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ON THE UNIQUENESS OF SOME DIFFERENTIAL  
INVARIANTS:  $d$ ,  $[\cdot, \cdot]$ ,  $\nabla$ 

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**1. Introduction.** It is known that Nijenhuis's theory of natural bundles [8], [10], [12] allows us to characterize differential-geometrical (= invariant) operations and mappings as natural transformations of some liftings, or equivalently, as certain differential invariants in the sense of [6]. Moreover, this theory, combined with the prolongation theory of liftings [4], [5], is an important tool for computation of differential-geometrical operations – either by “infinitesimal”, or “direct” methods (see e.g. [7], [9]).

The purpose of this paper is to apply Nijenhuis's theory to the problem of uniqueness of the exterior derivative of forms, the Lie bracket of vector fields, and the Levi-Civita connection of metric fields.

As a motivation, consider the exterior derivative  $\omega \rightarrow d\omega$  of forms on a differential manifold  $X$ . It is known that the exterior derivative may be defined by various properties. For example (see [1]), one may require (1) if  $\omega = f$  is a function, then  $df$  is the differential of  $f$ , (2) the domain of definition of  $d\omega$  coincides with the domain of definition of  $\omega$ , (3) the mapping  $\omega \rightarrow d\omega$  is  $R$ -linear, (4) for each  $p$ -form  $\omega$  and each  $q$ -form  $\eta$ ,  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$ , and (5)  $d(d\omega) = 0$  for each  $\omega$ . There is also another definition of  $d$  (see [11]), based on its behaviour under mappings:  $d$  is the unique  $R$ -linear mapping such that for each local diffeomorphism of  $X$ ,

$$(1.1) \quad d\alpha^*\omega = \alpha^*d\omega.$$

Let  $A^pTX$  be the bundle of  $p$ -forms over  $X$ ,  $j^r A^pTX$  its  $r$ -jet prolongation,  $\mathcal{D}_n$  the category of smooth  $n$ -dimensional manifolds and their embeddings,  $\mathcal{D}_X$  its subcategory of morphisms  $\alpha: U \rightarrow V$ , where  $U, V \subset X$ , and  $\mathcal{FB}(A^pTX)$  the category of local bundle homomorphisms of  $A^pTX$  whose projections are morphisms of  $\mathcal{D}_X$ . If  $\tau_p: \mathcal{D}_X \rightarrow \mathcal{FB}(A^pTX)$  denotes the natural lifting and  $j^1\tau_p: \mathcal{D}_X \rightarrow \mathcal{FB}(j^1 A^pTX)$  its 1-jet prolongation, then  $d$  may be regarded as a natural transformation of  $j^1\tau_p$  to  $\tau_{p+1}$  or, which is equivalent, a differential invariant  $d_0: T_n^1 A^p R^n \rightarrow A^{p+1} R^n$ , where

$T_n^1 A^p R^n$  ( $A^{p+1} R^n$ ) is the type fiber of  $j^1 A^p TX$  ( $A^{p+1} TX$ , respectively); the differential invariant  $d_0$  obtained in this way is a linear mapping. Thus the problem of uniqueness consists in the uniqueness of the differential invariant  $d_0$  or, if we allow higher order liftings, in the uniqueness of  $d_0$ :  $T_n^r A^p R^n \rightarrow A^{p+1} R^n$ , where  $r \geq 1$  is arbitrary.

The problem of uniqueness of the Lie bracket and the Levi-Civita connection may analogously be formulated in terms of appropriate liftings, their natural transformations, and the corresponding differential invariants. The methods we use are, however, more general, and should be compared with [11] and, in the case of the Levi-Civita connection, with [3], [13].

**2. Basic geometrical structures.** Throughout this paper,  $R$  denotes the field of real numbers,  $R^n$  the real,  $n$ -dimensional Euclidean space. If  $R^n$  is considered with its natural vector-space structure, its dual vector space is denoted by  $R^{n*}$ .  $X$  denotes a smooth  $n$ -dimensional differential manifold. Our notations given below follow [5], [7].

Recall that the  $r$ th differential group  $L_n^r$  of  $R^n$  is the group of invertible  $r$ -jets with source and target at the origin  $0 \in R^n$ . Let  $K_n^r$  denote the kernel of the canonical projection  $L_n^r \rightarrow L_n^1$ , and let  $\iota_r: L_n^1 \rightarrow L_n^r$  be the canonical homomorphism of the Lie groups. It is known that  $K_n^r$  is a nilpotent normal subgroup of  $L_n^r$ , diffeomorphic with some Euclidean space, and that  $L_n^r = L_n^1 \times_s K_n^r$  (= semi-direct product).

The canonical (global) coordinates  $b_{j_1}^i, b_{j_2}^i, \dots, b_{j_1 \dots j_r}^i$ ,  $1 \leq i \leq n$ ,  $1 \leq j_1 \leq \dots \leq j_r \leq n$ , on  $L_n^r$  are defined as follows. Let  $j_0^r \alpha \in L_n^r$ , where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  is a local diffeomorphism of  $R^n$ , let  $\alpha^{-1} = (\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_n^{-1})$  be the inverse diffeomorphism. We set

$$(2.1) \quad b_{j_1}^i(j_0^r \alpha) = D_{j_1} \alpha_i^{-1}(0), \dots, b_{j_1 \dots j_r}^i(j_0^r \alpha) = D_{j_1} \dots D_{j_r} \alpha_i^{-1}(0),$$

where  $D_j$  is the partial derivative operator with respect to the  $j$ -th variable in  $R^n$ . We define  $a_j^i(j_0^r \alpha) = b_j^i(j_0^r \alpha^{-1})$ ; then  $a_p^i b_p^j = \delta_j^i$  (= the Kronecker symbol).

We denote by  $TX$  the tangent bundle of  $X$ , by  $A^p TX$  ( $j^p A^p TX$ ) the bundle of  $p$ -forms (the  $r$ -jet prolongation of the bundle of  $p$ -forms, respectively) over  $X$ . The type fiber of  $A^p TX$  ( $j^p A^p TX$ ) is denoted by  $A^p R^n$  ( $T_n^p A^p R^n$ , respectively), and the canonical coordinates on  $A^p R^n$  ( $T_n^p A^p R^n$ ) are denoted by  $\omega_{i_1 i_2 \dots i_p}$ ,  $1 \leq i_1 < i_2 < \dots < i_p \leq n$  ( $\omega_{i_1 i_2 \dots i_p}, \omega_{i_1 i_2 \dots i_p, k_1}, \dots, \omega_{i_1 i_2 \dots i_p, k_1 k_2 \dots k_r}$ ,  $1 \leq i_1 < i_2 < \dots < i_p \leq n$ ,  $1 \leq k_1 \leq k_2 \leq \dots \leq k_r \leq n$ , respectively).  $A^p R^n$  ( $T_n^p A^p R^n$ ) has a natural structure of an  $L_n^1$ -module ( $L_n^{r+1}$ -module, respectively).

Put

$$(2.2) \quad Q = R^n \otimes (R^{n*} \odot R^{n*}),$$

where  $\odot$  denotes the symmetric tensor product, and denote by  $\Gamma_{jk}^i$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq k \leq n$ , the canonical coordinates on  $Q$ .  $Q$  may be endowed with the structure of an  $L_n^2$ -module as follows. For each  $(j_0^2 \alpha, q) \in L_n^2 \times Q$  we set

$$(2.3) \quad \Gamma_{jk}^i(j_0^2 \alpha \cdot q) = a_p^i(j_0^2 \alpha) (b_j^p(j_0^2 \alpha) b_k^p(j_0^2 \alpha) \Gamma_{qp}^p(q) + b_{jk}^p(j_0^2 \alpha))$$

or, with the obvious meaning of the symbols,

$$(2.4) \quad \Gamma_{jk}^i = a_p^i (b_j^q b_k^r \Gamma_{qr}^p + b_{jk}^p).$$

The mapping  $(j_0^2 \alpha, q) \rightarrow j_0^2 \alpha$ .  $q$  introduced by either of these formulas, defines a left action of  $L_n^2$  on  $Q$ .

Consider the  $r$ -jet prolongation  $T_n^r Q$  of  $Q$ , i.e. the set of  $r$ -jets with source  $0 \in R^n$  and target in  $Q$ , with its natural manifold structure. Let  $\Gamma_{jk}^i, \Gamma_{jk, m_1}^i, \dots, \Gamma_{jk, m_1 \dots m_r}^i$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq k \leq n$ ,  $1 \leq m_1 \leq m_2 \leq \dots \leq m_r \leq n$ , be the canonical coordinates on  $T_n^r Q$ . Recall that for each  $p = 1, 2, \dots, r$ ,

$$(2.5) \quad \Gamma_{jk, m_1 \dots m_p}^i (j_0^r q) = D_{m_1} \dots D_{m_p} (\Gamma_{jk}^i q) (0).$$

The  $L_n^{r+1}$ -module structure of  $T_n^r Q$  may be easily described by formal differentiation of (2.4). We obtain

$$(2.6) \quad \Gamma_{jk, m}^i = a_s^i a_p^t b_{tm}^s (b_j^q b_k^r \Gamma_{qr}^p + b_{jk}^p) + \\ + a_p^i ((b_{jm}^q b_k^r + b_j^q b_{km}^r) \Gamma_{qr}^p + b_j^q b_k^t b_m^r \Gamma_{qr, t}^p + b_{jkm}^p),$$

etc.

**3. The uniqueness of exterior derivative.** The fact that the exterior derivative as defined above (see (1.1)) is unique was proved by Palais [11]. Our proof is based on a generalization of the following lemma proved in [7]: The left  $L_n^{r+2}$ -manifold  $T_n^r Q$  has the structure of a left principal  $K_n^{r+2}$ -bundle. This left principal  $K_n^{r+2}$ -bundle is trivial, and its base is diffeomorphic with some Euclidean space. The canonical projection  $\varrho_r: T_n^r Q \rightarrow T_n^r Q / K_n^{r+2}$  is equivariant in the sense that for each  $j_0^{r+2} \alpha \in L_n^{r+2}$ ,  $j_0^r q \in T_n^r Q$ ,

$$(3.1) \quad \varrho_r(j_0^{r+2} \alpha \cdot j_0^r q) = j_0^1 \alpha \cdot \varrho_r(j_0^r q).$$

Let us reformulate this lemma for spaces arising by prolongations of tensor bundles. Let  $E$  be the space of tensors of type  $(p, q)$  on  $R^n$ , let  $t_{j_1 \dots j_q}^{i_1 \dots i_p}$  be the canonical coordinates on  $E$ . Consider the product  $T_n^{r-1} Q \times T_n^r E$  ( $T_n^r E =$  the space of  $r$ -jets with source  $0 \in R^n$  and target in  $E$ ), endowed with the natural structure of an  $L_n^{r+1}$ -manifold.

**Lemma.** *The left  $L_n^{r+1}$ -manifold  $T_n^{r-1} Q \times T_n^r E$  has the structure of a left principal  $K_n^{r+1}$ -bundle. This left principal  $K_n^{r+2}$ -bundle is trivial, and its base is diffeomorphic with some Euclidean space. The canonical projection  $\pi_r: T_n^{r-1} Q \times T_n^r E \rightarrow (T_n^{r-1} Q \times T_n^r E) / K_n^{r+1}$  is equivariant in the sense that for each  $j_0^{r+1} \alpha \in L_n^{r+1}$ ,  $(j_0^{r-1} q, j_0^r \xi) \in T_n^{r-1} Q \times T_n^r E$ ,*

$$(3.2) \quad \pi_r(j_0^{r+1} \alpha \cdot (j_0^{r-1} q, j_0^r \xi)) = j_0^1 \alpha \cdot \pi_r(j_0^{r-1} q, j_0^r \xi).$$

**Proof.** Following [7] denote by  $R_{jkm}^i$  the components of the formal curvature tensor on  $T_n^r Q$ , by  $R_{jkm; s_1}^i, (R_{jkm; s_1; s_2}^i)$  the components of its first (second, respectively)

formal covariant derivative, etc. Put

$$(3.3) \quad \Gamma_{jkm_1 \dots m_s}^i = \Gamma_{(jk, m_1 \dots m_s)}^i,$$

where  $(jk, m_1 \dots m_s)$  means the symmetrization of (2.5), and denote by  $t_{j_1 \dots j_q}^{i_1 \dots i_p}$ ,  $t_{j_1 \dots j_q; m_1}^{i_1 \dots i_p}, \dots, t_{j_1 \dots j_q; m_1; \dots; m_r}^{i_1 \dots i_p}$  the formal covariant derivatives of  $t_{j_1 \dots j_q}^{i_1 \dots i_p}$ . For example,

$$(3.4) \quad t_{j_1 \dots j_q; m}^{i_1 \dots i_p} = t_{j_1 \dots j_q, m}^{i_1 \dots i_p} + \Gamma_{sm}^{i_1} t_{j_1 \dots j_q}^{i_2 \dots i_p} + \dots + \Gamma_{sm}^{i_p} t_{j_1 \dots j_q}^{i_1 \dots i_{p-1}s} - \Gamma_{m_j_1}^s t_{j_1 \dots j_q}^{i_1 \dots i_p} - \dots - \Gamma_{m_{j_q}}^s t_{j_1 \dots j_q}^{i_1 \dots i_p}.$$

Then the system of functions  $\Gamma_{j_1 j_2}^i, \dots, \Gamma_{j_1 \dots j_{r+1}}^i, 1 \leq i \leq n, 1 \leq j_1 \leq \dots \leq j_{r+1} \leq n, R_{jkl}^i, R_{jkl; m_1}^i, \dots, R_{jkl; m_1; \dots; m_{r-2}}^i, t_{j_1 \dots j_q}^{i_1 \dots i_p}, t_{j_1 \dots j_q; m_1}^{i_1 \dots i_p}, \dots, t_{j_1 \dots j_q; m_1; \dots; m_r}^{i_1 \dots i_p}$  contains a subsystem defining a global chart on  $T_n^{r-1}Q \times T_n^r E$ . The coordinate functions belonging to this chart will be referred to as the adapted coordinates. In terms of this chart, the action of  $L_n^{r+1}$  on  $T_n^{r-1}Q \times T_n^r E$  is expressed by the formulas

$$(3.5) \quad \bar{F}_{m_1 \dots m_{s+2}}^i = \Gamma_{m_1 \dots m_{s+2}}^i + s_{m_1 \dots m_{s+2}}^i + b_{m_1 \dots m_{s+2}}^i, \\ \bar{R}_{jkl; m_1; \dots; m_s}^i = R_{jkl; m_1; \dots; m_s}^i, \quad \bar{t}_{j_1 \dots j_q; m_1; \dots; m_s}^{i_1 \dots i_p} = t_{j_1 \dots j_q; m_1; \dots; m_s}^{i_1 \dots i_p},$$

where  $s_{m_1 \dots m_{s+2}}^i$  is a polynomial in the canonical coordinates on  $K_n^{s+1}$  and in the adapted coordinates on  $T_n^{s-1}Q$ . Consequently, the  $K_n^{r+1}$ -orbits in  $T_n^{r-1}Q \times T_n^r E$  are defined by the equations

$$(3.6) \quad R_{jkl}^i = a_{jkl}^i, \dots, R_{jkl; m_1; \dots; m_{r-2}}^i = a_{jklm_1 \dots m_{r-2}}^i, \\ t_{j_1 \dots j_q}^{i_1 \dots i_p} = b_{j_1 \dots j_q}^{i_1 \dots i_p}, \dots, t_{j_1 \dots j_q; m_1; \dots; m_r}^{i_1 \dots i_p} = b_{j_1 \dots j_q m_1 \dots m_r}^{i_1 \dots i_p},$$

with arbitrary constants on the right-hand sides.

To prove the lemma it is sufficient to show that the equivalence relation "there exists  $j_0^{r+1} \alpha \in L_n^{r+1}$  such that  $(j_0^{r-1} q_1, j_0^r \xi_1) = j_0^{r+1} \alpha \cdot (j_0^{r-1} q_2, j_0^r \xi_2)$ " is a closed submanifold of the product  $(T_n^{r-1}Q \times T_n^r E) \times (T_n^{r-1}Q \times T_n^r E)$ , and that the action of  $K_n^{r+1}$  on  $T_n^{r-1}Q \times T_n^r E$  is free. In the adapted coordinates on the first and the second factor, this equivalence has the equations  $R_{jkl}^i = \bar{R}_{jkl}^i, \dots, R_{jkl; m_1; \dots; m_{r-2}}^i = \bar{R}_{jkl; m_1; \dots; m_{r-2}}^i, t_{j_1 \dots j_q}^{i_1 \dots i_p} = \bar{t}_{j_1 \dots j_q}^{i_1 \dots i_p}, \dots, t_{j_1 \dots j_q; m_1; \dots; m_r}^{i_1 \dots i_p} = \bar{t}_{j_1 \dots j_q; m_1; \dots; m_r}^{i_1 \dots i_p}$  and is therefore closed. Equations (3.5) imply that  $K_n^{r+1}$  acts freely on  $T_n^{r-1}Q \times T_n^r E$ . The remaining assertions are verified in the same manner as in [7].

We are now in position to prove the uniqueness of the exterior derivative. To this purpose we put  $E = \Lambda^p R^n$  in the lemma, and denote by  $\omega_{j_1 \dots j_p}$  the canonical coordinates on  $\Lambda^p R^n$ .

**Theorem 1.** *There exists a unique, up to a multiplicative constant factor, linear differential invariant from  $T_n \Lambda^p R^n$  to  $\Lambda^{p+1} R^n$ . In canonical coordinates, this differential invariant is expressed by the equations*

$$(3.7) \quad \omega_{j_1 \dots j_p j_{p+1}} = \omega_{[j_1 \dots j_p, j_{p+1}]},$$

where  $[j_1 \dots j_p, j_{p+1}]$  means antisymmetrization.

Proof. 1) Existence. Using the structure of an  $L_n^{r+1}$ -manifold on  $T_n^r A^p R^n$  one directly verifies that (3.7) defines an  $L_n^{r+1}$ -equivariant mapping from  $T_n^r A^p R^n$  to  $A^{p+1} R^n$ .

2) Uniqueness. Let  $d_r: T_n^r A^p R^n \rightarrow A^{p+1} R^n$  be a linear differential invariant. Consider the product  $T_n^{r-1} Q \times T_n^r A^p R^n$  together with its structure of a principal  $K_n^{r+1}$ -bundle. We have the diagram

$$(3.8) \quad \begin{array}{ccc} T_n^{r-1} Q \times T_n^r A^p R^n & \xrightarrow{\text{proj}} & T_n^r A^p R^n \xrightarrow{d_r} A^{p+1} R^n \\ \downarrow \pi_r & & \uparrow d_{r,0} \\ (T_n^{r-1} Q \times T_n^r A^p R^n) / K_n^{r+1} & \text{-----} & \end{array}$$

where  $\text{proj}$  denotes the projection onto the second factor. Since the mapping  $d_r \circ \text{proj}$  is  $K_n^{r+1}$ -equivariant, it can be factored through the projection  $\pi_r$ . Denote by  $[j_0^{r-1} q, j_0^r \omega]$  the equivalence class of a pair  $(j_0^{r-1} q, j_0^r \omega) \in T_n^{r-1} Q \times T_n^r A^p R^n$ . Then by definition

$$(3.9) \quad d_{r,0}([j_0^{r-1} q, j_0^r \omega]) = d_r(j_0^r \omega).$$

Since  $d_r$  is linear, we have for each  $j_0^{r-1} q, j_0^r \omega_1, j_0^r \omega_2$

$$(3.10) \quad d_{r,0}([j_0^{r-1} q, j_0^r \omega_1 + j_0^r \omega_2]) = d_{r,0}([j_0^{r-1} q, j_0^r \omega_1]) + d_{r,0}([j_0^{r-1} q, j_0^r \omega_2]).$$

In coordinates,  $d_{r,0}$  is a mapping  $(R_{jkl}^i, \dots, R_{jkl; m_1; \dots; m_{r-2}}^i, \omega_{i_1 \dots i_p}, \dots, \omega_{i_1 \dots i_p; m_1; \dots; m_r}) \rightarrow (\omega_{j_1 \dots j_{p+1}}(R_{jkl}^i, \dots, R_{jkl; m_1; \dots; m_{r-2}}^i, \omega_{i_1 \dots i_p}, \dots, \omega_{i_1 \dots i_p; m_1; \dots; m_r}))$ , and (3.10) means that for each fixed  $R_{jkl}^i, \dots, R_{jkl; m_1; \dots; m_{r-2}}^i, \omega_{j_1 \dots j_{p+1}}$  depends linearly on  $\omega_{i_1 \dots i_p}, \dots, \omega_{i_1 \dots i_p; m_1; \dots; m_r}$ . Hence

$$(3.11) \quad \bar{\omega}_{j_1 \dots j_{p+1}} = A_{j_1 \dots j_{p+1}}^{i_1 \dots i_p} \omega_{i_1 \dots i_p} + \dots + A_{j_1 \dots j_{p+1}}^{i_1 \dots i_p; m_1; \dots; m_r} \omega_{i_1 \dots i_p; m_1; \dots; m_r}$$

where the coefficients depend on  $R_{jkl}^i, \dots, R_{jkl; m_1; \dots; m_{r-2}}^i$ . Since  $\omega_{j_1 \dots j_{p+1}}$  is an  $L_n^1$ -equivariant mapping and  $\omega_{i_1 \dots i_p}, \dots, \omega_{i_1 \dots i_p; m_1; \dots; m_r}$  are tensors, the coefficients in (3.11) must also be tensors: take for example  $\omega_{i_1 \dots i_p; m_1} = 0, \dots, \omega_{i_1 \dots i_p; m_1; \dots; m_r} = 0$ ; then after a transformation  $j_0^1 \alpha \in L_n^1$ ,

$$(3.12) \quad \begin{aligned} \bar{\omega}_{j_1 \dots j_{p+1}} &= b_{j_1}^{k_1} \dots b_{j_{p+1}}^{k_{p+1}} \omega_{k_1 \dots k_{p+1}} = b_{j_1}^{k_1} \dots b_{j_{p+1}}^{k_{p+1}} A_{k_1 \dots k_{p+1}}^{i_1 \dots i_p} \omega_{i_1 \dots i_p} = \\ &= \bar{A}_{j_1 \dots j_{p+1}}^{i_1 \dots i_p} \bar{\omega}_{i_1 \dots i_p} = \bar{A}_{j_1 \dots j_{p+1}}^{i_1 \dots i_p} b_{i_1}^{l_1} \dots b_{i_p}^{l_p} \omega_{l_1 \dots l_p} \end{aligned}$$

which is possible if and only if

$$(3.13) \quad \bar{A}_{j_1 \dots j_{p+1}}^{s_1 \dots s_p} = a_{i_1}^{s_1} \dots a_{i_p}^{s_p} b_{j_1}^{k_1} \dots b_{j_{p+1}}^{k_{p+1}} A_{k_1 \dots k_{p+1}}^{i_1 \dots i_p}.$$

We now apply the condition that the composed mapping  $d_r \circ \text{proj} = d_{r,0} \circ \pi_r$  should be independent of  $j_0^{r-1} q$ . In coordinates this means that the expression (3.11)

rewritten in the canonical coordinates on  $T_u^{r-1}Q \times T_n^r A^r R^n$ , does not depend on  $\Gamma_{jk}^i, \dots, \Gamma_{jk, m_1 \dots m_{r-1}}^i$ . Applying the formulas for the formal covariant derivatives analogous to (3.4) one easily obtains that  $A_{j_1 \dots j_{p+1}}^{i_1 \dots i_p; m_1 \dots m_r}$  (= the coefficient at  $\omega_{i_1 \dots i_p, m_1 \dots m_r}$ ) is independent of  $\Gamma_{jk}^i, \dots, \Gamma_{jk, m_1 \dots m_{r-1}}^i$  (unless  $r = 1$ ); the classical invariant theory then says, however, that  $A_{j_1 \dots j_{p+1}}^{i_1 \dots i_p; m_1 \dots m_r} = 0$  (see e.g. [2]). Repeating this argument we obtain

$$(3.14) \quad A_{j_1 \dots j_{p+1}}^{i_1 \dots i_p} = 0, \dots, A_{j_1 \dots j_{p+1}}^{i_1 \dots i_p; m_1; m_2} = 0.$$

Hence

$$(3.15) \quad \begin{aligned} \omega_{j_1 \dots j_{p+1}} &= A_{j_1 \dots j_{p+1}}^{i_1 \dots i_p; m} \omega_{i_1 \dots i_p; m} = \\ &= A_{j_1 \dots j_{p+1}}^{i_1 \dots i_p; m} (\omega_{i_1 \dots i_p, m} - \Gamma_{i_1 m}^s \omega_{s i_1 \dots i_p} - \Gamma_{i_p m}^s \omega_{i_1 \dots i_p s}). \end{aligned}$$

Hence  $A_{j_1 \dots j_{p+1}}^{i_1 \dots i_p; m}$  is a constant tensor. Since it is antisymmetric in the subscripts, it must be a multiple of the permutation tensor  $\varepsilon_{j_1 \dots j_{p+1}}^{i_1 \dots i_p; m}$ , and we have

$$\omega_{j_1 \dots j_{p+1}} = c \cdot \omega_{[j_1 \dots j_p; j_{p+1}]}$$

This completes the proof.

**4. The uniqueness of the Lie bracket.** Using the notation of the fundamental lemma of Section 3, put  $E = R^n \times R^n$ , and examine the bilinear differential invariants from  $T_n^r(R^n \times R^n)$  to  $R^n$ . The canonical coordinates on  $T_n^r(R^n \times R^n)$  (on  $R^n$ ) are denoted by  $\xi^i, \zeta^i, \xi_{j_1}^i, \zeta_{j_1}^i, \dots, \xi_{j_1 \dots j_r}^i, \zeta_{j_1 \dots j_r}^i$  ( $\lambda^i$ , respectively). As before, we shall also consider the left principal  $K_n^{r+1}$ -bundle  $\pi_r: T_n^{r-1}Q \times T_n^r(R^n \times R^n) \rightarrow (T_n^{r-1}Q \times T_n^r(R^n \times R^n))/K_n^{r+1}$  and denote by  $\xi^i, \zeta^i, \xi_{j_1}^i, \zeta_{j_1}^i, \dots, \xi_{j_1 \dots j_r}^i, \zeta_{j_1 \dots j_r}^i$  the adapted coordinates on the total space of this bundle.

**Theorem 2.** *There exists a unique, up to a multiplicative constant, bilinear differential invariant from  $T_n^r(R^n \times R^n)$  to  $R^n$ . In canonical coordinates, this differential invariant is expressed by the equations*

$$(4.1) \quad \lambda^i = \xi^k \zeta_k^i - \zeta^k \xi_k^i.$$

Proof. 1) Existence. The mapping  $(j_0^r \xi, j_0^r \zeta) \rightarrow \lambda(j_0^r \xi, j_0^r \zeta)$  defined by (4.1) has all the properties required.

2) Uniqueness. Let  $\lambda_r: T_n^r(R^n \times R^n) \rightarrow R^n$  be a bilinear differential invariant. In the diagram

$$(4.2) \quad \begin{array}{ccccc} T_n^{r-1}Q \times T_n^r(R^n \times R^n) & \xrightarrow{\text{proj}} & T_n^r(R^n \times R^n) & \xrightarrow{\lambda_r} & R^n \\ \downarrow \pi_r & & & \uparrow \lambda_{r,0} & \\ (T_n^{r-1}Q \times T_n^r(R^n \times R^n))/K_n^{r+1} & \dashrightarrow & & & \end{array}$$

the mapping  $\lambda_r \circ \text{proj}$  can be factored through  $\pi_r$ . Hence  $\lambda_r$  is, in the sense of the classical invariant theory, a vector (= element of  $R^n$  endowed with the standard action of  $L_n^1$ ) depending on the tensors  $R_{jkl}^i, \dots, R_{jkl;m_1, \dots, m_{r-2}}^i, \zeta^i, \zeta^i, \dots, \zeta_{j_1, \dots, j_r}^i, \zeta_{j_1, \dots, j_r}^i$ ; by the bilinearity assumption, the components  $\lambda^k$  of  $\lambda_r$  are linear combinations of tensors  $\zeta^i, \zeta^j, \zeta_{i_1, i_2}^j, \dots, \zeta_{i_1, i_2, i_3, i_4}^j, \dots, \zeta_{i_1, i_2, i_3, i_4, i_5}^j, \zeta_{i_1, i_2, i_3, i_4}^j, \dots$ , whose coefficients are constant tensors. Therefore the only non-vanishing coefficients are at the tensors  $\zeta_{i_1, i_2}^j, \zeta_{i_1}^j, \zeta^i$ , and  $\lambda^k$  must have the form

$$(4.3) \quad \lambda^k = A_{ij\zeta}^{kl} \zeta_{i_1}^j + B_{ij\zeta}^{kl} \zeta_{i_1}^j = A_{ij\zeta}^{kl} (\zeta_{i_1}^j + \Gamma_{lm}^j \zeta^m) + B_{ij\zeta}^{kl} (\zeta_{i_1}^j + \Gamma_{lm}^j \zeta^m).$$

Since  $\lambda^k$  does not depend on  $\Gamma_{jk}^i, \dots, \Gamma_{jk, i_1, \dots, i_{r-1}}^i$ , we have

$$(4.4) \quad A_{ij}^{kl} \delta_p^i \delta_q^m + B_{ij}^{kl} \delta_q^i \delta_p^m + A_{ij}^{km} \delta_p^i \delta_q^l + B_{ij}^{km} \delta_q^i \delta_p^l = 0.$$

But

$$(4.5) \quad A_{ij}^{kl} = a \delta_i^k \delta_j^l + b \delta_j^k \delta_i^l, \quad B_{ij}^{kl} = a' \delta_i^k \delta_j^l + b' \delta_j^k \delta_i^l$$

for some  $a, b, a', b' \in R$  so that

$$(4.6) \quad \begin{aligned} & (a \delta_i^k \delta_j^l + b \delta_j^k \delta_i^l) \delta_p^i \delta_q^m + (a' \delta_i^k \delta_j^l + b' \delta_j^k \delta_i^l) \delta_q^i \delta_p^m + \\ & + (a \delta_i^k \delta_j^m + b \delta_j^k \delta_i^m) \delta_p^i \delta_q^l + (a' \delta_i^k \delta_j^m + b' \delta_j^k \delta_i^m) \delta_q^i \delta_p^l = \\ & = a(\delta_j^k \delta_p^m \delta_q^l + \delta_p^k \delta_j^m \delta_q^l) + b(\delta_j^k \delta_p^m \delta_q^l + \delta_j^k \delta_p^m \delta_q^l) + \\ & + a'(\delta_q^k \delta_j^l \delta_p^m + \delta_q^k \delta_j^m \delta_p^l) + b'(\delta_j^k \delta_q^l \delta_p^m + \delta_j^k \delta_q^m \delta_p^l) = \\ & = (b + b')(\delta_j^k \delta_p^m \delta_q^l + \delta_j^k \delta_p^m \delta_q^l) + a \delta_p^k (\delta_j^l \delta_q^m + \delta_j^m \delta_q^l) + a' \delta_q^k (\delta_j^l \delta_p^m + \delta_j^m \delta_p^l) = 0. \end{aligned}$$

If  $n = 1$ , then (4.3) is rewritten in the form  $\lambda = A\zeta(\zeta' + \Gamma\zeta) + B\zeta(\zeta' + \Gamma\zeta) = A\zeta\zeta' + B\zeta\zeta' + (A + B)\Gamma\zeta\zeta$ , and the independence of  $\Gamma$  means that  $A + B = 0$ , i.e.,  $\lambda_r$  is of the form (4.1). Let  $n > 1$ . Take for example  $j = k = 1, p = q = m = l = 2$ . Then (4.6) gives

$$(4.7) \quad b + b' = 0.$$

Similarly take  $k = p = 1, l = j = m = q = 2$ . In this way we get

$$(4.8) \quad a = 0, \quad a' = 0.$$

Returning back to (4.3) we get

$$(4.9) \quad \lambda^k = A_{ij\zeta}^{kl} \zeta_{i_1}^j + B_{ij\zeta}^{kl} \zeta_{i_1}^j = b \delta_j^k \delta_{i_1}^l \zeta_{i_1}^j + b' \delta_j^k \delta_{i_1}^l \zeta_{i_1}^j = b(\zeta^l \zeta_{i_1}^k - \zeta_{i_1}^l \zeta^k).$$

This completes the proof.

**5. The uniqueness of the Levi-Civita connection.** Let us consider the space  $E = R^{n*} \odot R^{n*}$  of tensors of type  $(0, 2)$  on  $R^n$  and denote by  $g_{ij}, 1 \leq i \leq j \leq n$ , the



canonical coordinates on  $E$ . Let  $E_0 = E \setminus \{g \in E \mid \det(g_{ij}(g)) = 0\}$  be the subset of *regular* tensors;  $E_0$  is endowed with the natural structure of an  $L_n^1$ -manifold. It is of great interest to know all differential invariants from  $L_n^{k+1}$ -invariant open subsets of  $T_n^r E_0$  to  $Q$  which correspond in a well-known manner to linear connections which are geometric objects (= concominants) of the metric tensor and its derivatives of order  $\leq r$ .

An example of a connection of this type is the Levi-Civita connection  $\nabla$ , for which  $r = 1$ . For  $r > 1$ , however, some other examples may be given which differ from the Levi-Civita connection. For this reason we shall consider the problem of uniqueness only for  $r = 1$ . We note that the problem of uniqueness of the Levi-Civita connection has been stated and solved in a different way in [3] and [13].

Let  $g_{ij}, g_{ij,k}$  be the canonical coordinates on  $T_n^r E_0$ , and let  $(g^{ij})$  denote the inverse matrix of  $(g_{ij})$ . As before, let  $\Gamma_{jk}^i$  be the canonical coordinates on  $Q$ .

**Theorem 3.** *There exists a unique differential invariant from  $T_n^1 E_0$  to  $Q$ . In canonical coordinates, this differential invariant is expressed by the equations*

$$(5.1) \quad \Gamma_{jk}^i = \frac{1}{2} g^{im} (g_{mj,k} + g_{mk,j} - g_{jk,m}).$$

Proof. The action of the group  $L_n^2$  on  $T_n^1 E_0$  and  $Q$  is expressed by

$$(5.2) \quad \bar{g}_{ij} = b_i^p b_j^q g_{pq}, \\ \bar{g}_{ij,m} = (b_{im}^p b_j^q + b_i^p b_{jm}^q) g_{pq} + b_i^p b_j^q b_m^l g_{pq,l}$$

and by (2.4). The fundamental vector fields on  $T_n^1 E_0$  relative to this action, are

$$(5.3) \quad \xi_p^q = \left( \frac{\partial \bar{g}_{ij}}{\partial b_p^q} \right)_e \frac{\partial}{\partial g_{ij}} + \left( \frac{\partial \bar{g}_{ij,m}}{\partial b_p^q} \right)_e \frac{\partial}{\partial g_{ij,m}} = \\ = (\delta_i^q g_{pj} + \delta_j^q g_{ip}) \frac{\partial}{\partial g_{ij}} + (\delta_i^q g_{pj,m} + \delta_j^q g_{ip,m} + \delta_m^q g_{ij,p}) \frac{\partial}{\partial g_{ij,m}}, \\ \xi_p^{qr} = \left( \frac{\partial \bar{g}_{ij,m}}{\partial b_p^q} \right)_e \frac{\partial}{\partial g_{ij,m}} = g_{ip} \left( \frac{\partial}{\partial g_{iq,r}} + \frac{\partial}{\partial g_{ir,q}} \right),$$

and the fundamental vector fields on  $Q$  relative to (2.4), are

$$(5.4) \quad \Xi_p^q = \left( \frac{\partial \Gamma_{jk}^i}{\partial b_p^q} \right)_e \frac{\partial}{\partial \Gamma_{jk}^i} = (-\delta_p^i \Gamma_{jk}^q + \delta_j^q \Gamma_{pk}^i + \delta_k^q \Gamma_{jp}^i) \frac{\partial}{\partial \Gamma_{jk}^i}, \\ \Xi_p^{qr} = \left( \frac{\partial \Gamma_{jk}^i}{\partial b_p^q} \right)_e \frac{\partial}{\partial \Gamma_{jk}^i} = \frac{\partial}{\partial \Gamma_{qr}^p},$$

where  $e$  denotes  $j_0^2 \text{id}$  (= the identity of  $L_n^2$ ). Hence each  $L_n^2$ -equivariant mapping

$(g_{ij}, g_{ij,k}) \rightarrow (\Gamma_{qr}^p(g_{ij}, g_{ij,k}))$  of  $T_n^1 E_0$  to  $Q$  must obey the system of equations

$$(5.5) \quad (\delta_a^q g_{pb} + \delta_b^q g_{ap}) \frac{\partial \Gamma_{jk}^i}{\partial g_{ab}} + (\delta_a^q g_{pb,m} + \delta_b^q g_{ap,m} + \delta_n^q g_{ab,p}) \frac{\partial \Gamma_{jk}^i}{\partial g_{ab,m}} = \\ = -\delta_p^i \Gamma_{jk}^q + \delta_j^q \Gamma_{pk}^i + \delta_k^q \Gamma_{jp}^i,$$

$$(5.6) \quad g_{ap} \frac{\partial \Gamma_{jk}^i}{\partial g_{aq,r}} + \frac{\partial \Gamma_{jk}^i}{\partial g_{ar,q}} = \frac{1}{2} \delta_p^i (\delta_j^q \delta_k^r + \delta_k^q \delta_j^r).$$

Consider the system (5.6). Multiplying both sides by  $g^{ps}$  and rewriting the system for cyclic permutations of the subscripts of the independent variable one gets

$$(5.7) \quad \frac{\partial \Gamma_{jk}^i}{\partial g_{sq,r}} + \frac{\partial \Gamma_{jk}^i}{\partial g_{sr,q}} = \frac{1}{2} g^{is} (\delta_j^q \delta_k^r + \delta_k^q \delta_j^r), \\ \frac{\partial \Gamma_{jk}^i}{\partial g_{rs,q}} + \frac{\partial \Gamma_{jk}^i}{\partial g_{rq,s}} = \frac{1}{2} g^{ir} (\delta_j^s \delta_k^q + \delta_k^s \delta_j^q), \\ -\frac{\partial \Gamma_{jk}^i}{\partial g_{qr,s}} - \frac{\partial \Gamma_{jk}^i}{\partial g_{qs,r}} = \frac{1}{2} g^{iq} (\delta_j^r \delta_k^s + \delta_k^r \delta_j^s),$$

which implies

$$(5.8) \quad \frac{\partial \Gamma_{jk}^i}{\partial g_{sr,q}} = \frac{1}{4} (g^{is} (\delta_j^q \delta_k^r + \delta_k^q \delta_j^r) + g^{ir} (\delta_j^s \delta_k^q + \delta_k^s \delta_j^q) - g^{iq} (\delta_j^r \delta_k^s + \delta_k^r \delta_j^s)).$$

Hence

$$(5.9) \quad \Gamma_{jk}^i = \frac{1}{4} (g^{is} (g_{sk,j} + g_{sj,k}) + g^{ir} (g_{jr,k} + g_{kr,j}) - \\ - g^{iq} (g_{kj,q} + g_{jk,q})) + \gamma_{jk}^i = \frac{1}{2} g^{is} (g_{sj,k} + g_{sk,j} - g_{jk,s}) + \gamma_{jk}^i,$$

where  $\gamma_{jk}^i$  does not depend on  $g_{pq,r}$ . Substituting this expression into (5.5) one obtains

$$(5.10) \quad \frac{\partial \gamma_{jk}^i}{\partial g_{aq}} = 0,$$

which means that  $\gamma_{jk}^i \in R$ . This shows that each differential invariant from  $T_n^1 E_0$  to  $Q$  must belong to the family of mappings (5.9), where  $\gamma_{jk}^i$  are real numbers. Consider the transformation properties of these mappings. It is directly seen that among these mappings there is one and only one which is  $L_n^2$ -equivariant; this is the mapping for which  $\gamma_{jk}^i = 0$ . This completes the proof.

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