On the uniqueness of factors of amalgamated products

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Let $G = A *_C B$ be an amalgamated product. We study conditions under which the factor B is determined by A and C, i.e. under which the existence of another splitting of G as an amalgamated product $G = A *_C D$ implies $B = D$ or $B \cong D$. We also describe the structure of the family of amalgamated products along a given malnormal subgroup.

There are situations where $D \ncong B$. We describe them and show that they are the only ones after giving a short account of controlled subgroups as introduced in [FRW]. Mary Jones has independently constructed an example of the type below and Ilya Kapovich has provided an insightful example to a related question. This work was motivated by the first author's work on splittings of surface groups [B].

1 Examples and main results

The two types of examples are similar in that B splits over a subgroup of C and that D is obtained by conjugating a boundary monomorphism with an element of A such that this conjugation cannot be done in B. We describe the amalgamated product and the HNN-extension case.

Case 1. Suppose that $B = B_1 *_{C_1} B_2$ where $C_1 \leqslant C \leqslant B_1$ and $a^{-1}C_1a \leqslant C$ for some $a \in A$. Then $A *_C B = A *_C D$ where

$$D = B_1 *_{a^{-1}C_1a} a^{-1}B_2a.$$

This holds since $G = A *_C B = A *_C (B_1 *_{C_1} B_2) = (A *_C B_1) *_{C_1} B_2 = (A *_C B_1) *_{a^{-1}C_1a} a^{-1}B_2a = A *_C (B_1 *_{a^{-1}C_1a} a^{-1}B_2a) = A *_C D$.

Case 2. Suppose that $B = \langle B_1, t \,|\, tC_1t^{-1} = C_1'\rangle$ is an HNN-extension with base group B_1 and associated subgroups C_1 and C_1' where $C_1 \leqslant C \leqslant B_1$ and $a^{-1}C_1a \leqslant C$ for some $a \in A$. Then $A *_C B = A *_C D$ where

$$D = \langle B_1, ta \,|\, (ta) \cdot a^{-1}C_1a \cdot (ta)^{-1} = C_1'\rangle.$$

This holds since $G = A *_C B = A *_C \langle B_1, t \,|\, tC_1t^{-1} = C_1'\rangle = \langle A *_C B_1, t \,|\, tC_1t^{-1} = C_1'\rangle = \langle A *_C B_1, ta \,|\, (ta) \cdot a^{-1}C_1a \cdot (ta)^{-1} = C_1'\rangle = A *_C \langle B_1, ta \,|\, (ta) \cdot a^{-1}C_1a \cdot (ta)^{-1} = C_1'\rangle = A *_C D$.

In the first case we say that the splitting $A *_C D$ is obtained from the splitting $A *_C B$ by a *move of type 1*, in the second case by a *move of type 2*.

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It is easy to see that there are situations where such moves yield non-isomorphic B and D. A simple example for a move of type 1 is the following. Suppose that $A = BS(1,2) = \langle a, x | a^{-1}xa = x^2\rangle$, that $C = B_1 = \langle x|-\rangle$, that $B_2 = \langle y|-\rangle$ and that $C_1 = \langle x\rangle = \langle y^2\rangle$. We then have $B = B_1 *_{C_1} B_2 = \langle x|-\rangle *_{\langle x=y^2\rangle} \langle y|-\rangle \cong \mathbb{Z}$ but $D = B_1 *_{a^{-1}C_1a} a^{-1}B_2a = \langle x|-\rangle *_{\langle x^2=a^{-1}y^2a\rangle} \langle a^{-1}ya|-\rangle$ which is the fundamental group of the Klein bottle.

On the other hand there is a situation where we can guarantee that B and D are isomorphic, namely in the case that the conjugation by the element could have been done by an element of B_1, i.e. if there exists an element $b \in B_1$ such that $b^{-1}c_1b = a^{-1}c_1a$ for all $c_1 \in C_1$. This is clear since then $D = B_1 *_{a^{-1}C_1a} a^{-1}B_2a \cong B_1 *_{b^{-1}C_1b} b^{-1}B_2b = B$. It is further clear that in this case there exists an isomorphism $\phi : B \to D$ such that $\phi|_{B_1} = \mathrm{Id}_{B_1}$, namely the extension of the map $\phi|_{B_1} = \mathrm{Id}_{B_1}$ and the map $\phi|_{b^{-1}B_2b}$ that maps $b^{-1}b_2b$ to $(a^{-1}b)b^{-1}b_2b(b^{-1}a) = a^{-1}b_2a$. These two maps extend to a homomorphism $\phi : B \to D$ since by assumption $a^{-1}c_1a = b^{-1}c_1b$ for all $c_1 \in C_1$, i.e. they coincide when restricted to the amalgam. This extension is clearly an isomorphism.

Our main result is the following.

Theorem 1. *Suppose that $G = A *_C B = A *_C D$ where G and C are finitely generated. Then the splitting $A *_C D$ can be obtained from the splitting $A *_C B$ by a finite number of moves of type 1 or 2.*

If any conjugation of subgroups of C in A can already be done in C, then our observation above immediately yields the following corollary.

Corollary. *Let $G = A *_C B$ where G and C are finitely generated and suppose that for any $C_1 \leqslant C$ and $a \in A$ with $aC_1a^{-1} \leqslant C$ there exists an element $c \in C$ such that $ac_1a^{-1} = cc_1c^{-1}$ for all $c_1 \in C_1$.*

Then $G = A *_C D$ implies that there exists an isomorphism $\phi : B \to D$ such that $\phi|_C = \mathrm{Id}_C$.

The hypothesis of the Corollary is clearly fulfilled if C is a *malnormal* subgroup of A, i.e. if $aCa^{-1} \cap C = 1$ for all $a \in A - C$. In the case of a malnormal subgroup we are able to give the following stronger result.

Theorem 2. *Let G be a finitely generated group and $C \neq 1$ be a malnormal subgroup that does not lie in a proper free factor of G. If G is a nontrivial amalgamated product over C, then there exists a unique decomposition of type $G = \mathop{*_C}\limits_{i=1}^{n} G_i$ where $C \neq G_i$, such that for any splitting $G = A *_C B$ we have that $A = \mathop{*_C}\limits_{i \in I_1} G_i$ and $B = \mathop{*_C}\limits_{i \in I_2} G_i$ where $I_1 \cup I_2 = \{1, \ldots, n\}$ and $I_1 \cap I_2 = \emptyset$.*

2 Controlled subgroups

We will assume familiarity of the reader with the Bass-Serre theory. Details can be found in [Sr]. Suppose that G acts minimally, simplicially and without inversion on a simplicial tree T. Let U be a subgroup of G. We define T_U to be a vertex fixed under the action of U if U acts with a global fixed point and to be the minimal U-invariant subtree of T otherwise. *The induced splitting* of U is the splitting corresponding to the action of U on T_U. For a graph Γ we denote the set of vertices of Γ by $V\Gamma$ and the set of edges by $E\Gamma$.

For $x, y \in VT$ we denote by $[x, y]$ the geodesic segment joining x and y and by (x, y) the segment without its boundary. For $x \in VT \cup ET$ we denote by $\mathrm{Stab}(x)$ the stabilizer of x in G.

We follow [FRW] and describe a situation where the induced splitting can be read off information that comes with U. In the definition below the tree T_1 corresponds to a lift of a maximal subtree of $U \backslash T_U$ to T_U, the tree T_2 corresponds to a subtree in T_U whose edge set projects bijectively onto the edge set of $U \backslash T_U$ and the elements t_v to the stable letters.

We say that the subgroup U of G is *controlled* by the tuple

$$(T_1, T_2, \{G_v | v \in VT_2\}, \{t_v | v \in VT_2 - VT_1\})$$

if

1. T_1 and T_2 are subtrees of T, $T_1 \subseteq T_2$.
2. For any $v \in VT_2 - VT_1$ there exists an edge $e_v \in ET_2$ with initial vertex v and terminal vertex in VT_1.
3. $G_v \leqslant G$ and $G_v v = v$ for all $v \in VT_2$.
4. U is generated by the G_v and the t_v.
5. $G_v \cap \mathrm{Stab}\ (e) = G_w \cap \mathrm{Stab}\ (e)$ for all edges $e \in ET_2$ with initial vertex v and terminal vertex w.
6. For every $v \in VT_2 - VT_1$ there exists a $x_v \in VT_1$ such that $t_v v = x_v$ and $G_v = t_v^{-1} G_{x_v} t_v$.
7. $t_v e_v \neq t_w e_w$, $t_v e_v \notin T_2$ and $t_w e_w \notin T_2$ for all $v, w \in VT_2 - VT_1$ and $v \neq w$.
8. All edges of $T_3 := T_2 \cup \{t_v e_v \,|\, v \in VT_2 - VT_1\}$ emanating at a vertex $x \in VT_1$ are G_x-inequivalent.
9. There exists no vertex $x \in VT_1$ such that a component C of $T_1 - \{x\}$ is also a component of $T_3 - \{x\}$ and that $G_v \leqslant G_x$ for all $v \in VC$.

There is a simple way to read the induced splitting of U off the above tuple. This graph of groups has as the underlying graph the graph obtained from T_2 by identifying v and x_v for every $v \in VT_2 - VT_1$. The vertex and edge groups are simply the groups G_v and G_e given above. The number of edges of the induced splitting coincides with the number of edges of T_2 and the groups G_v coincide with $\mathrm{Stab}(v) \cap U$ for all v; see [FRW] for details. Conversely for every subgroup U of G their exists such a tuple that makes U controlled.

We say that a subtree T^U of T is a *generating tree* for U if there exists a generating set M of U such that $mT^U \cap T^U \neq \emptyset$ for all $m \in M$. It is clear that UT^U is connected and U-invariant. In particular UT^U contains the minimal U-invariant subtree T_U, it follows that UT^U has at least as many U-equivalence classes of edges as T_U. It is further clear that gT^U is a generating tree of gUg^{-1} iff T^U is a generating tree for U. We have shown the following lemma.

Lemma 1. *Let T^U be a generating tree for U. Then the induced splitting of U has at most as many edges as T^U.*

3 The proofs

Proof of Theorem 1. We study the action of D on the Bass–Serre tree T associated to the splitting $G = A *_C B$. We assume that the number of edges of the induced splitting of D with respect to the action on T is minimal among all groups D' such that the splitting $A *_C D'$ can be obtained from $A *_C D$ by a finite numbers of moves of type 1 or 2. Such D' must exist since any splitting of a finitely generated group contains only finitely many edges and D is finitely generated since G and C are finitely generated.

Let x be the vertex fixed under the action of A, y be the vertex fixed under the action of B and $e = [x, y]$ be the edge fixed under the action of C. Let further $T_D \subset T$ be as in section 2. We distinguish the cases that T_D lies in the component of $T - \{x\}$ containing y, that T_D lies in a component of $T - \{x\}$ not containing y and that T_D contains x where the last case has the subcases that T_D contains y and that it doesn't.

Case 1: T_D lies in the component of $T - \{x\}$ containing y. Choose the vertex z of T_D that has minimal distance to x, possibly we have $y = z$, but by assumption $z \neq x$. Choose a tuple $(T_1, T_2, \{G_v | v \in VT_2\}, \{t_v | v \in VT_2 - VT_1\})$ that makes D controlled such that $z \in VT_1$. It is clear that $C \leqslant G_z$, otherwise $T_D \cap cT_D = \emptyset$ for any $c \in C - G_z$ as c fixes x but not z. This contradicts the D-invariance of T_D.

It is easy to check that the subgroup generated by A and D is controlled by the tuple $(T_1', T_2', \{G_v | v \in VT_2'\}, \{t_v | v \in VT_2' - VT_1'\})$ where T_i' is the tree spanned by T_i and x, the t_v are as before, $G_x = A$, $G_v = C$ for all $v \in VT_2' - VT_2$ with $v \neq x$ and all other G_v are as before. This subgroup however is G by assumption. It follows that the induced splitting has two vertices and therefore $VT_1' = VT_2' = \{x, y\}$ which implies that D fixes y, i.e. $D \leqslant B$, and therefore $D = B$ since otherwise A and D do not generate G.

Case 2: T_D lies in a component of $T - \{x\}$ not containing y. Choose z and $(T_1, T_2, \{G_v | v \in VT_2\}, \{t_v | v \in VT_2 - VT_1\})$ as in the first case.

We argue as in the first case to show that D fixes a vertex z adjacent to x. Clearly $y \neq z$. So, there is an $a \in A$ such that $ae = [x, z]$. We have $D \leqslant \mathrm{Stab}(z) = aBa^{-1}$ and $C \leqslant \mathrm{Stab}(ae) = aCa^{-1}$. Now, $G = A *_C D = (A *_C C) *_C D = (A *_{aCa^{-1}} aCa^{-1}) *_C D = A *_{aCa^{-1}} (aCa^{-1} *_C D)$. We also cleary have $G = A *_C B = A *_{aCa^{-1}} aBa^{-1}$. Note, that the equality $X *_Z Y = X *_Z Y_1$ with $Y \leqslant Y_1$ implies $Y = Y_1$ for any amalgamated product. Therefore $aCa^{-1} *_C D = aBa^{-1}$. Hence $B = C *_{a^{-1}Ca} a^{-1}Da$.

This implies that the splitting $A *_C D$ is obtained from the splitting $A *_C B$ by a move of type 1.

Case 3: T_D contains x and y. Note that in this case D is not elliptic with respect to the action on T because T_D is by definition the minimal D-invariant subtree. Since minimal subtrees cannot contain valence 1 vertices it follows that T_D must contain an edge $ae = a[x, y] = [ax, ay] = [x, z]$ different from e. It follows that D splits over the subgroup $C_1 := \mathrm{Stab}\ (ae) \cap D = \mathrm{Stab}\ (ae) \cap (D \cap \mathrm{Stab}\ (x)) = aCa^{-1} \cap C$. Since $D \cap \mathrm{Stab}\ (x) = D \cap A = C$ we further get that ae is not $(D \cap \mathrm{Stab}\ (x))$-equivalent to e.

In the case that D splits as an amalgamated product $D = D_1 *_{C_1} D_2$ we can choose a tuple $(T_1, T_2, \{G_v | v \in VT_2\}, \{t_v | v \in VT_2 - VT_1\})$ such that $ae \in ET_1$ and $e \in ET_2$ and that the two components T^1 and T^2 of $T_2 - (x, z)$ are generating trees for D_1 and D_2, where we choose D_1 to be the group corresponding to the component containing x and therefore also e. This implies in particular that $C \leqslant D_1$. We define $\bar{D}_2 := a^{-1}D_2a$ and $\bar{D} = \langle D_1, \bar{D}_2 \rangle$. The subtree $\bar{T} := T^1 \cup a^{-1}T^2$ is connected since a^{-1} maps $z \in T^2$

to $y \in T^1$, it follows that $\bar{T}$ is a generating tree for $\bar{D}$. Since $\bar{T}$ has less edges than T_1 it follows from Lemma 1 that the induced splitting of $\bar{D}$ has fewer edges than the induced splitting of D. In order to get a contradiction to the minimality assumption we have to show that $G = A *_C \bar{D}$ and that the splitting $G = A *_C \bar{D}$ can be obtained from the splitting $G = A *_C D$ by a move of type 1.

We have $G = A *_C D = A *_C (D_1 *_{C_1} D_2) = (A *_C D_1) *_{C_1} D_2 = (A *_C D_1) *_{a^{-1}C_1 a} a^{-1}D_2 a = (A *_C D_1) *_{C_2} \bar{D}_2 = A *_C (D_1 *_{C_2} \bar{D}_2) = A *_C \bar{D}$ where $C_2 = a^{-1}C_1 a = a^{-1}(C \cap aCa^{-1})a = a^{-1}Ca \cap C \leqslant C$. This calculation shows in particular that $\bar{D} = D_1 *_{C_2} \bar{D}_2$.

In the case that D splits as a HNN-extension $D = D_1 *_{C_1}$ we can choose a tuple $(T_1, T_2, \{G_v | v \in VT_2\}, \{t_v | v \in VT_2 - VT_1\})$ such that $ae = [x, z] \in ET_2 - ET_1$ and $e \in ET_2$. In particular $T^1 = T_2 - (x, z]$ is connected and is a generating tree for D_1. We have $D = \langle D_1, t_z \,|\, t_z c_1 t_z^{-1} = \psi(c_1) \text{ for all } c_1 \in C_1 \rangle$ where ψ maps C_1 isomorphically to a subgroup of D_1. We define $\bar{D} = \langle D_1, t_z a \rangle$. Since t_z maps z to a vertex x_z of T^1 it follows $t_z a$ maps y to x_z which implies that T^1 is a generating tree for $\bar{D}$ which implies that the induced splitting of $\bar{D}$ has fewer edges than the induced splitting of D. In order to get a contradiction to the minimality we have to check that $G = A *_C \bar{D}$ and that the splitting $G = A *_C \bar{D}$ can be obtained from the splitting $G = A *_C D$ by a move of type 2. The proof is analogous to the case of an amalgamated product.

We have $G = A *_C D = A *_C \langle D_1, t_z \,|\, t_z C_1 t_z^{-1} = \psi(C_1) \rangle = \langle A *_C D_1, t_z \,|\, t_z C_1 t_z^{-1} = \psi(C_1) \rangle = \langle A *_C D_1, t_z a \,|\, (t_z a) \cdot a^{-1} C_1 a \cdot (a^{-1} t_z^{-1}) = \psi(C_1) \rangle = A *_C \langle D_1, t_z a \,|\, (t_z a) \cdot a^{-1} C_1 a \cdot (a^{-1} t_z^{-1}) = \psi(C_1) \rangle$. We have shown that $G = A *_C D = A *_C \bar{D}$ where $\bar{D} = \langle D_1, t_z a \,|\, (t_z a) \cdot a^{-1} C_1 a \cdot (a^{-1} t_z^{-1}) = \psi(C_1) \rangle$.

Case 4: T_D contains x but not y. In this case we proceed as in case 3. In both cases the new group $\bar{D}$ has a generating tree containing y that has at most as many edges as the induced splitting of D. It follows that either $T_{\bar{D}}$ contains y in which the statement follows from one of the above cases or the induced splitting of $\bar{D}$ has less edges than the induced splitting of D which contradicts our minimality assumption. □

We conclude with the proof of Theorem 2.

Proof of Theorem 2. Corollary 1 of [W] states that rank $(G) - 1$ is an upper bound on the number of factors of a decomposition of G as an amalgamated product of type $\overset{n}{\underset{i=1}{*}}{}_C G_i$ provided that $C \neq 1$, $C \neq G_i$ and that $C \leqslant G$ is malnormal. This guarantees the existence of a decomposition $G = \overset{n}{\underset{i=1}{*}}{}_C G_i$ where $C \neq G_i$ with maximal n. This also follows from Z. Sela's acylindrical accessibility result [Sl]. Suppose $G = A *_C B$ is an arbitrary nontrivial splitting. In order to prove Theorem 1 it clearly suffices to show that for any $i \in \{1, \ldots, n\}$ either $G_i \leqslant A$ or $G_i \leqslant B$.

Choose T, x, y and e be as in the proof of Theorem 1. We have to show that G_i fixes either x or y. Define $T_i := T_{G_i}$. Let z be the vertex of T_i that is in minimal distance to e. As in the proof of Theorem 1 we see that $C \leqslant G_i \cap \text{Stab}\,(z)$. Since the action of G on T is 1-acylindrical, i.e. does not fix a segment of length 2, this implies that $z = x$ or that $z = y$. Without loss of generality we can assume that $z = x$. We have to show that G_i fixes x. As $C \leqslant \text{Stab}\,(x) \cap G_i$, it follows that C is a subgroup of a vertex group of the induced splitting of G_i. This implies that the induced splitting of G_i does not contain a trivial edge group since otherwise C is contained in a free factor of G_i and therefore in a free factor of G which contradicts the assumption.

We show that every edge of T_i is G_i-equivalent to e. If T_i contains an edge this implies that G_i splits as an amalgamated product over C which contradicts the maximality of the splitting $\underset{i=1}{\overset{n}{*}}{}_C G_i$. Note that in this case G_i cannot split as a HNN-extension since x and y are not G-equivalent. If T_i contains no edge then T_i consists of the vertex x and therefore fixes x which finishes the proof.

Let f be an edge of T_i and choose $w \in G$ such that $we = f$. Since all edge stabilizers are non-trivial there exists a nontrivial element $g \in G_i$ such that $gf = f$, in particular $g \in wCw^{-1}$, i.e. $g = wcw^{-1}$ for some $c \in C$. The malnormality of C implies the malnormality of G_i, i.e. $G_i \cap wG_iw^{-1} = 1$ for all $w \in G - G_i$. Since $g \in G_i$ and $c \in C \leqslant G_i$ it follows that $w \in G_i$. Thus e and f are G_i-equivalent. $\square$

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