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ON THE UNIQUENESS OF SOLUTIONS TO MARTINGALE PROBLEMS FOR DIFFUSION OPERATORS WITH PROGRESSIVELY MEASURABLE RANDOM COEFFICIENTS

MASAAKI TSUCHIYA*

Dedicated to the memory of Professor Hiroshi Kunita

Abstract. The uniqueness of solutions to martingale problems for diffusion operators with progressively measurable coefficients is studied and a uniqueness result is obtained: the uniqueness holds under the conditions of the boundedness and uniform ellipticity for the coefficients of the diffusion operators and under an additional condition for the diffusion coefficients. Construction of appropriate approximation consisting of simple functions to the diffusion coefficients plays a key role; the additional condition is used to ensure the simpleness and then the uniqueness follows from the result in the case of diffusion operators with simple type coefficients, which is due to Stroock and Varadhan.

1. Introduction

In investigations of weak solutions to stochastic differential equations based on Itô's stochastic integrals, the approach solving the corresponding martingale problems provides a powerful one, which is initiated by Stroock and Varadhan [10] (see also [11]); especially, their result on the uniqueness of solutions is notable. Although they have considered the existence of solutions to martingale problems for diffusion operators with progressively measurable coefficients defined on the space of continuous sample paths with values in a Euclidean space, they mainly investigate the uniqueness of solutions in the case of ordinary diffusion operators; that is, their coefficients are Borel measurable functions defined on a time-space.

Let Ω be the space $C([0,\infty) \to \mathbb{R}^d)$ of $\mathbb{R}^d$–valued continuous sample paths ω, $(\mathcal{M}_t)_{t\geq 0}$ the filtration generated by the coordinate process $\{x(t)\}_{t\geq 0}$ (i.e., $x(t,\omega) := \omega(t)$) defined on Ω and $\mathbb{S}^d_+$ the space of symmetric nonnegative definite $d\times d$–matrices. Take an $\mathbb{S}^d_+$–valued and $\mathbb{R}^d$–valued progressively measurable functions $a(t,\omega) \equiv (a^{ij}(t,\omega))_{1\leq i,j\leq d}$ and $b(t,\omega) \equiv (b^i(t,\omega))_{1\leq i\leq d}$ on $[0,\infty)\times\Omega$, respectively, where the progressive measurability is defined with respect to the filtration $(\mathcal{M}_t)_{t\geq 0}$. For the functions a and b, define the diffusion operator $L_t \equiv L_t(a,b) :=$

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$\frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,\omega)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(t,\omega)\frac{\partial}{\partial x_i}$. This paper concerns the uniqueness of solutions to the martingale problem for the diffusion operator $L_t(a,b)$ (say briefly, the $L_t(a,b)$*–martingale problem*). It is equivalent to discuss the uniqueness of the probability laws of weak solutions to the corresponding d–dimensional stochastic differential equation (see [4], Chap. II, §7; [9], p.160, Theorem (20.1); [11], Theorem 4.5.1, Theorem 4.5.2) .

The uniqueness result, Theorem 3.6, is obtained under the conditions of the boundedness of the coefficients a, b and of the uniform ellipticity of the diffusion operator $L_t(a,b)$ (that is, the uniformly positive definiteness of the diffusion matrix a), which is given as Condition 2.1, and under an additional condition for the diffusion matrix, which is given as Assumption 3.2.

Our approach to obtain the uniqueness result is to verify that any solution to the $L_t(a,0)$–martingale problem can be approximated by solutions to martingale problems for diffusion operators with progressively measurable *simple* type coefficients (the simpleness is defined in the beginning of §3); the verification is composed of three steps. The first step is based on the argument used in the proof for Theorem 7.1.4 of [11]: it gives an approximation of a given solution to the $L_t(a,0)$–martingale problem by the laws of the Itô processes with covariances $a_{\varepsilon,n}$ ($\varepsilon > 0, n \in \mathbb{N}$) which are provided by mollifier approximations for a and then by their discretization. The second step is to make some modification $\tilde{a}_{\varepsilon,n}$ of $a_{\varepsilon,n}$ in such a way that the laws are solutions to the $L_t(\tilde{a}_{\varepsilon,n},0)$–martingale problems. The third step is to ensure that the modifications $\tilde{a}_{\varepsilon,n}$ are simple functions under the additional condition. Then, by applying the uniqueness result in the case of diffusion operators with simple type coefficients (see Lemma 6.1.5 of [11]; the necessary part used in this paper is referred as Lemma 4.1 in Appendix), we have the uniqueness result.

Finally, we note that, in the first and third steps mentioned above, the theory of stochastic integrals with respect to martingales due to Kunita and Watanabe [6] and others plays a key role.

2. Diffusion Operator With Progressively Measurable Coefficients and the Associated Martingale Problem

Following Stroock and Varadhan [11], we provide the framework of martingale problems for diffusion operators.

We begin with introducing basic notations. Let $\Omega := C([0,\infty) \to \mathbb{R}^d)$ the space of $\mathbb{R}^d$–valued continuous sample paths ω defined on $\mathbb{R}_+ = [0,\infty)$; Ω equipped with the locally uniform convergence topology is a Polish space. The process $\{x(t)\}_{t\geq 0}$ defined by $x(t,\omega) := \omega(t)$ for $\omega \in \Omega$ is called the coordinate or canonical process. Hence, a generic element ω of Ω is denoted by $x(\cdot)$ sometimes. For a given $\omega \in \Omega$ and $s \geq 0$, define the stopped path ω_s of ω at s by $\omega_s(\cdot) := \omega(\cdot \wedge s)$, that is, $x(\cdot,\omega_s) = x(\cdot \wedge s, \omega)$. Consider σ-fields generated by the coordinate process: let $\mathcal{M} := \sigma\{x(t); 0 \leq t < \infty\}$ and $\mathcal{M}_t^s := \sigma\{x(r); s \leq r \leq t\}$ $(0 \leq s \leq t \leq \infty)$; $\mathcal{M}_t^0$ and $\mathcal{M}_\infty^s$ are simply written as $\mathcal{M}_t$ and $\mathcal{M}^s$, respectively. Note that $\mathcal{M} = \mathcal{B}(\Omega)$ (the Borel field of Ω) (see [1]). Finally, $\mathcal{P}(\Omega)$ stands for the space of probability measures on $(\Omega, \mathcal{M})$ with the Proholov metric. We also use the space of continuous sample paths on a compact time interval. For $T > 0$, let $\Omega_T := C([0,T]) \to \mathbb{R}^d)$

the space of $\mathbb{R}^d$–valued continuous sample paths defined on $[0,T]$. Then, $\mathcal{M}_{(T)}$ the σ–field on Ω_T and $\mathcal{P}(\Omega_T)$ the space of probability measures on $(\Omega_T, \mathcal{M}_{(T)})$ with the Prohorov metric are defined similarly (see [1] for details).

Let $a(t,\omega) \equiv (a^{ij}(t,\omega))_{1\le i,j\le d}$ and $b(t,\omega) \equiv (b^i(t,\omega))_{1\le i\le d}$ be an $\mathbb{S}^d_+$–valued and $\mathbb{R}^d$–valued progressively measurable functions on $[0,\infty)\times\Omega$ with respect to the filtration $(\mathcal{M}_t)_{t\ge 0}$, respectively. Then, $L_t = L_t(a,b)$ denotes the diffusion operator with coefficients a, b, as before:

$$L_t(a,b) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,\omega)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(t,\omega)\frac{\partial}{\partial x_i}.$$

For the diffusion operator $L_t = L_t(a,b)$, we consider the following condition on the coefficients:

Condition 2.1. (i) (Boundedness) The coefficients $a(t,\omega)$ and $b(t,\omega)$ are bounded, that is,

$$\sup_{(t,\,\omega)\in[0,\infty)\times\Omega} \|a(t,\omega)\| < \infty, \qquad \sup_{(t,\,\omega)\in[0,\infty)\times\Omega} |b(t,\omega)| < \infty.$$

(ii) (Uniform ellipticity) There are constants λ and Λ such that $0<\lambda\le\Lambda<\infty$ and

$$\lambda|\theta|^2 \le a(t,\omega)\theta\bullet\theta \le \Lambda|\theta|^2 \quad \text{for all } (t,\omega)\in[0,\infty)\times\Omega,\ \theta\in\mathbb{R}^d,$$

here the symbol "$\bullet$" indicates the inner product in $\mathbb{R}^d$.

Although martingale problems are treated on a general measurable space by Stroock and Varadhan, we restrict the space to the path space $(\Omega, \mathcal{M}; (\mathcal{M}_t)_{t\ge 0})$ with filtration.

To state the martingale problem for the diffusion operator $L_t(a,b)$, we recall the following equivalence (a part of Theorem 4.2.1 of [11]): In the following, we assume that Condition 2.1 (i) holds. Let $\{\xi(t)\}_{t\ge s}$ $(s\ge 0)$ be an $\mathbb{R}^d$–valued progressively measurable right continuous process on $(\Omega, \mathcal{M}; (\mathcal{M}_t)_{t\ge s})$ and suppose that, for a probability measure P on $(\Omega,\, \mathcal{M})$, the process is P–almost surely continuous. Then for the probability measure P, the following conditions are equivalent:

(i) For any $\theta\in\mathbb{R}^d$, $\left\{\exp\left(i\theta\bullet(\xi(t)-\xi(s)) - \int_s^t i\theta\bullet b(r)dr + \int_s^t \theta\bullet a(r)\theta dr\right)\right\}_{t\ge s}$ is a martingale on $(\Omega,\, \mathcal{M}, P; (\mathcal{M}_t)_{t\ge s})$.

(ii) For any $f\in C_b^2(\mathbb{R}^d)$, $\left\{f(\xi(t)) - f(\xi(s)) - \int_s^t L_r f(\xi(r))dr\right\}_{t\ge s}$ is a martingale on $(\Omega,\, \mathcal{M}, P; (\mathcal{M}_t)_{t\ge s})$.

(iii) For any $f\in C_b^{1,2}(\mathbb{R}_+\times\mathbb{R}^d)$, $\left\{f(t,\xi(t)) - f(s,\xi(s)) - \int_s^t \hat{L}_r f(r,\xi(r))dr\right\}_{t\ge s}$ is a martingale on $(\Omega,\, \mathcal{M}, P; (\mathcal{M}_t)_{t\ge s})$. Here, $\hat{L}_r := \frac{\partial}{\partial r} + L_r$.

If one of the equivalent conditions holds, $\{\xi(t)\}_{t\ge s}$ is called an Itô process with covariance a and drift b, and it is denoted by

$$\xi(\cdot) \sim \mathcal{I}^s(a,b) \text{ on } (\Omega,\, \mathcal{M}, P; (\mathcal{M}_t)_{t\ge s}).$$

Definition 2.2. ($L_t(a,b)$–martingale problem) A probability measure P on $(\Omega, \mathcal{M})$ is called a solution to the *martingale problem for the diffusion operator* $L_t(a,b)$ (say briefly, the $L_t(a,b)$*–martingale problem*) *starting from* $(s,x) \in [0,\infty) \times \mathbb{R}^d$, if P fulfills the following conditions

(i) for given s and x,

$$P(x(r) = x \text{ for all } r \in [0,s]) = 1;$$

(ii) $x(\cdot) \sim \mathcal{I}^s(a,b)$ on $(\Omega, \mathcal{M}, P; (\mathcal{M}_t)_{t \geq s})$.

3. Uniqueness Result for the $L_t(a,b)$–martingale Problem

To obtain the uniqueness result, we need to construct an appropriate approximation for an arbitrarily given solution to the $L_t(a,0)$–martingale problem. The first step of such approximation is to make an approximation of a by simple type progressive measurable functions. Here, a progressively measurable function $\xi : [0,\infty) \times \Omega \to \mathbb{R}^M$ is said to be *simple*, if for a subdivision $\Delta : 0 = t_0 < t_1 < \cdots \nearrow \infty$ of $[0,\infty)$ and bounded $\mathbb{R}^M$–valued $\mathcal{M}_{t_j}$–measurable functions ξ_j $(j = 0,1,\dots)$ it is given as the form

$$\xi(t,\omega) = \sum_{j=0}^{\infty} \xi_j(\omega)\, \mathbf{1}_{[t_j,t_{j+1})}(t) \quad (t \geq 0,\ \omega \in \Omega).$$

In the following, we need to consider a sequence of subdivisions $\{\Delta_n\}_{n=1}^{\infty}$ such that Δ_{n+1} is a refinement of Δ_n and the modulus of Δ_n goes to zero as $n \to \infty$: for the notational simplicity, we restrict it to the case of $\Delta_n = \{k2^{-n}\}_{k=0}^{\infty}$ $(n = 1,2,\dots)$.

In what follows, $a = a(t,\omega)$ is an arbitrarily fixed $\mathbb{S}_+^d$–valued progressively measurable function defined on $[0,\infty) \times \Omega$ satisfying Condition 2.1 (i), (ii). We will provide a family $\{\tilde{a}_{\varepsilon,n}\}_{\varepsilon>0,n\in\mathbb{N}}$ of simple functions such that any solution to the $L_t(a,0)$–martingale problem can be approximated by solutions to the $L_t(\tilde{a}_{\varepsilon,n},0)$–martingale problems as $n \to \infty$ and then $\varepsilon \searrow 0$; which plays a key role for the uniqueness result. Actually, the simpleness will be verified under Assumption 3.2 below.

We first consider approximation to an arbitrarily given solution to the $L_t(a,0)$–martingale problem; to this purpose, we recall the argument used in the proof of Theorem 7.1.4 of [11]. Take a function $\rho \in C_c^{\infty}(\mathbb{R}^1)$ with $\mathrm{supp}(\rho) \subset [0,1]$, $\rho \geq 0$ and $\int_{\mathbb{R}^1} \rho(t)dt = 1$. For the function, define a mollifier $\{\rho_\varepsilon\}_{\varepsilon>0}$ by $\rho_\varepsilon(t) := 1/\varepsilon\rho(t/\varepsilon)$ $(t \in \mathbb{R}^1)$ for each $\varepsilon > 0$. For each $\omega \in \Omega$, consider the mollifier approximation of $a(\cdot,\omega)$ extended as $a(t,\omega) = a(0,\omega)$ for $t < 0$:

$$a_\varepsilon(t,\omega) := \int_{-\infty}^{\infty} \rho_\varepsilon(t-r)a(r,\omega)dr \quad (t \geq 0,\ \omega \in \Omega).$$

Then, $a_\varepsilon : [0,\infty) \times \Omega \to \mathbb{S}_+^d$ is progressively measurable. In addition, for each integer $n \geq 1$, let $a_{\varepsilon,n}(t,\omega) := a_\varepsilon([2^n t]/2^n,\omega)$ $(t \geq 0, \omega \in \Omega)$. Accordingly, for each $\varepsilon > 0$ and $n \geq 1$, $a_{\varepsilon,n} : [0,\infty) \times \Omega \to \mathbb{S}_+^d$ is progressively measurable and simple. Hence $a_{\varepsilon,n}(t)$ is $\mathcal{M}_{[2^n t]/2^n}$–measurable for $t \geq 0$ and

$$\lambda|\theta|^2 \leq a_{\varepsilon,n}(t,\omega)\theta \bullet \theta \leq \Lambda|\theta|^2 \quad \text{for all } (t,\omega) \in [0,\infty) \times \Omega,\ \theta \in \mathbb{R}^d.$$

In the following, we treat an arbitrarily fixed solution P to the $L_t(a,0)$–martingale problem starting from (s,x); then for each $T>0$

$$E^P\left[\int_0^T \|a_{\varepsilon,\,n}(t)-a(t)\|^2 dt\right]\to 0 \tag{3.1}$$

as $n\to\infty$ and then $\varepsilon\searrow 0$. Let

$$\beta(t):=\int_s^t a^{-1/2}(r)dx(r)\quad (t\geq s). \tag{3.2}$$

Then $\{\beta(t)\}_{t\geq s}$ is a d–dimensional Brownian motion on $(\Omega,\mathcal{M},P;(\mathcal{M}_t)_{t\geq s})$ and further

$$x(t)=x(s)+\int_s^{t\vee s} a^{1/2}(r)d\beta(r)\quad (t\geq 0),\ P\text{-a.s.} \tag{3.3}$$

Here $a^{-1/2}$ denotes the inverse of $a^{1/2}$ and we refer to §5.2 of [11] for the measurability of $a^{1/2}$ and to §4.3 of [11] for the martingale property, the existence of moment, the quadratic variation and stochastic integrals with respect to $\{x(t)\}_{t\geq 0}$. Next define

$$\xi_{\varepsilon,\,n}(t):=x(s)+\int_s^{t\vee s} a_{\varepsilon,\,n}^{1/2}(r)d\beta(r)\quad (t\geq 0); \tag{3.4}$$

then $\{\xi_{\varepsilon,\,n}(t)\}_{t\geq 0}$ is a progressively measurable continuous process on $(\Omega,\mathcal{M},P;$ $(\mathcal{M}_t)_{t\geq 0})$ and $\xi_{\varepsilon,\,n}(\cdot)\sim\mathcal{I}^s(a_{\varepsilon,\,n},0)$ on $(\Omega,\,\mathcal{M},P;(\mathcal{M}_t)_{t\geq s})$. Using Doob's inequality and (3.1), we have, for every $T>0$,

$$E^P\left[\sup_{0\leq t\leq T}|\xi_{\varepsilon,\,n}(t)-x(t)|^2\right]\to 0\quad (n\to\infty \text{ and then } \varepsilon\searrow 0); \tag{3.5}$$

so that the probability law $P_{\varepsilon,\,n}$ on $(\Omega,\mathcal{M})$ induced by the process $\{\xi_{\varepsilon,\,n}(t)\}_{t\geq 0}$ converges to P in $\mathcal{P}(\Omega)$ as $n\to\infty$ and then $\varepsilon\searrow 0$. Because, for each $T>0$, letting $r_T:\Omega\to\Omega_T$ the restriction map (i.e., $r_T\omega:=\omega|_{[0,T]}$), we have $P_{\varepsilon,\,n}r_T^{-1}\to Pr_T^{-1}$ in $\mathcal{P}(\Omega_T)$ as $n\to\infty$ then $\varepsilon\searrow 0$ by (3.5). Therefore, it holds $P_{\varepsilon,\,n}\to P$ in $\mathcal{P}(\Omega)$ as $n\to\infty$ and then $\varepsilon\searrow 0$ (use the continuous sample space version of Theorem 16.7 of [1]).

In general, $P_{\varepsilon,\,n}$ is not a solution to the $L_t(a_{\varepsilon,n},0)$–martingale problem. Hence we modify the family $\{a_{\varepsilon,n}\}_{\varepsilon,n}$ to obtain the required family $\{\tilde{a}_{\varepsilon,n}\}_{\varepsilon,n}$ of progressively measurable functions as follows: For each $\varepsilon>0$ and $n\in\mathbb{N}$, let $\widetilde{\mathcal{M}}_u^t\equiv\widetilde{\mathcal{M}}_u^t(\xi_{\varepsilon,n}):=\sigma(\xi_{\varepsilon,n}(r);t\leq r\leq u)$ $(0\leq t\leq u)$ and set $\widetilde{\mathcal{M}}_t=\widetilde{\mathcal{M}}_t^0$. Then, $\widetilde{\mathcal{M}}_u^t\subset\mathcal{M}_u^t$. Consider the conditional expectation $E^P[a_{\varepsilon,n}(r,\cdot)\mid\widetilde{\mathcal{M}}_r]$. Define the map $\Psi_{\varepsilon,n}:\Omega\longrightarrow\Omega$ as $\Psi_{\varepsilon,n}(\omega):=\xi_{\varepsilon,n}(\cdot,\omega)\in\Omega$. Noting $\widetilde{\mathcal{M}}_r=\Psi_{\varepsilon,n}^{-1}(\mathcal{M}_r)$ for each $r\geq 0$ and applying Theorem 4.2.8 in [3], we see that there is an $\mathcal{M}_r$–measurable $\mathbb{S}_+^d$–valued function $\tilde{a}_{\varepsilon,n}(r,\cdot)$ on $(\Omega,\mathcal{M})$ such that $E^P[a_{\varepsilon,n}(r,\cdot)\mid\widetilde{\mathcal{M}}_r]=\tilde{a}_{\varepsilon,n}(r,\Psi_{\varepsilon,n}(\cdot))\equiv\tilde{a}_{\varepsilon,n}(r,\xi_{\varepsilon,n}(\cdot))$ P–a.s. Then, by Theorem 97 and Remark 98 (a) in Chap. IV of [2] (see also [9], p. 122), $\tilde{a}_{\varepsilon,n}$ is progressively measurable. Moreover, we see the following.

Lemma 3.1. *Let $a = a(t,\omega)$ be the $\mathbb{S}^d_+$–valued progressively measurable function satisfying Condition 2.1. Then, for each $\varepsilon > 0$ and $n \in \mathbb{N}$, the probability measure $P_{\varepsilon,\, n}$ is a solution to the $L_t(\tilde{a}_{\varepsilon,n}, 0)$–martingale problem starting from (s,x).*

Proof. For each $f \in C^2_c(\mathbb{R}^d)$, set

$$M_{\varepsilon,n;f}(t) := f(\xi_{\varepsilon,n}(t)) - f(\xi_{\varepsilon,n}(s)) - \int_s^t L_r f(\xi_{\varepsilon,n}(r))dr,$$
$$\tilde{M}_{\varepsilon,n;f}(t) := f(\xi_{\varepsilon,n}(t)) - f(\xi_{\varepsilon,n}(s)) - \int_s^t \tilde{L}_{\varepsilon,n;r} f(\xi_{\varepsilon,n}(r))dr;$$

where $L_r = L_r(a_{\varepsilon,n}, 0)$ and $\tilde{L}_{\varepsilon,n;r} := \frac{1}{2}\sum_{i,j=1}^d \tilde{a}^{ij}_{\varepsilon,n}(r, \xi_{\varepsilon,n}(\cdot))\frac{\partial^2}{\partial x_i \partial x_j}$. Noting that $\xi_{\varepsilon,\, n}(\cdot) \sim \mathcal{I}^s(a_{\varepsilon,\, n}, 0)$ on $(\Omega,\, \mathcal{M}, P; (\mathcal{M}_t)_{t\ge s})$ and $\widetilde{\mathcal{M}}_t \subset \mathcal{M}_t$, we have for $s \le t_1 < t_2$ and a bounded $\widetilde{\mathcal{M}}_{t_1}$–measurable function G

$$\begin{aligned}
0 &= E^P[(M_{\varepsilon,n;f}(t_2) - M_{\varepsilon,n;f}(t_1))G] \\
&= E^P[(f(\xi_{\varepsilon,n}(t_2)) - f(\xi_{\varepsilon,n}(t_1)))G] - \int_{t_1}^{t_2} E^P[L_r f(\xi_{\varepsilon,n}(r))G]dr \\
&= E^P[(f(\xi_{\varepsilon,n}(t_2)) - f(\xi_{\varepsilon,n}(t_1)))G] - \int_{t_1}^{t_2} E^P[E^P[L_r f(\xi_{\varepsilon,n}(r))G \mid \widetilde{\mathcal{M}}_r]]dr \\
&= E^P[(f(\xi_{\varepsilon,n}(t_2)) - f(\xi_{\varepsilon,n}(t_1)))G] - \int_{t_1}^{t_2} E^P[\tilde{L}_{\varepsilon,n;r} f(\xi_{\varepsilon,n}(r))G]dr \\
&= E^P[(\tilde{M}_{\varepsilon,n;f}(t_2) - \tilde{M}_{\varepsilon,n;f}(t_1))G];
\end{aligned}$$

hence $\{\tilde{M}_{\varepsilon,n;f}(t)\}_{t\ge s}$ is a martingale on $(\Omega,\, \mathcal{M}, P; (\widetilde{\mathcal{M}}_t)_{t\ge s})$.
Thus $x(\cdot) \sim \mathcal{I}^s(\tilde{a}_{\varepsilon,\, n}, 0)$ on $(\Omega,\, \mathcal{M}, P_{\varepsilon,n}; (\mathcal{M}_t)_{t\ge s})$. Therefore, combining with the fact $P(\xi_{\varepsilon,n}(r) = x(s), 0 \le r \le s) = 1$, we see that $P_{\varepsilon,n}$ is a solution to the $L_t(\tilde{a}_{\varepsilon,n}, 0)$–martingale problem starting from (s,x). Consequently, the lemma is proved. □

The function $\tilde{a}_{\varepsilon,n}$ is not simple in general; however we need the simpleness to apply Lemma 4.1 in Appendix; the lemma is used to show the uniqueness of solutions to the $L_t(\tilde{a}_{\varepsilon,n}, 0)$–martingale problem. Thus, we impose the following assumption on the diffusion matrix a to ensure the simpleness.

Assumption 3.2. There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is an $n = n(\varepsilon) \in \mathbb{N}$ satisfying the condition: if for $\omega,\ \omega' \in \Omega$ the Lebesgue measure of the set $\{t \ge 0 : a(t,\omega) \ne a(t,\omega')\}$ is positive, it holds $a_{\varepsilon,n}(\cdot,\omega) \ne a_{\varepsilon,n}(\cdot,\omega')$, that is, $a_\varepsilon(k2^{-n},\omega) \ne a_\varepsilon(k2^{-n},\omega')$ for some $k \in \mathbb{Z}_+$.

Since $\varDelta_{n+1} = \{k2^{-(n+1)}\}_{k=0}^\infty$ is a refinement of $\varDelta_n = \{k2^{-n}\}_{k=0}^\infty$, for each $\varepsilon < \varepsilon_0$ and $n \ge n(\varepsilon)$, it holds $a_{\varepsilon,n}(\cdot,\omega) \ne a_{\varepsilon,n}(\cdot,\omega')$ for $\omega,\ \omega' \in \Omega$ in Assumption 3.2.

When the set $\{t \ge 0 : a(t,\omega) \ne a(t,\omega')\}$ for $\omega,\ \omega' \in \Omega$ has a positive Lebesgue measure, ε_0 and $n(\varepsilon)$ mentioned above always exist, if they are allowed to depend on such ω and ω'. In Assumption 3.2, we suppose that ε_0 and $n(\varepsilon)$ can be taken to be independent of ω and ω'.

The following is an example of a satisfying Condition 2.1 and Assumption 3.2, here $\mathbb{S}^d_{++}$ denotes the set of positive definite matrices of $\mathbb{S}^d_+$ and $\|\omega\|_t := \max_{0\le r\le t}|\omega(r)|$.

Example 3.3. Given constant matrices $A,\ B \in \mathbb{S}^d_{++}$ with $A \neq B$ and a constant $K > 0$, let

$$a(t,\omega) = A\,\mathbf{1}_{\Xi}(t,\omega) + B\,\mathbf{1}_{\Xi^c}(t,\omega),$$

where $\Xi := \{(t,\omega) : \|\omega\|_t \le K\}$.

Lemma 3.4. *Suppose that Condition 2.1 and Assumption 3.2 are fulfilled. Then there exits a subset $\Omega_0 \in \mathcal{M}$ with $P(\Omega_0) = 1$ such that for $\varepsilon < \varepsilon_0$ and $n \ge n(\varepsilon)$, the restriction of $\Psi_{\varepsilon,n}$ into Ω_0 is a one-to-one measurable map. Hence $\Psi_{\varepsilon,n} : \Omega_0 \longrightarrow \Psi_{\varepsilon,n}(\Omega_0)$ is a Borel isomorphism.*

Proof. The measurability of the map $\Psi_{\varepsilon,n}$ is easily seen. In the following, we will show the one-to-one property. For the sake of simplicity, we consider the case of $s = 0$. The process defined by the Riemann sum with coefficients $a^{1/2}_{\eta,m}$

$$\left\{\sum_{k=0}^{\infty} a^{1/2}_{\eta,m}(k2^{-m})(\beta((k+1)2^{-m}\wedge t) - \beta(k2^{-m}\wedge t))\right\}_{t\ge 0}$$

converges to the process defined by the stochastic integral of the right hand side of (3.3) in probability uniformly on each compact interval as $m \to \infty$ and then $\eta \to 0$ Furthermore, the process

$$\left\{\sum_{k=0}^{\infty}(\beta((k+1)2^{-m}\wedge t) - \beta(k2^{-m}\wedge t))(\beta((k+1)2^{-m}\wedge t) - \beta(k2^{-m}\wedge t))^*\right\}_{t\ge 0}$$

also converges to the quadratic variation process $\{\langle\beta\rangle(t)\}_{t\ge 0}$ in probability uniformly on each compact interval as $m \to \infty$, where the superscript " $*$ " indicates the transposed operation for the matrices (see [11], §4.3; [5], Chap. I, §4; [8], Chap. IV, §1). That is, they converge to the limits in probability with respect to the metrics of $C([0,\infty) \to \mathbb{R}^d)$ and $C([0,\infty) \to \mathbb{S}^d_+)$, respectively. Therefore, they converge to the limits almost surely with respect to the metrics, respectively, via some subsequences $\{\eta_j\}_{j=1}^{\infty}$ with $\eta_j \searrow 0$ and $\{m_j\}_{j=1}^{\infty} (\subset \{m\}_{m=1}^{\infty})$ with $m_j \nearrow \infty$ as $j \to \infty$; that is, there exists a subset $\Omega_0 \in \mathcal{M}$ with full measure such that for each element of Ω_0 they converge to the limits with respect to the metrics via the subsequences, respectively. We note that if $\omega \in \Omega_0$, then $\omega_t \in \Omega_0$ and $\xi_{\varepsilon,n}(\cdot\,,\omega_t) = \xi_{\varepsilon,n}(\cdot \wedge t,\omega)$ for $t \ge 0$. For $\omega,\ \omega' \in \Omega_0$ with $\mathcal{L}^1(\{t \ge 0 : a(t,\omega) \neq a(t,,\omega')\}) = 0$, we see that $\beta(\cdot\,,\omega) = \beta(\cdot\,,\omega')$ (that is, $\xi_{\varepsilon,n}(\cdot\,,\omega) = \xi_{\varepsilon,n}(\cdot\,,\omega')$) implies $\omega = \omega'$ by (3.3) and (3.4), where $\mathcal{L}^1$ indicates the Lebesgue measure on $\mathbb{R}^1$. For $\omega,\ \omega' \in \Omega_0$ with $\mathcal{L}^1(\{t \ge 0 : a(t,\omega) \neq a(t,,\omega')\}) > 0$ (hence $\omega \neq \omega'$), it follows from Assumption 3.2 that $\xi_{\varepsilon,n}(\cdot\,,\omega) \neq \xi_{\varepsilon,n}(\cdot\,,\omega')$. Indeed, since $\langle\xi_{\varepsilon,n}\rangle(dt,\omega) = a_{\varepsilon,n}(t,\omega)dt$ and $\langle\xi_{\varepsilon,n}\rangle(dt,\omega') = a_{\varepsilon,n}(t,\omega')dt$, it holds $\langle\xi_{\varepsilon,n}\rangle(dt,\omega) \neq \langle\xi_{\varepsilon,n}\rangle(dt,\omega')$; hence $\xi_{\varepsilon,n}(\cdot\,,\omega) \neq \xi_{\varepsilon,n}(\cdot\,,\omega')$. Consequently, the restriction of $\Psi_{\varepsilon,n}$ into Ω_0 is a one-to-one measurable map. Therefore, noting $\mathcal{M} = \mathcal{B}(\Omega)$, the last assertion of the lemma follows immediately from the Kuratowski theorem (see Therem 3.9, Corollary 3.3 in [7]). $\square$

Remark 3.5. Under Condition 2.1 and Assumption 3.2, for $\varepsilon < \varepsilon_0$ and $n \geq n(\varepsilon)$, $\tilde{a}_{\varepsilon,n}$ is simple and further the family $\{\tilde{a}_{\varepsilon,n}\}$ consisting of such simple functions can be taken to be the same one for mutually absolutely continuous solutions to the $L_t(a,0)$–martingale problem. Because, in this case, $\tilde{a}_{\varepsilon,n}(t,\cdot) = a_{\varepsilon,n}(t,\Psi^{-1}_{\varepsilon,n}(\cdot))$ almost surely and further the Brownian motion $\{\beta(t)\}$ and the process $\{\xi_{\varepsilon,n}(t)\}$ are defined as the same ones for such solutions.

We have the following uniqueness result.

Theorem 3.6. *Suppose that Condition 2.1 and Assumption 3.2 are fulfilled. Then, the uniqueness of solutions to the $L_t(a,b)$–martingale problem holds; that is, for any $(s,x) \in [0,\infty)\times\mathbb{R}^d$, there is at most one solution to the $L_t(a,b)$–martingale problem starting from (s,x).*

Proof. Let $\mathcal{C}(s,x)$ be the set of solutions to the $L_t(a,b)$–martingale problem starting from (s,x). In the following, we suppose $\mathcal{C}(s,x) \neq \emptyset$. Since $\mathcal{C}(s,x)$ is a convex set, to show the uniqueness, it is enough to verify the equality $P = Q$ for mutually equivalent $P,\ Q \in \mathcal{C}(s,x)$. By the Cameron-Martin-Girsanov formula (see Lemma 6.4.1 in [11]), it is enough to examine the uniqueness in the case of $b = 0$, that is, the $L_t(a,0)$–martingale problem. We apply Lemma 3.1 to mutually equivalent solutions P and Q by noting Remark 3.5. Then for the approximating family $\{\tilde{a}_{\varepsilon,n}\}$, take approximating families $\{P_{\varepsilon,n}\}$ of P and $\{Q_{\varepsilon,n}\}$ of Q, respectively. Since, for each $\varepsilon < \varepsilon_0$ and $n \geq n(\varepsilon)$, $P_{\varepsilon,n}$ and $Q_{\varepsilon,n}$ are solutions to the $L_t(\tilde{a}_{\varepsilon,n},0)$–martingale problem starting from (s,x), $P_{\varepsilon,n} = Q_{\varepsilon,n}$ by Lemma 4.1 in Appendix. Because of $P_{\varepsilon,n} \to P$ and $Q_{\varepsilon,n} \to Q$ in $\mathcal{P}(\Omega)$ as $n \to \infty$ and then $\varepsilon \searrow 0$, we have $P = Q$; that is, the theorem is proved. □

4. Appendix

In Appendix, we recall some results of [11] connected with the result of this paper.

We first recall Lemma 6.1.1 of [11]. The lemma is stated as follows: Let P be a probability measure on $(\Omega,\mathcal{M}^s)$ and suppose that $P(x(s) = \eta(s)) = 1$ for some $\eta \in \Omega$. Then there exists a unique probability measure on $(\Omega,\mathcal{M})$, which is denoted by $\delta_\eta \otimes_s P$, satisfying the following:

$$\delta_\eta \otimes_s P(A \cap B) = \delta_\eta(A)P(B) \quad \text{for } A \in \mathcal{M}_s,\ B \in \mathcal{M}^s,$$

that is,

$$\begin{aligned} \delta_\eta \otimes_s P &= \delta_\eta \quad \text{on } \mathcal{M}_s, \\ \delta_\eta \otimes_s P &= P \quad \text{on } \mathcal{M}^s. \end{aligned}$$

We further need a fine property for conditional probabilities with respect to sub σ–fields related to the filtration $(\mathcal{M}_t)_{t\geq 0}$; hence, following [11], we recall the notion of a regular conditional probability distribution. Let P be a probability measure on $(\Omega,\mathcal{M})$ and τ a stopping time relative to the filtration $(\mathcal{M}_t)_{t\geq 0}$. Then, a family $\{Q_\omega\}_{\omega\in\Omega}$ of probability measures on $(\Omega,\mathcal{M})$ is called a *regular conditional probability distribution* (abbreviated as a *r.c.p.d.*) *of P given $\mathcal{M}_\tau$*, if it fulfills the following conditions:

(i) the function $\omega \longrightarrow Q_\omega(A)$ is $\mathcal{M}_\tau$–measurable for each $A \in \mathcal{M}$;

(ii) $Q_\omega(\tau(\cdot) = \tau(\omega),\ x(s,\cdot) = x(s,\omega),\ 0 \le s \le \tau(\omega)) = 1$, P–a.e. $\omega \in \Omega$;
(iii) for $A \in \mathcal{M}_\tau$ and $B \in \mathcal{M}$,

$$P(A \cap B) = \int_A Q_\omega(B) P(d\omega).$$

The family $\{Q_\omega\}_{\omega\in\Omega}$ gives a disintegration of the probability measure P. A converse result is obtained by Theorem 6.1.2 of [11] as follows: For a given probability measure P on $(\Omega, \mathcal{M})$, finite stopping time τ relative to the filtration $(\mathcal{M}_t)_{t\ge 0}$ and family $\{Q_\omega\}_{\omega\in\Omega}$ of probability measures on $(\Omega, \mathcal{M})$ with the measurability condition as in (i) above and the condition $Q_\omega(x(\tau(\omega), \cdot) = x(\tau(\omega), \omega)) = 1$, P–a.e. $\omega \in \Omega$ instead of (ii) above, there exists a unique probability measure R on $(\Omega, \mathcal{M})$ such that $R = P$ on $(\Omega, \mathcal{M}_\tau)$ and $\{\delta_\omega \otimes_{\tau(\omega)} Q_\omega\}_{\omega\in\Omega}$ is a *r.c.p.d.* of R given $\mathcal{M}_\tau$. Then the probability measure R is denoted by $P \otimes_{\tau(\cdot)} Q_\cdot$. Moreover, the theorem provides a characterization in terms of P, τ and $\{Q_\omega\}$ for which a progressively measurable right continuous process $\{\theta(t)\}_{t\ge s}$ on $(\Omega, \mathcal{M}, P \otimes_{\tau(\cdot)} Q_\cdot; (\mathcal{M}_t)_{t\ge s})$ is a martingale. The theorem plays a key role in the proof of Lemma 4.1 below.

Although the next result is a part of Lemma 6.1.5 of [11], we state it in an adapted form to this paper and recall the proof with adding some supplement on measurability for using it in the proof of the main theorem.

Lemma 4.1. *Assume that the coefficients of the diffusion operator $L_t(a, b)$ are simple: that is, for a subdivision $\Delta : 0 = t_0 < t_1 < \cdots \nearrow \infty$, it holds that $a(t) = a(t_j)$, $b(t) = b(t_j)$ if $t_j \le t < t_{j+1}$ $(j = 0, 1, \dots)$. Then, for each $(s, x) \in [0, \infty) \times \mathbb{R}^d$, there is at most one solution to the $L_t(a, b)$–martingale problem starting from (s, x).*

Proof. For the sake of simplicity, we consider the case of $s = 0$. For an arbitrary given solution P to the $L_t(a, b)$–martingale problem starting from $(0, x)$ and nonnegative integer k, we will show that for a *r.c.p.d.* $\{P_\omega\}_{\omega\in\Omega}$ of P given $\mathcal{M}_{t_k}$

$$P_\omega = \delta_\omega \otimes_{t_k} \mathcal{W}^{(a_k(*,\omega),\, b_k(*,\omega))}_{t_k,\, x(t_k,\omega)} \quad \text{on } \mathcal{M}_{t_{k+1}},\ P\text{–a.e. } \omega \in \Omega, \tag{4.1}$$

where, for an $\mathbb{S}^d_+$–valued and $\mathbb{R}^d$–valued bounded measurable functions α and β on $\mathbb{R}_+$, $\mathcal{W}^{(\alpha,\beta)}_{s,x} \equiv \mathcal{W}^{(\alpha(*),\,\beta(*))}_{s,x}$ denotes the unique solution to the $L_t(\alpha, \beta)$–martingale problem starting from (s, x) and $a_k(t, \omega) := a(t, \omega_{t_k})$, $b_k(t, \omega) := b(t, \omega_{t_k})$; from the equality (4.1), the uniqueness of solutions follows easily.

To show the equality (4.1), let us consider

$$Q := P \otimes_{t_{k+1}} \mathcal{W}^{(a_k(*,\cdot),\, b_k(*,\cdot))}_{t_{k+1},\, x(t_{k+1},\cdot)}.$$

Here, we applied Theorem 6.1.2 of [11] to defining Q; so that we have to verify the $\mathcal{M}_{t_{k+1}}$–measurability of $\mathcal{W}^{(a_k(*,\omega),\, b_k(*,\omega))}_{t_{k+1},\, x(t_{k+1},\omega)}$ with respect to ω, which is done later. Letting $L^{(k)}_t := L_t(a_k, b_k)$ and

$$\theta_{(k)}(t) := f(x(t)) - \int_0^t L^{(k)}_r f(x(r))dr \quad \text{for } f \in C^2_b(\mathbb{R}^d),$$

by definition, we see that $\{\theta_{(k)}(t) - \theta_{(k)}(t \wedge t_{k+1})\}_{t\ge 0}$ is a martingale on $(\Omega, \mathcal{M}, \mathcal{W}^{(a_k(*,\omega),\, b_k(*,\omega))}_{t_{k+1},\, x(t_{k+1},\omega)}; (\mathcal{M}_t)_{t\ge 0})$ for each $\omega \in \Omega$. On the other hand, since

$L_t \equiv L_t(a,b) = L_t^{(k)}$ for $0 \le t < t_{k+1}$, $\{\theta_{(k)}(t \wedge t_{k+1})\}_{t\ge 0}$ is a martingale on $(\Omega, \mathcal{M}, P; (\mathcal{M}_t)_{t\ge 0})$. Therefore, by Theorem 6.1.2 of [11], $\{\theta_{(k)}(t)\}_{t\ge 0}$ is a martingale on $(\Omega, \mathcal{M}, Q; (\mathcal{M}_t)_{t\ge 0})$, that is, $x(\cdot) \sim \mathcal{I}^0(a_k, b_k)$ on $(\Omega, \mathcal{M}, Q; (\mathcal{M}_t)_{t\ge 0})$. Denote by $\{Q_\omega\}_{\omega\in\Omega}$ a *r.c.p.d.* of Q given $\mathcal{M}_{t_k}$. Then, by Theorem 6.1.3 of [11], there is a Q–null set $N \in \mathcal{M}_{t_k}$ such that $x(\cdot) \sim \mathcal{I}^{t_k}(a_k(*,\omega), b_k(*,\omega))$ on $(\Omega, \mathcal{M}, Q_\omega; (\mathcal{M}_t)_{t\ge t_k})$ for all $\omega \notin N$. Hence, from the uniqueness of solutions to the $L_t(a_k(*,\omega), b_k(*,\omega))$–martingale problem, it follows that (if necessary, N is replaced by another null set)

$$Q_\omega = \delta_\omega \otimes_{t_k} \mathcal{W}_{t_k,\, x(t_k,\omega)}^{(a_k(*,\omega),\, b_k(*,\omega))} \quad \text{for every } \omega \notin N.$$

Since $P = Q$ on $\mathcal{M}_{t_{k+1}}$, we have

$$P_\omega = Q_\omega = \delta_\omega \otimes_{t_k} \mathcal{W}_{t_k,\, x(t_k,\omega)}^{(a_k(*,\omega),\, b_k(*,\omega))} \quad \text{on } \mathcal{M}_{t_{k+1}} \text{ for every } \omega \notin N;$$

hence we have the equality (4.1).

Finally, we verify the $\mathcal{M}_{t_{k+1}}$–measurability of $\mathcal{W}_{t_{k+1},\, x(t_{k+1},\omega)}^{(a_k(*,\omega),\, b_k(*,\omega))}$. Let $\boldsymbol{A}_k(t,\omega) := \int_0^t a_k(r,\omega)dr$. Then

$$\boldsymbol{A}_k(t,\omega) = \sum_{j=0}^{\infty} a_k(t_j,\omega)(t\wedge t_{j+1} - t\wedge t_j).$$

For $\boldsymbol{\xi} = (\xi_0, \xi_1, \dots) \in (\mathbb{S}_+^d)^{\mathbb{Z}_+}$, set

$$A(t,\boldsymbol{\xi}) := \sum_{j=0}^{\infty} \xi_j\,(t\wedge t_{j+1} - t\wedge t_j).$$

Then, $A(t,\boldsymbol{\xi})$ is a continuous function of $(t,\boldsymbol{\xi}) \in \mathbb{R}_+ \times (\mathbb{S}_+^d)^{\mathbb{Z}_+}$ and $\boldsymbol{A}_k(t,\omega) = A(t, \boldsymbol{\alpha}_k(\omega))$ with $\boldsymbol{\alpha}_k(\omega) = (a_k(t_0,\omega), a_k(t_1,\omega), \dots)$. We also consider $\boldsymbol{B}_k(t,\omega) := \int_0^t b_k(r,\omega)dr$ and define $B(t,\boldsymbol{\eta})$ for $(t,\boldsymbol{\eta}) \in \mathbb{R}_+ \times (\mathbb{R}^d)^{\mathbb{Z}_+}$ in the same way as above. Then $B(t,\boldsymbol{\eta})$ is a continuous function of $(t,\boldsymbol{\eta}) \in \mathbb{R}_+ \times (\mathbb{R}^d)^{\mathbb{Z}_+}$ and $\boldsymbol{B}_k(t,\omega) = B(t,\boldsymbol{\beta}_k(\omega))$ with $\boldsymbol{\beta}_k(\omega) = (b_k(t_0,\omega), b_k(t_1,\omega), \dots)$. For each $\boldsymbol{\xi} \in (\mathbb{S}_+^d)^{\mathbb{Z}_+}$ and $\boldsymbol{\eta} \in (\mathbb{R}^d)^{\mathbb{Z}_+}$, let

$$\alpha(t,\boldsymbol{\xi}) := \sum_{j=0}^{\infty} \xi_j\, \mathbf{1}_{[t_j,t_{j+1})}(t),$$

$$\beta(t,\boldsymbol{\eta}) := \sum_{j=0}^{\infty} \eta_j\, \mathbf{1}_{[t_j,t_{j+1})}(t).$$

By the continuity of $A(t,\boldsymbol{\xi})$ and $B(t,\boldsymbol{\eta})$ and by the uniqueness of solutions to the $L_t(\alpha(*,\boldsymbol{\xi}), \beta(*,\boldsymbol{\eta}))$–martingale problem, we see that $\mathcal{W}_{s,x}^{(\alpha(*,\boldsymbol{\xi}),\, \beta(*,\boldsymbol{\eta}))}$ is partially continuous in (s,x) and in $(\boldsymbol{\xi},\boldsymbol{\eta})$, respectively, by using the usual arguments based on the tightness and Skorohod's representation theorem. Therefore, it is measurable in $(s,x;\boldsymbol{\xi},\boldsymbol{\eta})$. Since $\mathcal{W}_{t_{k+1},\, x(t_{k+1},\omega)}^{(a_k(*,\omega),\, b_k(*,\omega))} = \mathcal{W}_{t_{k+1},\, x(t_{k+1},\omega)}^{(\alpha(*,\boldsymbol{\alpha}_k(\omega)),\, \beta(*,\boldsymbol{\beta}_k(\omega)))}$, the required measurability is verified. □

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