k-1} e^{-\alpha(t-\sigma-jp-p)}p\zeta\nu_o$$

$$= \left[1 + e^{-\alpha(t-\sigma-p)} \frac{1-e^{\alpha kp}}{1-e^{\alpha p}}\right] p\zeta\nu_o$$

$$= \left[1 + \frac{e^{\alpha p}}{e^{\alpha p}-1}\left\{e^{-\alpha(t-\sigma-kp)} - e^{-\alpha(t-\sigma)}\right\}\right] p\zeta\nu_o$$

$$\leq \frac{2e^{\alpha p}-1}{e^{\alpha p}-1}p\zeta\nu_o$$

$$= \frac{\zeta}{M_2(\beta,\zeta)}$$

or

$$M_2(\beta,\zeta)\int_{\sigma}^{t} e^{-\alpha(t-u)}\nu(u)du \leq \zeta. \tag{12}$$

Furthermore, since $M \leq (1-\zeta)M_2(\beta,\zeta)/2$, we have

$$M\int_{\sigma}^{t} e^{-\alpha(t-u)}\nu(u)du \leq \frac{M\zeta}{M_2(\beta,\zeta)} \leq \frac{(1-\zeta)\zeta}{2(1+\pi^*(0)+\varepsilon)} \leq \frac{(1-\zeta)\zeta}{2} < \zeta. \tag{13}$$

Let us write the variation of constants formula for the solution $x = x(\sigma,\phi)$ of (3) in the form

$$x_t = T(t,\sigma)\phi + \left(\int_\sigma^s + \int_s^t\right)[d_u B_t(\cdot,u)][G_1(\sigma,\phi) - G_1(u,x_u)]$$
$$+ \int_\sigma^t B_t(\cdot,u)F_1(u,x_u)du$$
$$+ \int_\sigma^t [d_u B_t(\cdot,u)][G_2(\sigma,\phi) - G_1(u,x_u)] \tag{14}$$
$$+ \int_\sigma^t B_t(\cdot,u)F_2(u,x_u)du$$

for $\sigma \leq s \leq t$.

Therefore, as long as $|x_t| \leq \delta_2(\varepsilon)$, it follows from (5) and the hypotheses on F,G, that

$$|x_t| \leq M(1+\pi^*(0) + \varepsilon)|\phi|e^{-\alpha(t-\sigma)}$$
$$+ M[\pi^*(\sigma)e^{-\alpha(t-s)} + \pi^*(s) + \varepsilon(1+\alpha^{-1}) + \int_\sigma^t e^{-\alpha(t-u)}\nu(u)du] \sup_{\sigma \leq u \leq t} |x_u|$$

for $\sigma \leq s \leq t$. T) $s = s_0 = \sigma + \beta$ and use our estimates on β, ε, M_0 and (9), (1 11), then

$$|x_t| \leq M(1+\pi^*(0) + \varepsilon)|\phi| + \eta \sup_{\sigma \leq u \leq t} |x_u|$$
$$\leq \frac{1-\zeta}{2} M_2|\phi| + \eta \sup_{\sigma \leq u \leq t} |x_u|$$

for $t \geq \sigma + 2\beta$ as long as $|x_t| \leq \delta_2(\varepsilon)$. If δ is chosen as above and $|\phi| < \delta$, then we know that

$$|x_t| \leq M_1(\beta)|\phi| \leq (1-\zeta)M_2(\beta,\varepsilon)|\phi|/2 \leq M_2(\beta,\varepsilon)|\phi| \leq \delta_2(\varepsilon)$$

for $\sigma \leq t \leq \sigma + 2\beta$. Therefore,

$$|x_t| \leq \frac{1-\zeta}{2} M_2(\beta,\varepsilon)|\phi| + \eta \sup_{\sigma \leq u \leq t} |x_u|$$

for all $t \geq \sigma$ for which $|x_t| \leq \delta_2(\varepsilon)$. Consequently, for $|x_t| \leq \delta_2(\varepsilon)$,

$$\begin{aligned} \sup_{\sigma \leq u \leq t} |x_u| &< \frac{1-\zeta}{2(1-\eta)} M_2(\beta,\varepsilon)|\phi| \\ &\leq \frac{M(1-\zeta)}{2M-1-\zeta} M_2(\beta,\varepsilon)|\phi| \qquad (15) \\ &\leq M_2(\beta,\varepsilon)|\phi| \leq \delta_2(\varepsilon) \end{aligned}$$

for $|\phi| < \delta$ since $M \geq 1$. The continuation theorem implies that $x(t)$ is defined for $t \geq \sigma - r$, (15) is satisfied for $t \geq \sigma$ and the solution $x = 0$ of (3) is uniformly stable.

For $s = s_o = \sigma + \beta$ in (14) and $t \geq \sigma + 2\beta$, it follows from (14) and the estimates (15) and (13) that

$$\begin{aligned} |x_t| &\leq M\Big\{(1+\pi^*(0) + \varepsilon)e^{-\alpha(t-\sigma)} + \pi^*(\sigma)Me^{-\alpha(t-s_o)} + \pi^*(s_o)M_2(\beta,\zeta) \\ &\qquad + M_2(\beta,\zeta)\varepsilon(1+\alpha^{-1}) + \zeta\Big\}|\phi| \\ &\leq M\Big\{(1+\pi^*(0) + \varepsilon)e^{-\alpha(t-\sigma)} + \eta\Big\}|\phi| \\ &\leq M\Big\{(1+\pi^*(0) + \varepsilon)e^{-\alpha(t-\sigma)} + \frac{1+\zeta}{2M}\Big\}|\phi|. \end{aligned}$$

For any δ_o, $(1+\zeta)/2 < \delta_o < 1$, choose $T \geq 2\beta$ so large that

$$Me^{-\alpha T}(1+\pi^*(0) + \varepsilon) + \frac{1+\zeta}{2} < \delta_o.$$

For $t \geq \sigma + T$, it follows that

$$|x_t| \leq \delta_o|\phi|.$$

Since T is independent of σ and ϕ, this clearly implies exponential asymptotic stability and proves the theorem.

In [1], asymptotic stability theorems of the above type were proved for systems which contained either terms of the form F_1, G_1 or F_2, G_2 but not both simultaneously. In addition to combining these results into one, the more significant part of the above theorem is the fact that uniform asymptotic stability is proved under the weak hypothesis (9). In [1], it was assumed that $\pi(t) \to 0$ as $t \to \infty$.

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Footnotes

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