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On the uniqueness of solutions of stochastic differential equations

By

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Introduction

In this paper, we shall discuss the uniqueness problem for solutions of stochastic differential equations.

The theory of stochastic differential equations, as is well known, was developed mainly by Ito and furnishes a very important tool of constructing diffusion processes. Skorohod [4] showed the existence of solutions under the condition that coefficients are only continuous and then, the problem of the uniqueness of solutions becomes important. In order to define a diffusion process through a solution of the stochastic differential equation, it is sufficient to verify *the uniqueness in the sense of the probability law of solutions*. It may be needless to say that there are many means to verify it; in analytic way, through the theory of differential equations (cf. Stroock-Varadhan [5]) and in probabilistic way through several transformations such as time change or the change of drift.

Here, we shall study mainly *the pathwise uniqueness of solutions*. In Ito's classical theory where the coefficients are assumed to be Lipschitz continuous, the pathwise uniqueness holds and the solution can be constructed on a given Brownian motion through successive approximation. The uniqueness in the sense of the probability law is obvious in this case. There are several examples where Ito's theory can not apply yet we can prove the pathwise uniqueness. Such examples were given by Skorohod [4] and Tanaka [6]. We will improve their results below. As we shall see, the pathwise uniqueness generally implies the uniqueness in the sense of the probability law and thus, the solution defines a unique diffusion process. In this way, we have some examples (besides the Ito's case) of constructing diffusion processes through solutions of stochastic differential equations by verifying the pathwise uniqueness. Other construction of such processes seems to be much more difficult.

Let $\sigma(x)=(\sigma^i_j(x))$ and $b(x)=(b^i(x))$, $i, j=1, 2, \cdots, n$, be defined on $\boldsymbol{R}^n$, Borel measurable in x such that σ is an $n\times n$-matrix and b is an $n\times 1$-matrix.*1) We consider the following Ito's stochastic differential equation;

$$(1.1)\qquad dx_t=\sigma(x_t)\,dB_t+b\,(x_t)\,dt$$

or in component wise,

$$(1.1)'\qquad dx^i_t=\sum_{j=1}^{n}\sigma^i_j(x_t)\,dB^j_t+b^i(x_t)\,dt,\qquad i=1, 2, \cdots, n.$$

A presice formulation is as follows; by a probability space $(\Omega, \mathscr{F}, P)$ with an increasing family of Borel fields $\mathscr{F}_t$, which is denoted as $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, we mean a probability space $(\Omega, \mathscr{F}, P)$ with a system $\{\mathscr{F}_t\}_{t\in[0,\infty)}$ of sub-Borel fields of $\mathscr{F}$ such that $\mathscr{F}_t\subset\mathscr{F}_s$ if $t<s$.

Definition 1. By a solution of the equation (1.1), we mean a probability space with an increasing family of Borel fields $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ and a family of stochastic processes $\mathfrak{X}=\{x_t=(x^1_t, x^2_t, \cdots, x^n_t), B_t=(B^1_t, B^2_t, \cdots, B^n_t)\}$ defined on it such that

*1) For the sake of simplicity, we consider the case when the coefficients are independent of t (i.e. temporally homogeneous case) but all the arguments below remain valid when the coefficients are time dependent, i.e. the case when $\sigma=\sigma(t, x)$ and $b=b(t, x)$.

(i) with probability one, x_t and B_t are continuous in t and $B_0=0$,

(ii) they are adapted to $\mathscr{F}_t$, i.e., for each t, x_t and B_t are $\mathscr{F}_t$-measurable,

(iii) B_t is a system of $\mathscr{F}_t$-martingales such that

$$<B^i, B^j>_t=\delta_{ij}\cdot t, \qquad i, j=1, 2, \cdots, n,^{*2)}$$

(iv) $\mathfrak{X}=\{x_t, B_t\}$ satisfies

$$(1.1)'' \qquad x_t-x_0=\int_0^t \sigma(x_s)\,dB_s+\int_0^t b(x_s)\,ds$$

where the integral by dB_s is understood in the sense of the stochastic integral.

Remark 1. As is well known (e.g. [3]), B_t is an n-dimensional Brownian motion such that B_t-B_s and $\mathscr{F}_s$ are independent $(t>s)$.

Now we shall introduce several notions of the uniqueness for solutions of (1.1).

Definitions 2. (*Pathwise uniqueness*) We shall say that the pathwise uniqueness holds for (1.1) if, for any two solutions $\mathfrak{X}=(x_t, B_t)$ and $\mathfrak{X}'=(x_t', B_t')$ defined on a same probability space $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, $x_0=x_0'$ and $B_t\equiv B_t'$ imply $x_t\equiv x_t'$.

When σ and b are Lipschitz continuous, then, as is well known by Ito's theory, the pathwise uniqueness holds and x_t is constructed on arbitrarily given Brownian motion B_t as a measurable function of x_0 and B_t.

On the other hand, Skorohod [4] proved the existence of solutions in the sense of Def. 1 for arbitrarily given distribution of x_0, when σ

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*2) $<B^i, B^j>_t$ is a continuous bounded variation process such that $B_t^i B_t^j-<B^i, B^j>_t$ is an $\mathscr{F}_t$-martingale, ([3]).

and b are only continuous and, in such a general case, the existence and the uniqueness need to be discussed separately.

Definition 3. (*Uniqueness in the sense of the probability law*) We shall say that the uniqueness in the sense of the probability law holds for (1.1) if, for any two solutions $\mathfrak{X}=(x_t, B_t)$ and $\mathfrak{X}'=(x_t', B_t')$*3) such that $x_0=x$ and $x_0'=x$ (*a.s.*) for some $x\in \boldsymbol{R}^n$, the probability law of the processes x_t and x_t' on the space $\{W, \mathscr{B}(W)\}$ coincides, where W is the Fréchet space of all $\boldsymbol{R}^n$-valued continuous functions on $[0, \infty)$ with the compact uniform topology and $\mathscr{B}(W)$ is the topological Borel field on W.

Stroock-Varadhan [5] proved the existence and the uniqueness in the sense of the probability law of solutions of (1.1) when σ is bounded continuous and uniformly elliptic and b is bounded and measurable.

Remark 2. A solution of (1.1) can be defined in a resticted sense as follows; let $Y\subset W$: by a Y-solution of (1.1), we mean a solution $\mathfrak{X}$ of (1.1) such that $P[x.(\omega)\in Y]=1$. The uniqueness (Definitions 2 and 3) is defined in the same way and all the propositions and corollaries below remain valid for Y-solutions.

Proposition 1. *The pathwise uniqueness implies the uniqueness in the sense of the probability law.*

Proof. Let $\mathfrak{X}=(x_t, B_t)$ and $\mathfrak{X}'=(x_t', B_t')$ be two solutions of (1.1), (which may be defined on different probability spaces) such that $x_0=x$ and $x_0'=x$ (*a.s.*). Let W be defined as in Def. 3 and let $P(dw_1\, dw_2)$ and $P'(dw_1\, dw_2)$ be the probability law of $\mathfrak{X}$ and $\mathfrak{X}'$ on the space $(W\times W, \mathscr{B}(W\times W))$ respectively.

Let $P_{w_2}(dw_1)$ be the regular conditional distribution of $P(dw_1\, dw_2)$ given w_2; i.e., (i) for each w_2, it is a probability measure on $(W, \mathscr{B}(W))$,

*3) They may be defined on different probability spaces.

(ii) for each $B\in\mathscr{B}(W)$, $P_{w_2}(B)$ is $\mathscr{B}(W)$-measurable in w_2, (iii) for any $B, B'\in\mathscr{B}(W)$, $P(B\times B')=\int_{B'}P_{w_2}(B)R(dw_2)$, where R is the probability law of B_t on $(W, \mathscr{B}(W))$, i.e., the Wiener measure on $(W, \mathscr{B}(W))$. Similarly $P'_{w_2}(dw_1)$ is defined for $P'(dw_1\, dw_2)$. Define a probability measure $Q(dw_1\, dw_2\, dw_3)$ on $(W\times W\times W, \mathscr{B}(W\times W\times W))$, by

$$(1.2)\qquad Q(dw_1\, dw_2\, dw_3)=P_{w_3}(dw_1)P'_{w_3}(dw_2)R(dw_3).$$

Let $\mathscr{B}_t(w)$ be the Borel algebra generated by $w(s)$, $s\leq t$. $\mathscr{B}_t(W\times W)$ and $\mathscr{B}_t(W\times W\times W)$ are defined similarly. We shall show that $[w_3(t), \mathscr{B}_t(W\times W\times W)]$ is a system of martingales with respect to Q such that $<w_3^i, w_3^j>_t=\delta_{ij}\cdot t$ i.e., w_3 is an n-dimensional Brownian motion such that $w_3(t)-w_3(s)$ is independent of $\mathscr{B}_s(W\times W\times W)$. For this, we need the following

Lemma 1. *If* $B\in\mathscr{B}_t(W)$, $P_w(B)$ $(P'_w(B))$ *is* $\mathscr{B}_t(W)$*-measurable in* w.

Proof. Let $P_w^t(\cdot)$ be the regular conditional distribution given $\mathscr{B}_t(W)$; (i) for each w, it is a probability measure on $(W, \mathscr{B}(W))$, (ii) for each $B\in\mathscr{B}(W)$, $P_w^t(B)$ is $\mathscr{B}_t(W)$-measurable, (iii) for each $B\in\mathscr{B}(W)$, $B'\in\mathscr{B}_t(W)$, $P(B\times B')=\int_{B'}P_w^t(B)R(dw)$. It is sufficient to prove that if $B\in\mathscr{B}_t(W)$ then $P_w(B)=P_w^t(B)$, *a.s.*. For this, it is enough to show that, if $F(w)$ is $\mathscr{B}(W)$-measurable and bounded,

$$\int_{W\times W}F(w_2)\, I_B(w_1)P(dw_1\, dw_2)=\int_W F(w)\, P_w^t(B)\, R(dw).$$

By the theory of multiple Wiener integrals ([1]) or by a result in [3], we may assume that

$$F(w_2)=c+\int_0^\infty \Phi_s(w_2)\, dw_2(s)\ ^{*4)}$$

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*4) $\int_0^\infty \Phi_s(w_2)\, dw_2=\sum_{i=1}^n\int_0^\infty \Phi_s^i(w_2)\, dw_s^i$, and $\Phi_s(w)$ is a measurable $\mathscr{B}_s(W)$-adapted process.

$$=c+\int_t^\infty \Phi_s(w_2)\,dw_2+\int_0^t \Phi_s(w_2)\,dw_2 \qquad a.a.w_2\,(R(dw_2)),$$

where c is a constant and the integral by dw_2 is a stochastic integral. Since $(B_t, \mathscr{F}_t)$ is a martingale, it is clear that $(w_2(t), \mathscr{B}_t(W\times W), P(dw_1\,dw_2))$ is a martingale. Now,

$$\int_{W\times W} F(w_2)\,I_B(w_1)\,P(dw_1\,dw_2)$$

$$=c\int_{W\times W} I_B(w_1)\,P(dw_1\,dw_2)+\int_{W\times W}\Big(\int_t^\infty \Phi_s(w_2)\,dw_2\Big)I_B(w_1)\,P(dw_1\,dw_2)$$

$$+\int_{W\times W}\Big(\int_0^t \Phi_s(w_2)\,dw_2\Big)I_B(w_1)\,P(dw_1\,dw_2)$$

and the second term is 0 since $(w_2(t), \mathscr{B}_t(W\times W))$ is a martingale and $I_B(w_1)$ is $\mathscr{B}_t(W\times W)$-measurable. Thus, the above integral is equal to $c\int_W P_w^t(B)\,R(dw)+\int_W\Big(\int_0^t \Phi_s(w)\,dw\Big)P_w^t(B)\,R(dw)=\int_W F(w)\,P_w^t(B)\,R(dw)$.

Now we return to the proof of the proposition. If F_1, F_2, F_3 are $\mathscr{B}_s(W)$-measurable bounded functions, then

$$\int_{W\times W\times W}[w_3^i(t)-w_3^i(s)]\,F_1(w_1)\,F_2(w_2)\,F_3(w_3)\,Q(dw_1\,dw_2\,dw_3)$$

$$=\int_W[w^i(t)-w^i(s)]\Big(\int_W F_1(w_1)\,P_w(dw_1)\Big)$$

$$\times\Big(\int_W F_2(w_2)\,P'_w(dw_2)\Big)F_3(w)\,R(dw).$$

Since $\int_W F_1(w_1)\,P_w(dw_1)$ and $\int_W F_2(w_2)\,P'_w(dw_2)$ are $\mathscr{B}_s(W)$-measurable in w by the above lemma, the above integral is 0.

Similarly, we can prove that

$$\int_{W\times W\times W}\{[w_3^i(t)-w_3^i(s)][w_3^j(t)-w_3^j(s)]-\delta_{ij}(t-s)\}$$

$$\times F_1(w_1)\,F_2(w_2)\,F_3(w_3)\,Q(dw_1\,dw_2\,dw_3)=0.$$

Thus we have proved that $[w_3(t), \mathscr{B}_t(W\times W\times W), Q]$ is a system of martingales such that $<w_3^i, w_3^j>_t=\delta_{ij}\cdot t$. Since (x_t, B_t) and (w_1, w_3) are the equivalent processes and so are (x'_t, B'_t) and (w_2, w_3), we have two solutions (w_1, w_3) and (w_2, w_3) on the same probability space $(W\times W\times W, \mathscr{B}(W\times W\times W), Q; \mathscr{B}_t(W\times W\times W))$. Since $w_1(0)=w_2(0)=x$ *a.s.* (Q), the pathwise uniqueness implies $w_1(t)=w_2(t)$ *a.s.* (Q). This implies that $P(dw_1\, dw_2)=P'(dw_1\, dw_2)$ and hence the uniqueness in the sense of probability law holds. Another consequence is that, for *a.a.* $w_3\,(R(dw_3))$, $P_{w_3}\times P_{w_3}(w_1(t)=w_2(t))=1$ and this implies that there exists $F(w)$ such that $w_1=w_2=F(w_3)$. By Lemma 1, the mapping $w \rightsquigarrow F(w)$ is $\mathscr{B}_t(W)/\mathscr{B}_t(W)$ measurable. Thus we have

Corollary 1. *If the pathwise uniqueness holds and if a solution* (x_t, B_t) *exists such that* $x_0=x\in \mathbf{R}^n$, *then there exists a function* $F(w)$; $w\in W \rightsquigarrow F(w)\in W$ *such that it is* $\mathscr{B}_t(W)/\mathscr{B}_t(W)$*-measurable and* $x_.=F(B_.)$ *a.s..*

Remark 3. The uniqueness in the sense of probability law does not necessarily imply the pathwise uniqueness. The following example is due to Tanaka. Let $n=1$, $\sigma(x)=1$ for $x\geqq 0$ and $=-1$ for $x<0$, and $b(x)\equiv 0$. The existence of a solution (1.1) is shown in the following way; let $B(t)$ be a one dimensional Brownian motion and $x(0)$ be a real random variable such that they are mutually independent. Let $x(t)=x(0)+B(t)$ and $\tilde{B}(t)=\int_0^t \sigma(x(s))\, dB(s)$. Then $\tilde{B}(t)$ is a Brownian motion and $[x(t), \tilde{B}(t)]$ is clearly a solution. The uniqueness of the probability law is obvious since, for any solution, $\int_0^t \sigma(x(s))\, dB(s)$ is a Brownian motion. The pathwise uniqueness does not hold since, for $x(0)=0$, if $x(t)$ is a solution then $-x(t)$ is also a solution. Similarly, if $n=1$, $\sigma(x)=\operatorname{sgn} x\,|x|^\beta$, $0<\beta<\frac{1}{2}$ and $b(x)\equiv 0$, and if solutions are restricted to Y-solutions where $Y=\{w; \int_0^t I_{[s;w(s)=0]}\, ds=0,\ {}^\forall t>0\}$, the uniqueness in the sense of probability law holds but the pathwise uniqueness does not hold.

Proposition 2. *Suppose the uniqueness in the sense of the probability law holds. Suppose further that a solution* (x_t, B_t) *of* (1.1) *exists such that* $x_0 = x$ *a.s. for every* $x \in \boldsymbol{R}^n$ *and if* P_x *is the probability law of* x_t *on* $(W, \mathscr{B}(W))$ *which is unique by the first assumption,* $x \rightsquigarrow P_x(B)$ *is universally measurable for every* $B \in \mathscr{B}(W)$. *Then* $\{P_x, x \in \boldsymbol{R}^n\}$ *has the strong Markov property.*

Proof. This was proved essentially in Stroock-Varadhan [5]. Let (x_t, B_t) be a solution on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ and σ be an $\mathscr{F}_t$-stopping time such that $P(\sigma < \infty) = 1$. We assume, as we may, that (x_t, B_t) is given as function space-type, i.e., $(\Omega, \mathscr{F}) = (W \times W, \mathscr{B}(W \times W))$. Let $P(\ /\mathscr{F}_\sigma)$ be the regular conditional distribution given $\mathscr{F}_\sigma$. Let $\tilde{x}_t = x_{t+\sigma}$, $\tilde{B}_t = B_{t+\sigma} - B_\sigma$ and $\tilde{\mathscr{F}}_t = \mathscr{F}_{t+\sigma}$. Since by Doob's optional sampling theorem,

$$E(B^i_{t+\sigma} - B^i_{s+\sigma};\ A \cap B) = 0 \qquad \text{for every } A \in \mathscr{F}_{s+\sigma} \text{ and } B \in \mathscr{F}_\sigma,$$

we have $E(\tilde{B}^i_t - \tilde{B}^i_s;\ A/\mathscr{F}_\sigma) = 0$.

Thus, $(\tilde{B}_t, \tilde{\mathscr{F}}_t, P(\ /\mathscr{F}_\sigma))$ is a system of martingales. Similarly, we can prove

$$E[\{(\tilde{B}^i_t - \tilde{B}^i_s)(\tilde{B}^j_t - \tilde{B}^j_s) - \delta_{ij}\cdot(t-s)\};\ A/\mathscr{F}_\sigma] = 0 \qquad {}^\forall A \in \tilde{\mathscr{F}}_s.$$

Thus, $(\tilde{x}_t, \tilde{B}_t)$ is a solution on $(\Omega, \mathscr{F}, P(\ /\mathscr{F}_\sigma); \tilde{\mathscr{F}}_t)$ and the uniqueness in the sense of probability law implies $P(\tilde{x}. \in B/F_\sigma) = P_{x_\sigma}(B)$, ${}^\forall B \in \mathscr{B}(W)$.

Corollary 2. *Suppose that the uniqueness in the sense of the probability law holds and that, for every Borel probability measure* μ *on* $\boldsymbol{R}^n$, *a solution* (x_t, B_t) *of* (1.1) *exists such that* $P[x_0 \in dx] = \mu(dx)$. *Then* $P_x(B)$, $B \in \mathscr{B}(W)$, *is universally measurable in* x *and the probability law* Q *of* x_t *on* $(W, \mathscr{B}(W))$ *is given by* $Q(B) = \int P_x(B)\mu(dx)$, $B \in \mathscr{B}(W)$. *Thus* $\{P_x, x \in \boldsymbol{R}^n\}$ *has the strong Markov property.*

Proof. By the proof of Prop. 2, we have

$$P(x.\in B/\mathscr{F}_0)=P_{x_0}(B), \qquad {}^\forall B\in\mathscr{B}(W).$$

Thus, $P_x(B)$ is $\overline{\mathscr{B}}^\mu(\boldsymbol{R}^n)$*5) measurable and since this holds for every Borel probability measure μ, $P_x(B)$ is $\bigcap_\mu \overline{\mathscr{B}}^\mu(\boldsymbol{R}^n)$ measurable, that is, universally measurable. Obviously, $Q(B)=P(x.\in B)=\int P_x(B)\,\mu(dx)$.

Corollary 3. *Suppose that the pathwise uniqueness holds and that, for every Borel probability measure μ on $\boldsymbol{R}^n$, a solution of* (1.1) *exists such that $P(x_0\in dx)=\mu(dx)$. Then, there exists a function $F(x, w)$; $(x, w)\in\boldsymbol{R}^n\times W \rightsquigarrow F(x, w)\in W$ such that, for every t, it is $\bigcap_\mu \overline{\mathscr{B}(\boldsymbol{R}^n)\times\mathscr{B}_t(W)}^{\mu\times R}/\mathscr{B}_t(W)$-measurable and every solution (x_t, B_t) of* (1.1) *satisfies $x.=F(x_0, B.)$ where R is the Wiener measure (i.e., the probability law of B) on $(W, \mathscr{B}(W))$.*

Proof. As is proved in Prop. 1, if $x_0=x$ *a.s.*, there exists $F(w)\equiv F(x, w)$: $w\rightsquigarrow F(x, w)\in W$ such that it is $\mathscr{B}_t(W)/\mathscr{B}_t(W)$-measurable and

$$x.=F(x, B.), \quad a.s..$$

Let (x_t, B_t) be a solution on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$.
Since (x_t, B_t) is also a solution on $(\Omega, \mathscr{F}, P(\cdot/\mathscr{F}_0); \mathscr{F}_t)$, we have $x.=F(x_0, B)$ *a.s.*. Now the measurability of F can be proved easily.

Finally, we shall give some non-trivial example of the pathwise uniqueness. Skorohod ([4]) and Tanaka ([6]) proved the pathwise uniqueness of the solution, in one-dimensional case, of the equation

$$dx_t=\sigma(x_t)\,dB_t \tag{1.3}$$

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*5) $\overline{\mathscr{B}}^\mu(\boldsymbol{R}^n)$ is the completion of $\mathscr{B}(\boldsymbol{R}^n)$ (=the set of all Borel subsets of $\boldsymbol{R}^n$) by the measure μ.

if $\sigma(x)$ satisfies $|\sigma(x)-\sigma(y)|\leqq K|x-y|^{\alpha}$, ${}^{\forall}x,\ y\in \boldsymbol{R}^1$ for some constants $K>0$ and $\alpha>\frac{1}{2}$.

This can be strengthened and the uniqueness holds for $\alpha\geqq\frac{1}{2}$.

In fact, we can prove the following

Theorem 1. *Let*

$$dx_t=\sigma(x_t)\,dB_t+b(x_t)\,dt, \tag{1.4}$$

where

$$\sigma(x)=\begin{pmatrix}\sigma_1(x_1) & & 0\\ & \sigma_2(x_2) & \\ & & \ddots \\ 0 & & \sigma_n(x_n)\end{pmatrix},\quad b(x)=(b^1(x),\ b^2(x),\cdots,b^n(x))^{*6)}$$

such that

(i) *there exists a positive increasing function* $\rho(u)$, $u\in(0,\infty)$ *such that*

$$|\sigma_i(\xi)-\sigma_i(\eta)|\leq\rho(|\xi-\eta|),\qquad {}^{\forall}\xi,\ \eta\in\boldsymbol{R}^1,\quad i=1,2,\cdots,n$$

$$\textit{and}\ \int_{0+}\rho^{-2}(u)\,du=+\infty,$$

(ii) *there exists a positive increasing concave function* $\kappa(u)$, $u\in(0,\infty)$, *such that*

$$|b_i(x)-b_i(y)|\leq\kappa(|x-y|),\qquad {}^{\forall}x,\ y\in\boldsymbol{R}^n,\quad i=1,2,\cdots,n$$

$$\textit{and}\ \int_{0+}\kappa^{-1}(u)\,du=+\infty.$$

Then the pathwise uniqueness holds for (1.4).

Proof. Let $a_0=1>a_1>a_2>\cdots>a_k\to 0$ be defined by

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*6) $x=(x_1,x_2,\cdots,x_n)\in\boldsymbol{R}^n$

$$\int_{a_1}^{a_0}\rho^{-2}(u)\,du=1,\ \int_{a_2}^{a_1}\rho^{-2}(u)\,du=2,\ \cdots,\ \int_{a_k}^{a_{k-1}}\rho^{-2}(u)\,du=k,\ \cdots.$$

Then there exists a twice continuously differentiable function $\varphi_k(u)$ on $[0, \infty)$ such that $\varphi_k(0)=0$,

$$\varphi_k'(u)=\begin{cases} 0, & 0\leq u\leq a_k \\ \text{between 0 and 1}, & a_k<u<a_{k-1} \\ 1, & u\geq a_{k-1} \end{cases}$$

and

$$\varphi_k''(u)=\begin{cases} 0, & 0\leq u\leq a_k \\ \text{between 0 and } \frac{2}{k}\rho^{-2}(u), & a_k<u<a_{k-1} \\ 0, & u\geq a_{k-1}. \end{cases}$$

We extend $\varphi_k(u)$ on $(-\infty, \infty)$ symmetrically, i.e., $\varphi_k(u)=\varphi_k(|u|)$. Clearly $\varphi_k(u)$ is a twice continuously differentiable function on $(-\infty, \infty)$ such that $\varphi_k(u)\uparrow |u|$, $k\to\infty$.

Now let $(x(t), B(t))$ and $(x'(t), B'(t))$ be two solutions of (1.4) on the same probability space such that $x(0)=x'(0)$ and $B(t)\equiv B'(t)$. Then,

$$x^i(t)-x'^i(t)=\int_0^t[\sigma_i(x^i(s))-\sigma_i(x'^i(s))]\,dB^i(s)$$
$$+\int_0^t[b_i(x(s))-b_i(x'(s))]\,ds, \qquad i=1, 2, \cdots, n,$$

and by Ito's formula,

$$\varphi_k(x^i(t)-x'^i(t))=\int_0^t\varphi_k'(x^i(s)-x'^i(s))\,[\sigma_i(x^i(s))-\sigma_i(x'^i(s))]$$
$$\times\, dB^i(s)+\int_0^t\varphi_k'(x^i(s)-x'^i(s))\,[b_i(x(s))-b_i(x'(s))]\,ds$$
$$+\frac{1}{2}\int_0^t\varphi_k''(x^i(s)-x'^i(s))\,[\sigma_i(x^i(s))-\sigma_i(x'^i(s))]^2\,ds$$

$$=I_1+I_2+I_3, \quad \text{say.}$$

Then, $E[I_1]=0$

and, since φ_k' is uniformly bounded,

$$|E[I_2]|\leq K_1\int_0^t E[\kappa(|x(s)-x'(s)|)]\,ds\leq K_1\int_0^t \kappa(E|x(s)-x'(s)|)\,ds^{*7)}$$

by Jensen's inequality.

We have, for I_3,

$$|I_3|\leq\frac{1}{2}\int_0^t \varphi_k''(x^i(s)-x'^i(s))\,\rho^2(|x^i(s)-x'^i(s)|)\,ds$$

$$\leq\frac{1}{2}\,t\cdot\max_{a_k\leq|u|\leq a_{k-1}}[\varphi_k''(u)\,\rho^2(|u|)]\leq\frac{1}{2}\,t\cdot\frac{2}{k}\to 0, \qquad k\to\infty.$$

Also, $\varphi_k(x^i(t)-x'^i(t))\uparrow|x^i(t)-x'^i(t)|$ as $k\to\infty$. Thus we have

$$E|x^i(t)-x'^i(t)|\leq K_1\int_0^t \kappa(E|x(s)-x'(s)|)\,ds, \qquad i=1, 2, \cdots, n$$

and hence, we have

$$E|x(t)-x'(t)|\leq K_2\int_0^t \kappa(E|x(s)-x'(s)|)\,ds.$$

As is well known, this implies $E|x(t)-x'(t)|\equiv 0$ and therefore $x(t)\equiv x'(t)$. Thus, the pathwise uniqueness holds for solutions of (1.4).

Q.E.D.

As a corollary, the pathwise uniqueness holds for (1.4) if σ is Hölder continuous of order $\frac{1}{2}$ and b is Lipschitz continuous.

The condition $\int_{0+}\rho^{-2}(u)\,du=\infty$ is, in a certain sense, best possible. For, consider the equation (1.3) when $n=1$. If $\sigma(x_0)=0$ and $\int_{\varepsilon>|x-x_0|>0}\sigma^{-2}(x)\,dx<\infty$ for some $\varepsilon>0$ there are infinitely many solutions

*7) K_1 and K_2 are positive constants.

of (1.3) just as the example of Girsanov [2].

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References

[1] Ito, K., Multiple Wiener Integral, J. Math. Soc. Japan **3** (1951) 157-169.
[2] Girsanov, I. V., An example of non-uniqueness of the solution of the stochastic equation of K. Ito. (English translation) Theory of Prob. and its Appl. **7** (1962) 325-331.
[3] Kunita, H. and S. Watanabe, On square integrable martingales. Nagoya Math. J. **30** (1967), 209-245.
[4] Skorohod, A. V., Studies in the theory of random processes, Addison-Wisley 1965 (Originally published in Kiev)
[5] Stroock, D. W. and S. S. R. Varadhan, Diffusion processes with continuous coefficients, I, II, Comm. Pure Appl. Math. **12** (1969) 345-400 and 479-530
[6] Tanaka, H. and M. Hasegawa, Stochastic differential equations, Vol. 19 of Seminar on Probability (in Japanese) 1964.