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ON THE UNIQUENESS OF THE SOLUTION OF AN EVOLUTION FREE BOUNDARY PROBLEM IN THEORY OF LUBRICATION

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Abstract. In this paper we study a time dependent free boundary problem related to the cavitation modeling in a lubricated device. We prove some continuity properties and the uniqueness of a weak solution.

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INTRODUCTION

This problem and related others problems were previously studied in many papers ([1], [2], [3], [4], [6], [7], [9], [10]...). However, most of these papers are related to stationary case where existence and uniqueness results are available. The evolution problem was studied in [1] and [9] where, using different formulations, both authors proved existence of solutions. In [9], a numerical approach of the evolution problem is also given. And, in [5] the authors studied the corresponding semi-discretized problem to approximate the solution of the stationary problem.

Our goal in this paper is to prove comparison and uniqueness of a solution of the evolution problem. The proof of this result is based on the inequality stated in theorem 3.1; and the tools leading to this inequality consists in some techniques already developed by the first author to prove a relative result in the stationary case [3]. However, the dependence of all variables x,y and t, for all functions of the problem and the less regularity of the solution, makes the problem more difficult and the analysis used to deduce comparison and then uniqueness from the inequality of theorem 3.1 is quite different from the stationary case.

In section 1 we present the mathematical formulation of the problem. It corresponds to the Elrod-Adams model for cavitation in lubrication ([1], [6], [7], [9]); the set of equations defines a moving nonlinear free boundary problem. Then we recall the previous results about existence of solutions.

In section 2, lemma 2.1 and corollary 2.2 are technical statements in order to handle rigorously test functions involving the solution. The main results of this section are the properties stated in theorem 2.3 and theorem 2.4.

Section 3 is devoted to prove a comparison principle for the solutions of the problem, when we can compare their values on the upper boundary. The uniqueness is easily deduced from this result.

1. The model problem

The study of the journal-bearing device gives place to a mathematical formulation in the domain $Q = (0,T) \times \Omega$ where $T > 0$ and $\Omega = (0, 2\pi) \times (0,1)$. The unknowns of the problem are
the pressure p and the relative content γ of the oil film. In this geometry, the supply of lubricant is made throught the top boundary $\Sigma_a = (0,2\pi) \times \{1\} \times (0,T)$ where we take $p = p_a$. On the boundary $\Sigma_0 = (0,2\pi) \times \{0\} \times (0,T)$ we suppose $p = 0$ (see Figure 1).

When the lubrication takes place by an incompressible fluid the pressure satisfies the moving Reynolds equation:

$$\frac{\partial h}{\partial t} - div(h^3 \nabla p) = -\frac{dh}{dx} \quad \gamma = 1 \quad if \quad p > 0$$

and γ satisfies the conservation law

$$\frac{\partial(\gamma h)}{\partial x} + \frac{\partial(\gamma h)}{\partial t} = 0 \quad 0 \leq \gamma \leq 1 \quad if \quad p = 0.$$

On the free boundary Σ, we have $p = 0$ and the conservation condition of the flux

$$h^3 \frac{\partial p}{\partial n} = (1-\gamma) h \cos(n, i)$$

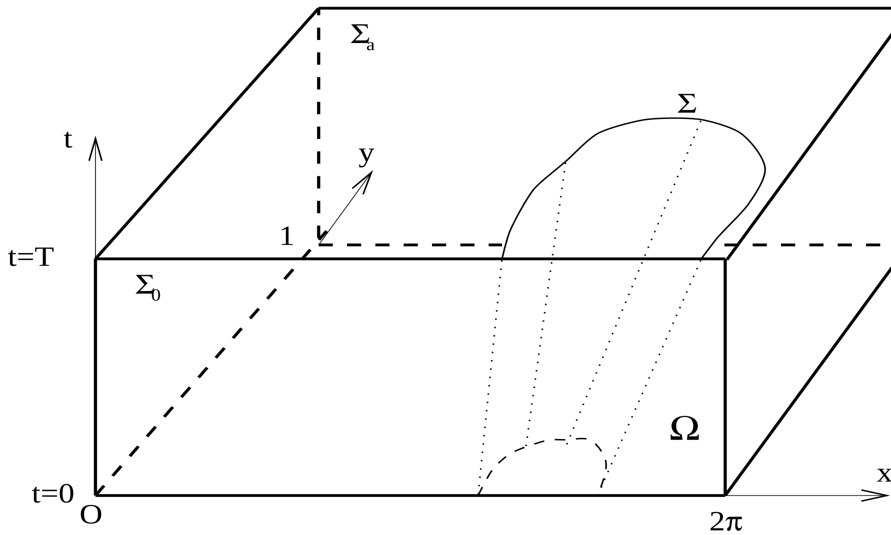

Figure 1: Domain Q

being n the normal vector to the free boundary Σ, and (n, i) the angle between n and i (i is the unity vector in the x-direction).
The function $h(x, y, t)$ that represents the gap, belongs to $C^\infty(Q)$, it is periodic in the x variable and it satisfies:

$$h(x, y, 0) = h(x, y, T) \quad \forall (x, y) \in (0, 2\pi) \times (0, 1)$$

The Reynolds equation and the Conservation laws lead to the following equation valid both in the cavitation and the lubricated region:

$$\frac{\partial (h\gamma)}{\partial t} - div(h^3 \nabla p) = -\frac{\partial (h\gamma)}{\partial x} \quad in \quad \mathcal{D}'(Q)$$

and the problem can be formulated in the following weak formulation:

Problem $\mathcal{P}$

$$\begin{array}{ll} & Find \quad (p, \gamma) \in L^2(0, T; H^1(\Omega)) \times L^\infty(Q) \quad such \quad that \\ i) & p \geq 0 \quad and \quad \gamma \in H(p) \quad a.e \quad in \quad Q \\ ii) & -\int_Q h(x, y, t)\gamma(x, y, t)\xi_t + \int_Q h^3(x, y, t)\nabla p(x, y, t)\nabla \xi \\ & = \int_Q h(x, y, t)\gamma(x, y, t)\xi_x \qquad \forall \xi \in V \\ iii) & p = 0 \quad on \quad \Sigma_0; \quad p = p_a \quad on \quad \Sigma_a \end{array}$$

Where H is the Heaviside graph, and

$$V = \{\xi \in H^1(Q) / \xi \quad 2\pi - x - periodic, \quad \xi = 0 \quad on \quad \Sigma_0 \cup \Sigma_a, \quad \xi(x, y, 0) = \xi(x, y, T) = 0\}$$

Existence of solutions for this and others related problems is well known ([1],[7],[9]), and we have:

Theorem 1.1. *Problem $\mathcal{P}$ has at least one solution.*

Proof. This result is stated by using elliptic regularization techniques introduced in [11]. The Heaviside graph is approximated by a smooth function F_ϵ depending on a parameter ϵ and the problem is approximated by a family of elliptic regularized problems. A priori estimates of the solution p_ϵ of the regularized problems are obtained by means of a suitable election of F_ϵ. Then the convergences $p_\epsilon \to p$ (weakly) in $L^2(0,T;H^1(\Omega))$ and $F_\epsilon(p_\epsilon) \to \gamma$ (weakly) in $L^\infty(Q)$ are obtained. □

2. Some properties of the solutions

In the sequel, we will need to handle some test functions involving the solution p, such as $\zeta = min(\frac{p}{\delta}, \xi)$ being ξ a smooth function. This test functions belong to $L^2(0,T;H^1(\Omega))$ and the derivative $\dfrac{\partial \zeta}{\partial t}$ is not a function. To solve this difficulty, we give the following lemma for general test functions involving p. A similar result related to the Dam problem is given in [8].

Lemma 2.1. *Let (p,γ) be a solution of problem $\mathcal{P}$, $\phi = p_a y$, $\xi \in \mathcal{D}(\overline{\Omega} \times (0,T))$ with $\xi \geq 0$ and $F \in W^{1,\infty}_{loc}(\mathbb{R}^2)$, such hat:*
(i) $F(p,\xi) \in L^2(0,T,H^1(\Omega))$
(ii) $F(\phi,\xi) \in H^1(Q)$
(iii)$F(0,\xi) \in H^1(Q)$
(iv)$F(z_1,z_2) \geq 0$ a.e $z_1 \in \mathbb{R}$ and $z_2 \geq 0$
(v) either $\dfrac{\partial F}{\partial z_1} \geq 0$ a.e in $\mathbb{R}^2$, or $\dfrac{\partial F}{\partial z_1} \leq 0$ a.e in $\mathbb{R}^2$
Then, for all $\xi \in \mathcal{D}(\overline{\Omega \times (0,T)})$, we have

$$\begin{aligned}
&\int_Q h^3 \nabla p \nabla \big(F(p,\xi)\big) - \int_Q h\gamma \big(F(p,\xi)\big)_x - \int_Q h\gamma\big(F(0,\xi)\big)_t \\
&= \int_Q h^3 \nabla p \nabla \big(F(\phi,\xi)\big) - \int_Q h\gamma\big(F(\phi,\xi)\big)_x \\
&- \int_Q h\gamma\big(F(\phi,\xi)\big)_t - \int_Q h_t\big(F(p,\xi) - F(0,\xi)\big)
\end{aligned}$$

Particulary, if $F(\phi,\xi) \in V$, then

$$\begin{aligned}
&\int_Q h^3 \nabla p \nabla \big(F(p,\xi)\big) - \int_Q h\gamma \big(F(p,\xi)\big)_x - \int_Q h\gamma\big(F(0,\xi)\big)_t \\
&= -\int_Q h_t\big(F(p,\xi) - F(0,\xi)\big)
\end{aligned}$$

Proof. Let ζ be a smooth function such that $\zeta = 0$ on $\Sigma_0 \cup \Sigma_a$, 2π-x periodic with $\text{supp}(\zeta) \subset \mathbb{R}^2 \times (\tau_0, T-\tau_0)$ for some τ_0. Then $\forall \tau \in (-\tau_0, \tau_0)$ the function $(x,t) \longmapsto \zeta(x, t-\tau)$

defined in $\Omega \times (0,T)$ is a test function of problem $\mathcal{P}$. Then $\forall \tau \in (-\tau_0, \tau_0)$ we have:

$$\begin{aligned} 0 &= \int_Q h^3(x,t)\nabla p(x,t)\nabla\zeta(x,t-\tau) - \int_Q h(x,t)\gamma(x,t)\zeta_x(x,t-\tau) - \int_Q h(x,t)\gamma(x,t)\zeta_t(x,t-\tau) \\ &= \int_Q h^3(x,t)\nabla p(x,t)\nabla\zeta(x,t-\tau) - \int_Q h(x,t)\gamma(x,t)\zeta_x(x,t-\tau) + \int_Q h(x,t)\gamma(x,t)\zeta_\tau(x,t-\tau) \\ &= \int_Q h^3(x,t)\nabla p(x,t)\nabla\zeta(x,t-\tau) - \int_Q h(x,t)\gamma(x,t)\zeta_x(x,t-\tau) + \frac{\partial}{\partial\tau}\Big[\int_Q h(x,t)\gamma(x,t)\zeta(x,t-\tau)\Big] \end{aligned}$$

since $\dfrac{\partial\zeta}{\partial t}(x,t-\tau) = -\dfrac{\partial\zeta}{\partial\tau}(x,t-\tau)$ and $h(x,t)$ and $\gamma(x,t)$ does not depend of τ.
Therefore, $\forall \tau \in (-\tau_0, \tau_0)$ we have

$$\begin{aligned} 0 &= \int_Q h^3(x,t+\tau)\nabla p(x,t+\tau)\nabla\zeta(x,t) - \int_Q h(x,t+\tau)\gamma(x,t+\tau)\zeta_x(x,t) \\ &+ \frac{\partial}{\partial\tau}\Big[\int_Q h(x,t+\tau)\gamma(x,t+\tau)\zeta(x,t)\Big] \end{aligned}$$

It is easy to see that this identity holds for any $\zeta \in L^2(0,T;V)$ such that $\zeta = 0$ a.e in $\Omega \times ((0,\tau_0) \cup (T-\tau_0,T))$, $\zeta = 0$ on $\Sigma_0 \cup \Sigma_a$ and 2π-x periodic.
Now, if we consider $\xi \in \mathcal{D}(\overline{\Omega} \times (\tau_0, T-\tau_0))$, $\xi \geq 0$, $2\pi - x$ periodic, and set $\zeta = F(p,\xi) - F(\phi,\xi)$, for all $\tau \in (-\tau_0,\tau_0)$ we have:

$$\begin{aligned} &\int_Q h^3(x,t+\tau)\nabla p(x,t+\tau)\nabla\big[F(p,\xi) - F(\phi,\xi)\big] - \int_Q h(x,t+\tau)\gamma(x,t+\tau)\big[F(p,\xi) - F(\phi,\xi)\big]_x \\ &\qquad = -\frac{\partial}{\partial\tau}\int_Q h(x,t+\tau)\gamma(x,t+\tau)\big[F(p,\xi) - F(\phi,\xi)\big] \end{aligned}$$

and then

$$\left\{\begin{aligned} &\int_Q h^3(x,t+\tau)\nabla p(x,t+\tau)\nabla F(p,\xi) - \int_Q h(x,t+\tau)\gamma(x,t+\tau)\big[F(p,\xi)\big]_x \\ &-\int_Q h(x,t)\gamma(x,t)\big[F(0,\xi(x,t-\tau))\big]_t - \int_Q h^3(x,t+\tau)\nabla p(x,t+\tau)\nabla F(\phi,\xi) \\ &+\int_Q h(x,t+\tau)\gamma(x,t+\tau)\big[F(\phi,\xi)\big]_x + \int_Q h(x,t)\gamma(x,t)\big[F(\phi,\xi(x,t-\tau))\big]_t \\ &= -\frac{\partial}{\partial\tau}\int_Q h(x,t+\tau)\gamma(x,t+\tau)\big[F(p,\xi) - F(\phi,\xi)\big] \\ &-\int_Q h(x,t)\gamma(x,t)\big[F(0,\xi(x,t-\tau))\big]_t + \int_Q h(x,t)\gamma(x,t)\big[F(\phi,\xi(x,t-\tau))\big]_t \end{aligned}\right. \tag{2.1}$$

The two last integrals in the right hand side can be written:

$$\begin{aligned} \int_Q h(x,t)\gamma(x,t)\big[F(0,\xi(x,t-\tau))\big]_t &= -\int_Q h(x,t)\gamma(x,t)\big[F(0,\xi(x,t-\tau))\big]_\tau \\ &= -\frac{\partial}{\partial\tau}\Big[\int_Q h(x,t)\gamma(x,t)F(0,\xi(x,t-\tau))\Big] \\ &= -\frac{\partial}{\partial\tau}\Big[\int_Q h(x,t+\tau)\gamma(x,t+\tau)F(0,\xi)\Big] \end{aligned}$$

and

$$\begin{aligned}\int_Q h(x,t)\gamma(x,t)\big[F(\phi,\xi(t-\tau))\big]_t &= -\int_Q h(x,t)\gamma(x,t)\big[F(\phi,\xi(x,t-\tau))\big]_\tau \\ &= -\frac{\partial}{\partial\tau}\Big[\int_Q h(x,t)\gamma(x,t)F(\phi,\xi(x,t-\tau)\Big] \\ &= -\frac{\partial}{\partial\tau}\Big[\int_Q h(x,t+\tau)\gamma(x,t+\tau)F(\phi,\xi)\Big]\end{aligned}$$

Hence the equality (2.1) can be written:

$$\begin{cases}\displaystyle\int_Q h^3(x,t+\tau)\nabla p(x,t+\tau)\nabla F(p,\xi) - \int_Q h(x,t+\tau)\gamma(x,t+\tau)\big[F(p,\xi)\big]_x \\ \displaystyle -\int_Q h(x,t)\gamma(x,t)\big[F(0,\xi(x,t-\tau))\big]_t - \int_Q h^3(x,t+\tau)\nabla p(x,t+\tau)\nabla F(\phi,\xi) \\ \displaystyle +\int_Q h(x,t+\tau)\gamma(x,t+\tau)\big[F(\phi,\xi)\big]_+ \int_Q h(x,t)\gamma(x,t)\big[F(\phi,\xi(x,t-\tau))\big]_t \\ \displaystyle = -\frac{\partial G(\tau)}{\partial\tau}\end{cases} \tag{2.2}$$

with

$$G(\tau) = \int_Q h(x,t+\tau)\gamma(x,t+\tau)\big[F(p,\xi)-F(0,\xi)\big]$$

The integrals on the left-hand side of equality (2.2) are continuous functions of τ, therefore, the function G belongs to $C^1(-\tau_0,\tau_0)$.
If $\dfrac{\partial F}{\partial z_1}\geq 0$, we have:

$$h(x,t+\tau)(\gamma(x,t+\tau)-1)\big[F(p,\xi)-F(0,\xi)\big]\leq 0.$$

Then, the function H defined by:

$$H(\tau) = \int_Q h(x,t+\tau)(\gamma(x,t+\tau)-1)\big[F(p,\xi)-F(0,\xi)\big].$$

satisfies $H(\tau)\leq 0 = H(0), \forall\tau\in(-\tau_0,\tau_0)$. Therefore it has a maximum in $\tau=0$.
Analogously, if $\dfrac{\partial F}{\partial z_1}\leq 0$, we deduce that H has a minimum in $\tau=0$.
By other hand, we have:

$$G(\tau) = H(\tau) + \int_Q h(x,t+\tau)\big[F(p,\xi)-F(0,\xi)\big] \tag{2.3}$$

then $H\in C^1(-\tau_0,+\tau_0)$ and we get:

$$\begin{aligned}\frac{\partial G}{\partial\tau}(\tau) &= \frac{\partial H}{\partial\tau}(\tau) + \int_Q \frac{\partial h}{\partial\tau}(x,t+\tau)\big[F(p,\xi)-F(0,\xi)\big] \\ &= \frac{\partial H}{\partial\tau}(\tau) + \int_Q \frac{\partial h}{\partial t}(x,t+\tau)\big[F(p,\xi)-F(0,\xi)\big]\end{aligned}$$

Now, since $\dfrac{\partial H}{\partial \tau}(0) = 0$ we deduce

$$\frac{\partial G}{\partial \tau}(0) = \int_Q \frac{\partial h}{\partial t}(x,t)\big[F(p,\xi) - F(0,\xi)\big]$$

Finally, letting $\tau \to 0$ in (2.2) we get the result. □

Corollary 2.2. *Let (p,γ) be a solution of problem $\mathcal{P}$ then*

$$\int_Q h^3\nabla p \nabla\Big(min\big(\frac{(p-k)^+}{\epsilon},\xi\big)\Big) - \int_Q h\gamma\Big(min\big(\frac{(p-k)^+}{\epsilon},\xi\big)\Big)_x = -\int_Q h_t min\big(\frac{(p-k)^+}{\epsilon},\xi\big)$$

$\forall\epsilon > 0$, $\forall \xi \in \mathcal{D}(\mathbb{R}^2 \times (0,T))$, 2π- x- periodic, $\xi = 0$ on Σ_a.

Proof. We apply the above lemma with $F(z_1,\xi) = min\big(\dfrac{(z_1-k)^+}{\epsilon},\xi\big)$. Note that $F(0,\xi) = 0$ and $F(\phi,\xi) = 0$ on $\Sigma_0 \cup \Sigma_a$. □

The following theorem provides an inequality that generalizes a monotonicity result from [2] that can be used to obtain some qualitative properties of the free boundary.

Theorem 2.3. *Let (p,γ) be a solution of the problem and χ the characteristic function of the set $[p > 0]$. Then*

$$\big(h\gamma\big)_t + \big(h\gamma\big)_x - (h_x + h_t)\chi \geq 0$$

Proof. Let $\xi \in \mathcal{D}(Q)$, $\xi \geq 0$, $\phi = min(\dfrac{p}{\epsilon},\xi) \in L^2(0,T,H^1(\Omega))$. From the above corollary, by choosing $k = 0$ we have

$$\begin{aligned}
0 &= \int_Q h^3\nabla p\nabla min(\frac{p}{\epsilon},\xi) - \int_Q h\gamma\Big(min(\frac{p}{\epsilon},\xi)\Big)_x + \int_Q h_t\Big(min(\frac{p}{\epsilon},\xi)\Big) \\
&= \int_Q h^3\nabla p\nabla min(\frac{p}{\epsilon},\xi) + \int_Q (h_x+h_t) min(\frac{p}{\epsilon},\xi) \\
&= \int_{[p>\epsilon\xi]} h^3\nabla p\nabla\xi + \frac{1}{\epsilon}\int_{[p\leq\epsilon\xi]} h^3|\nabla p|^2 + \int_Q (h_x+h_t) min(\frac{p}{\epsilon},\xi)
\end{aligned}$$

Letting $\epsilon \to 0$ and by using Lebesgue theorem we get

$$\int_Q h^3\nabla p\nabla\xi + \int_{[p>0]}(h_x+h_t)\xi \leq 0$$

From the equality

$$\int_Q h^3\nabla p\nabla\xi - \int_Q h\gamma\xi_t - \int_Q h\gamma\xi_x = 0$$

we get

$$\int_Q h\gamma\xi_t + \int_Q h\gamma\xi_x + \int_Q (h_x+h_t)\chi\xi \leq 0$$

for all $\xi \geq 0$; and then we deduce the result. □

Now, by using some techniques similar to the developed in [8], we prove the following strong continuity for γ.

Theorem 2.4. *Let (p,γ) be a solution of problem $\mathcal{P}$, then we have*

$$h\gamma \in C^0([0,T],L^p(\Omega)), \quad \forall p \in [1,\infty)$$

Proof. For simplicity, we shall denote in this proof by x the spacial variable. Let W' be the topological dual of the set

$$W=\{\xi\in H^1(\Omega):\xi=0 \quad on \quad (0,2\pi)\times\{0\}\cup(0,2\pi)\times\{1\}, \quad \xi \quad 2\pi-periodic\}$$

we have $h\gamma\in C^0((0,T),W')$ ([11]). Therefore

$$\big(h\gamma\big)(x,t+k)\underset{k\to 0}{\to}\big(h\gamma\big)(x,t) \text{ strongly in } W'.$$

Being $(h\gamma)(x,t)\in L^\infty(Q)$, we deduce

$$\big(h\gamma\big)(x,t+k)\underset{k\to 0}{\to}\big(h\gamma\big)(x,t) \text{ weakly* in } L^\infty(\Omega),\ \forall p\in(1,+\infty) \tag{2.4}$$

Now, let $p\in(1,+\infty)$, $\delta>0$ and $e=(1,0)$. From theorem 2.3 the function

$$k\longmapsto\int_\Omega\big(h\gamma\big)(x+ke,t+k)\xi(x)-\int_{\Omega\cap[p>0]}h(x+ke,t+k)\xi(x)$$

defined in $[-\delta,\delta]$, is an increasing function, for any positive function $\xi\in L^p(\Omega)$.
If k_n is a positive decreasing sequence, $0<k_n<\delta$ we get:

$$\begin{aligned}
&\int_\Omega\big(h\gamma\big)(x+k_ne,t+k_n)\xi(x)-\int_{\Omega\cap[p>0]}h(x+k_ne,t+k_n)\xi(x)\geq\\
&\geq\lim_{n\to\infty}\Big[\int_\Omega\big(h\gamma\big)(x+k_ne,t+k_n)\xi(x)-\int_{\Omega\cap[p>0]}h(x+k_ne,t+k_n)\xi(x)\Big]\\
=&\lim_{n\to\infty}\Big[\int_{\Omega-ek_n}\big(h\gamma\big)(x,t+k_n)\xi(x-ek_n)-\int_{(\Omega\cap[p>0])-ek_n}h(x,t+k_n)\xi(x-ek_n)\Big]\\
=&\int_\Omega\big(h\gamma\big)(x,t)\xi(x)-\int_{\Omega\cap[p>0]}h(x,t)\xi(x)
\end{aligned}$$

and we deduce

$$\big(h\gamma\big)(x+k_ne,t+k_n)-\chi(x,t)h(x+k_ne,t+k_n)\geq\big(h\gamma\big)(x,t)-\big(\chi h\big)(x,t). \tag{2.5}$$

If k_n is an increasing negative sequence, $-\delta<k_n<0$ we obtain

$$\big(h\gamma\big)(x+k_ne,t+k_n)-\chi(x,t)h(x+k_ne,t+k_n)\leq\big(h\gamma\big)(x,t)-\big(\chi h\big)(x,t).$$

Now consider p such that $1<p<\infty$

$$\begin{aligned}
\Big|\int_\Omega\big(h\gamma\big)^p(x,t)-\big(h\gamma\big)^p(x,t+k_n)\Big|\leq&\Big|\int_\Omega\big(h\gamma\big)^p(x,t)-\int_{\Omega-ek_n}\big(h\gamma\big)^p(x-ek_n,t)\Big|\\
&+\Big|\int_{\Omega-ek_n}\big(h\gamma\big)^p(x-ek_n,t)-\big(h\gamma\big)^p(x,t+k_n)\Big|\\
&+\Big|\int_\Omega\big(h\gamma\big)^p(x,t+k_n)-\int_{\Omega-ek_n}\big(h\gamma\big)^p(x,t+k_n)\Big|
\end{aligned}$$

It is not difficult to show that, the first and the last terms in the right side converges to 0, when $k_n\to 0$. Using the following inequality:
$\Big|\Big((h\gamma)(a)\Big)^p-\Big((h\gamma)(b)\Big)^p\Big|\leq p\Big(\max_{\bar{Q}}(h)\Big)^{p-1}\Big|\Big(h\gamma)(a)\Big)-\Big((h\gamma)(b)\Big)\Big|$, $\forall p>1$,

the second integral can be written:

$$\Big|\int_{\Omega-ek_n}(h\gamma)^p(x-ek_n,t)-(h\gamma)^p(x,t+k_n)\Big|=\Big|\int_{\Omega}(h\gamma)^p(x,t)-(h\gamma)^p(x+ek_n,t+k_n)\Big|$$

$$\leq\ p\Big(\max_{\bar{Q}}(h)\Big)^{p-1}\int_{\Omega}\Big|(h\gamma)(x,t)-(h\gamma)(x+ek_n,t+k_n)\Big|$$

$$\leq\ p\Big(\max_{\bar{Q}}(h)\Big)^{p-1}\Big[\int_{\Omega}\Big|\Big(h\gamma\Big)(x,t)-\Big(h\chi\Big)(x,t)-\Big(h\gamma\Big)(x+ek_n,t+k_n)+\chi(x,t)h(x+k_ne,t+k_n)\Big|$$

$$+\int_{\Omega}\Big|\chi(x,t)h(x+k_ne,t+k_n)-\chi(x,t)h(x,t)\Big|\Big]$$

Since the function $h(x,t)$ is Lipschitz continuous, the last integral on the right hand side converges to 0 when $k_n\to 0$. From the relation (2.5) we deduce that the first integral on the right hand side converges to 0. Hence we get:

$$\int_{\Omega}(h\gamma)^p(x,t+k_n)\to\int_{\Omega}(h\gamma)^p(x,t)\quad\forall p,\quad 1<p<\infty. \tag{2.6}$$

From (2.4) and (2.6) we deduce

$$\Big(h\gamma\Big)(x,t+k_n)\to\Big(h\gamma\Big)(x,t)\text{ strongly in}L^p(\Omega),1<p<\infty$$

and thus

$$\Big(h\gamma\Big)(x,t+k_n)\to\Big(h\gamma\Big)(x,t)\text{ strongly in}L^p(\Omega),1\leq p<\infty.$$

□

Theorem 2.5. *Let p be the solution of the problem; it satisfies*

$$\int_0^T\int_0^{2\pi}h^3(x,y,t)p(x,y,t)dxdt=p_ay\int_0^T\int_0^{2\pi}h^3(x,y,t)dxdt$$

Proof. Let $\phi(y)\in\mathcal{D}(0,1)$; taking ϕ as a test function in the equation of problem $\mathcal{P}$, being $\phi_t=0$,and $\phi_x=0$ we get

$$\int_Q h^3\nabla p\nabla\phi=0$$

by integrating by parts, and setting $F(y)=\displaystyle\int_0^{2\pi}\int_0^T h^3p$ we get

$$\int_0^1 F(y)\phi''=0$$

Then we obtain

$$\frac{d^2F(y)}{dy^2}=0\quad in\quad\mathcal{D}'(0,1)$$

And $F(y)=Ay+b$.
By considering the conditions over $y=0$ and $y=1$, we get:

$$F(0)=B=\int_0^{2\pi}\int_0^T h^3(x,0,t)p(x,0,t)dxdt=0$$

$$F(1)=A=\int_0^{2\pi}\int_0^T h^3(x,1,t)p(x,1,t)dxdt=p_a\int_0^{2\pi}\int_0^T h^3(x,1,t)dxdt$$

and, consequently:

$$\int_0^{2\pi}\int_0^T h^3(x,y,t)p(x,y,t)dxdt = p_a y\int_0^{2\pi}\int_0^T h^3(x,y,t)dxdt$$

$\square$

3. COMPARISON AND UNIQUENESS

We shall prove a comparison principle for functions satisfying the integral equality in Problem $\mathcal{P}$ when we can compare their values on the boundary Σ_a . This result implies the uniqueness of the solution of the problem.

Let be (p_1,γ_1) and (p_2,γ_2) two pairs satisfying:

$$(p_i,\gamma_i)\in L^2(0,T,H^1(\Omega))\times L^\infty(Q) \quad p_i \text{ is } 2\pi x\text{-periodic} \tag{3.1}$$

$$p_i\geq 0 \quad and \quad \gamma_i\in H(p_i) \quad a.e \quad in \quad Q \tag{3.2}$$

$$-\int_Q h(x,y,t)\gamma_i\xi_t+\int_Q h^3(x,y,t)\nabla p_i\nabla\xi=\int_Q h(x,y,t)\gamma_i\xi_x \quad \forall\xi\in V \tag{3.3}$$

$$p_i|_{\Sigma_a}=p_a^i \quad and \quad p_i|_{\Sigma_0}=0 \tag{3.4}$$

We shall suppose $p_a^1\leq p_a^2$. (3.5)

Theorem 3.1. *Let $(p_i,\gamma_i)(i=1,2)$ be two pairs satisfying (3.1)-(3.5). Then for all $\xi\in\mathcal{D}^+(0,1)$ we have:*

$$\int_Q (p_1-p_2)^+\frac{\partial(h^3\xi')}{\partial y}dxdydt\geq 0$$

Proof. We consider (X_1,t) and (X_2,s) two pairs of variables in the following way:

$$\begin{aligned}(p_1,\gamma_1):(X_1,t,X_2,s) &\longmapsto (p_1(X_1,t),\gamma_1(X_1,t))\\ (p_2,\gamma_2):(X_1,t,X_2,s) &\longmapsto (p_2(X_2,s),\gamma_2(X_2,s))\end{aligned}$$

with $X_1=(x_1,y_1)$ and $X_2=(x_2,y_2)$. Let $\xi\in\mathcal{D}(0,1)$, $\xi\geq 0$ and $\rho_\epsilon(r)=\frac{1}{\epsilon}\rho(\frac{r}{\epsilon})$, $\hat\rho_{\epsilon'}(r)=\frac{1}{\epsilon'}\hat\rho(\frac{r}{\epsilon'})$, $\tilde\rho_{\epsilon''}(r)=\frac{1}{\epsilon''}\tilde\rho(\frac{r}{\epsilon''})$ positive real functions in $\mathcal{D}(\mathbb{R})$, where ρ, $\hat\rho$, $\hat\rho$ are functions with supports in $(-1,1)$.

For $(X_1,t,X_2,s)\in\overline{Q\times Q}$, let $\phi(X_1,t,X_2,s)$ be defined as follows:

$$\phi(X_1,t,X_2,s)=\xi\Big(\frac{y_1+y_2}{2}\Big)\rho_\epsilon\Big(\frac{y_1-y_2}{2}\Big)\hat\rho_{\epsilon'}\Big(\frac{x_1-x_2}{2}\Big)\tilde\rho_{\epsilon''}\Big(\frac{t-s}{2}\Big)$$

If ϵ is small enough ($0<\epsilon<$ dist (supp ξ, $\partial[0,1]$); the functions $\phi(.,.,X_2,s)$ and $\phi(X_1,t,.,.)$ defined for any fixed (X_2,s) and any fixed (X_1,t), vanish on the boundary $\Sigma_0\cup\Sigma_a$.

By other hand, the function ϕ is identically zero when (x_1,x_2) does not belong to the set $B_{\epsilon'}=\{(x_1,x_2)\in\overline{(0,2\pi)\times(0,2\pi)}:|x_1-x_2|\leq 2\epsilon'\}$. In order to get a 2π-periodic function in the independent variables x_1 and x_2, we choose an even function $\hat\rho_{\epsilon'}$ and redefine it when (x_1,x_2) belongs to the set

$$T_{\epsilon'}\cup S_{\epsilon'}=\Big\{(x_1,x_2)\in\overline{(0,2\pi)\times(0,2\pi)}:|x_1-x_2|\geq 2\pi-2\epsilon'\Big\}$$

by setting

$$\hat{\rho}_{\epsilon'}\Big(\frac{x_1-x_2}{2}\Big)=\hat{\rho}_{\epsilon'}\Big(\frac{|x_1-x_2|-2\pi}{2}\Big).$$

Also, we can redefine $\tilde{\rho}_{\epsilon''}$ in the subset

$$T_{\epsilon''}\cup S_{\epsilon''}=\Big\{(t,s)\in\overline{(0,T)\times(0,T)}:|t-s|\leq 2\epsilon'' \quad or \quad |t-s|\geq T-2\epsilon''\Big\}$$

in order to have $\tilde{\rho}_{\epsilon''}(\frac{t-0}{2})=\tilde{\rho}_{\epsilon''}(\frac{0-s}{2})=0$ and $\tilde{\rho}_{\epsilon''}(\frac{t-T}{2})=\tilde{\rho}_{\epsilon''}(\frac{T-s}{2})=0$ for any $t\in[0,T]$ and any $s\in[0,T]$ by setting for all $0<\nu\leq\epsilon''$,

$$\tilde{\rho}_{\nu,\epsilon''}(r)=\frac{\nu}{\epsilon''^2}\tilde{\rho}(\frac{r}{\nu}).$$

For a new parameter $\delta>0$, we consider the function

$$\eta(X_1,t,X_2,s)=min\Big[\frac{(p_1(X_1,t)-p_2(X_2,s))^+}{\delta},\phi(X_1,t,X_2,s)\Big]=min\Big[\frac{(p_1-p_2)^+}{\delta},\xi\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}\Big].$$

Taking into account the above considerations, the functions $\eta(.,.,X_2,s)$ and $\eta(X_1,t,.,.)$ for fixed (X_2,s) and fixed (X_1,t) satisfy test functions boundary conditions.
We shall denote by Q_1, ∇_1 the domain and the gradient vector for the variables (X_1,t); and Q_2, ∇_2 the domain and the gradient vector for the variables (X_2,s).
For any $(X_2,s)\in Q_2$ we can apply lemma 2.1 choosing $F(p_1,\phi)=min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)$ and we get

$$\int_{Q_1}h^3(X_1,t)\nabla_1p_1\nabla_1min\big(\frac{(p_1-p_2)^+}{\delta},\phi\big)dX_1dt-\int_{Q_1}h(X_1,t)\gamma_1(X_1,t)min\big(\frac{(p_1-p_2)^+}{\delta},\phi\big)_{x_1}dX_1dt$$
$$=-\int_Q h_t(X_1,t)min\big(\frac{(p_1-p_2)^+}{\delta},\phi\big)dX_1dt$$

By integrating the above equality in the remaining variables and since $\gamma_1=1$ in the support of the function $min(\frac{(p_1-p_2)^+}{\delta},\phi)$ we get:

$$\int_{Q_1\times Q_2}h^3(X_1,t)\nabla_1p_1\nabla_1min\big(\frac{(p_1-p_))^+}{\delta},\phi\big)-\int_{Q_1\times Q_2}h(X_1,t)min\big(\frac{(p_1-p_2)^+}{\delta},\phi\big)_{x_1}$$
$$=-\int_{Q_1\times Q_2}h_t(X_1,t)min\big(\frac{(p_1-p_2)^+}{\delta},\phi\big) \tag{3.6}$$

Analogously, for any $(x_1,t)\in Q_1$, by taking $F(p_2,\phi)=min\big(\frac{(p_1-p_2))^+}{\delta},\phi\big)$ in the lemma 2.1, we get:

$$\int_{Q_2}h^3(X_2,s)\nabla_2p_2\nabla_2min\big(\frac{(p_1-p_2))^+}{\delta},\phi\big)dX_2ds-\int_{Q_2}h(X_2,s)\gamma_2min\big(\frac{(p_1-p_2)^+}{\delta},\phi\big)_{x_2}dX_2ds$$
$$=\int_{Q_2}h(X_2,s)\gamma_2min\big(\frac{p_1}{\delta},\phi\big)_s dX_2ds-\int_{Q_2}h_s(X_2,s)\Big[min\big(\frac{(p_1-p_2)^+}{\delta},\phi\big)-min\big(\frac{p_1}{\delta},\phi\big)\Big]dX_2ds$$

By integrating in the variables (X_1, t) we get:

$$\begin{aligned}
&\int_{Q_1\times Q_2} h^3(X_2,s)\nabla_2 p_2 \nabla_2 min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big) - \int_{Q_1\times Q_2} h(X_2,s)\gamma_2 min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)_{x_2}\\
&= \int_{Q_1\times Q_2} h(X_2,s)\gamma_2 min\Big(\frac{p_1}{\delta},\phi\Big)_s - \int_{Q_1\times Q_2} h_s(X_2,s)\Big[min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)\Big] \qquad (3.7)\\
&+ \int_{Q_1\times Q_2} h\Big[min\Big(\frac{p_1}{\delta},\phi\Big)\Big]_s
\end{aligned}$$

Then, substracting (3.7) from (3.6), we get:

$$\begin{aligned}
&\int_{Q_1\times Q_2} \Big[h^3(X_1,t)\nabla_1 p_1 \nabla_1 min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big) - h^3(X_2,s)\nabla_2 p_2 \nabla_2 min\Big(\frac{(p_1-p_2))^+}{\delta},\phi\Big)\Big]\\
&- \int_{Q_1\times Q_2} h(X_1,t)\Big(min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)\Big)_{x_1} - h(X_2,s)\gamma_2\Big(min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)\Big)_{x_2} \qquad (3.8)\\
&= \int_{Q_1\times Q_2} h(X_2,s)(1-\gamma_2)\Big(min\Big(\frac{p_1}{\delta},\phi\Big)\Big)_s - \int_{Q_1\times Q_2} \Big[h_t(X_1,t)-h_s(X_2,s)\Big] min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)
\end{aligned}$$

Moreover, from periodicity of $min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)$ and the boundary conditions, we have

$$\begin{aligned}
&\int_{Q_1\times Q_2} h^3(X_1,t)\nabla_1 p_1 \nabla_2 min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big) = 0\\
&\int_{Q_1\times Q_2} h^3(X_2,s)\nabla_2 p_2 \nabla_1 min\Big(\frac{(p_1-p_2)+}{\delta}\phi\Big) = 0\\
&\int_{Q_1\times Q_2} h(X_1,t)\Big(min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)\Big)_{x_2} = 0\\
&\int_{Q_1\times Q_2} h(X_2,s)\gamma_2\Big(min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)\Big)_{x_1} = 0\\
&\int_{Q_1\times Q_2} h(X_2,s)(1-\gamma_2)\Big(min\Big(\frac{p_1}{\delta},\phi\Big)\Big)_t = 0
\end{aligned}$$

By introducing the above relations in (3.8) we get:

$$\begin{aligned}
&\int_{Q_1\times Q_2} \Big[h^3(X_1,t)(\nabla_1+\nabla_2)p_1 - h^3(X_2,s)(\nabla_1+\nabla_2)p_2\Big](\nabla_1+\nabla_2) min\Big(\frac{(p_1-p_2))^+}{\delta},\phi\Big)\\
&- \int_{Q_1\times Q_2} \Big(h(X_1,t)-h(X_2,s)\gamma_2\Big)\Big(min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)\Big)_{x_1+x_2} \qquad (3.9)\\
&= \int_{Q_1\times Q_2} h(X_2,s)(1-\gamma_2)\Big(min\Big(\frac{p_1}{\delta},\phi\Big)\Big)_{t+s}\\
&- \int_{Q_1\times Q_2} \Big(h_t(X_1,t)-h_s(X_2,s)\Big) min\Big(\frac{(p_1-p_2)^+}{\delta},\phi\Big)
\end{aligned}$$

Now we make the following change of variables:

$$z = \frac{X_1+X_2}{2} \quad \sigma = \frac{X_1-X_2}{2} \quad \tau = \frac{t+s}{2} \quad \theta = \frac{t-s}{2}.$$

In the new variables we have $p_1(X_1,t) = p_1(z+\sigma,\tau+\theta)$ and $p_2(X_2,s) = p_2(z-\sigma,\tau-\theta)$; the test function is now

$$min\Big[\frac{(p_1-p_2)^+}{\delta},\xi(z_2)\rho_\epsilon(\sigma_2)\hat{\rho}_{\epsilon'}(\sigma_1)\tilde{\rho}_{\epsilon''}(\theta)\Big]$$

and equality (3.9) can be written as:

$$\begin{aligned}
&\int_{Q_1\times Q_2}\Big[h^3\nabla_z p_1(z+\sigma,\tau+\theta)-h^3\nabla_z p_2(z-\sigma,\tau-\theta)\Big]\nabla_z\Big(min(\frac{(p_1-p_2))^+}{\delta},\phi)\Big)\\
&-\int_{Q_1\times Q_2}\Big[h(z+\sigma,\tau+\theta)-h(z-\sigma,\tau-\theta)\gamma_2\Big]\Big(min(\frac{(p_1-p_2))^+}{\delta},\phi)\Big)_{z_1}\\
&=\int_{Q_1\times Q_2}h(X_2,s)(1-\gamma_2)\Big(min(\frac{p_1}{\delta},\phi)\Big)_\tau\\
&-\int_{Q_1\times Q_2}\Big[h_\tau(z+\sigma,t)-h_\tau(z-\sigma,\tau-\theta)\Big]min(\frac{(p_1-p_2)^+}{\delta},\phi)
\end{aligned}\tag{3.10}$$

Where we are omitting the constant due to the coordinates transform. For the sake of clearness, we enumerate the integrals and transform this expression in

$$J+I=K+L.$$

Now we shall study the integrals of equality (3.10) and their behaviour when limits are taken in the parameters δ, ϵ, ϵ' and ϵ''. For this, let us consider the sets:

$$A^\delta=\big[(p_1-p_2)^+>\delta\xi\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}\big]\quad and\quad B^\delta=\big[0<p_1-p_2<\delta\xi\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}\big].$$

By separating the different regions in the integral I we get

$$\begin{aligned}
I &= -\int_{B^\delta}\Big[h(z+\sigma,\tau+\theta)-h(z-\sigma,\tau-\theta)\gamma_2\Big](\frac{p_1-p_2}{\delta})_{z_1}\\
&\quad-\int_{A^\delta}\Big[h(z+\sigma,\tau+\theta)-h(z-\sigma,\tau-\theta)\gamma_2\Big]\Big(\xi(z_2)\rho_\epsilon(\sigma_2)\hat{\rho}_{\epsilon'}(\sigma_1)\tilde{\rho}_{\epsilon''}(\theta)\Big)_{z_1}\\
&= I_1+I_2
\end{aligned}$$

Since the function $\phi=\xi(z_2)\rho_\epsilon(\sigma_2)\hat{\rho}_{\epsilon'}(\sigma_1)\tilde{\rho}_{\epsilon''}(\theta)$ does not depend of variable z_1, we have $I_2=0$. The integral I_1 can be decomposed in:

$$\begin{aligned}
I_1 &= -\int_{B^\delta}\Big[h(z+\sigma,\tau+\theta)-h(z-\sigma,\tau-\theta)\Big](\frac{p_1-p_2}{\delta})_{z_1}-\int_{B^\delta}h(z-\sigma,\tau-\theta)\big(1-\gamma_2\big)(\frac{p_1-p_2}{\delta})_{z_1}\\
&= I_1^1+I_1^2
\end{aligned}$$

Since ϕ is independent of variable z_1, we can write the first integral as

$$I_1^1 = -\int_{Q_1\times Q_2}\Big[h(z+\sigma,\tau+\theta)-h(z-\sigma,\tau-\theta)\Big]\Big(min(\frac{(p_1-p_2))^+}{\delta},\phi)\Big)_{z_1}$$

By integrating by parts, letting $\delta\to 0$, and using the Lebesgue theorem, we get:

$$\lim_{\delta\to 0}I_1^1=\int_{Q_1\times Q_2}\Big[\frac{\partial h}{\partial z_1}(z+\sigma,\tau+\theta)-\frac{\partial h}{\partial z_1}(z-\sigma,\tau-\theta)\Big]\chi[p_1>p_2]\xi(z_2)\rho_\epsilon(\sigma_2)\hat{\rho}_{\epsilon'}(\sigma_1)\tilde{\rho}_{\epsilon''}(\theta)$$

Now, as $\dfrac{\partial h}{\partial z_1}$ is a Lipschitz continuous function there exists a constant c such that for any (z,τ) and any (σ,θ) it satisfies:

$$\begin{aligned}\Big|\big(\frac{\partial h}{\partial z_1}(z+\sigma,\tau+\theta)-\frac{\partial h}{\partial z_1}(z-\sigma,\tau-\theta)\Big| &\leq c\|(z+\sigma,\tau+\theta)-(z-\sigma,\tau-\theta)\| \\ &= 2c\|(\sigma,\theta)\| \\ &\leq c_1(|\sigma_1|+|\sigma_2|+|\theta|) \end{aligned} \tag{3.11}$$

Since $supp(\rho_\epsilon)=[-\epsilon,\epsilon]$, $supp(\hat\rho_{\epsilon'})=[-\epsilon',\epsilon']$ and $supp(\tilde\rho_{\epsilon''})=[-\epsilon'',\epsilon'']$, we get

$$|\lim_{\delta\to 0} I_1^1| \leq C(\epsilon+\epsilon'+\epsilon'')\int_{Q_1\times Q_2}\xi(z_2)\rho_\epsilon(\sigma_2)\hat\rho_{\epsilon'}(\sigma_1)\tilde\rho_{\epsilon''}(\theta)$$

Being the integral bounded, we obtain:

$$\lim_{\epsilon,\epsilon',\epsilon''\to 0}[\lim_{\delta\to 0} I_1^1]=0$$

Using the old variables in the second integral:

$$\begin{aligned} I_1^2 &= \int_{B^\delta} h(X_2,s)\big(1-\gamma_2\big)\Big[\big(\frac{p_1-p_2}{\delta}\big)_{x_1}+\big(\frac{p_1-p_2}{\delta}\big)_{x_2}\Big] \\ &= \int_{B^\delta} h(X_2,s)\big(1-\gamma_2\big)\big(\frac{p_1}{\delta}\big)_{x_1} \end{aligned}$$

since $1-\gamma_2=0$ when $p_2>0$ and $(p_1-p_2)_s=0$ if $p_2=0$. Being $h(X_2,s)\big(1-\gamma_2\big)$ independent of variable x_1, we can write

$$\begin{aligned} I_1^2 &= \int_{Q_1\times Q_2} h(X_2,s)\big(1-\gamma_2\big)min\Big[\frac{p_1}{\delta},\xi\rho_\epsilon\hat\rho_{\epsilon'}\tilde\rho_{\epsilon''}\Big]_{x_1} - \int_{A^\delta} h(X_2,s)\big(1-\gamma_2\big)\big(\xi\rho_\epsilon\hat\rho_{\epsilon'}\tilde\rho_{\epsilon''}\big)_{x_1} \\ &= \int_{B^\delta} h(X_2,s)\big(1-\gamma_2\big)\big(\xi\rho_\epsilon\hat\rho_{\epsilon'}\tilde\rho_{\epsilon''}\big)_{x_1} \end{aligned}$$

Being the function $h(X_2,s)\big(1-\gamma_2\big)\big(\xi\rho_\epsilon\hat\rho_{\epsilon'}\tilde\rho_{\epsilon''}\big)_{x_1}$ bounded for each ϵ, ϵ' and ϵ'', we conclude

$$\lim_{\delta\to 0}|I_1^2| \leq \lim_{\delta\to 0} C|B^\delta|=0.$$

Finally for the second integral in the equality (3.10) we get

$$\lim_{\epsilon,\epsilon',\epsilon''\to 0}\big(\lim_{\delta\to 0} I\big)=0 \tag{3.12}$$

In the same way as for integral I_1^1 being the derivative h_τ a Lipschitz continuous function, we deduce easily:

$$\lim_{\epsilon,\epsilon',\epsilon''\to 0}\big(\lim_{\delta\to 0}|L|\big)=0 \tag{3.13}$$

For the first integral on the right hand side, we can write

$$\begin{aligned} K &= \int_{Q_1\times Q_2} h(X_2,s)\big(1-\gamma_2\big)\Big(min\big(\frac{p_1}{\delta},\phi\big)\Big)_\tau \\ &= \int_{\big[0<p_1<\delta\phi\big]} h(X_2,s)\big(1-\gamma_2\big)\big(\frac{p_1}{\delta}\big)_t + \int_{\big[p_1>\delta\phi\big]} h(X_2,s)\big(1-\gamma_2\big)\phi_\tau \\ &= \int_{\big[0<p_1<\delta\phi\big]} h(X_2,s)\big(1-\gamma_2\big)\big(\frac{p_1}{\delta}\big)_t \end{aligned}$$

since the function ϕ does not depend on the variable τ. But

$$\begin{aligned}
\int_{\left[0<p_1<\delta\phi\right]} h(X_2,s)\big(1-\gamma_2\big)\frac{p_1}{\delta}_t &= \int_{Q_1\times Q_2} h(X_2,s)\big(1-\gamma_2\big)\Big(min\big(\frac{p_1}{\delta},\phi\big)\Big)_t - \int_{\left[p_1>\delta\phi\right]} h(X_2,s)(1-\gamma_2)\phi_t \\
&= -\int_{\left[p_1>\delta\phi\right]} h(X_2,s)\big(1-\gamma_2\big)\phi_t \\
&= -\int_{Q_1\times Q_2} h(X_2,s)\big(1-\gamma_2\big)\phi_t + \int_{\left[0<p_1<\delta\phi\right]} h(X_2,s)\big(1-\gamma_2\big)\phi_t \\
&= \int_{\left[0<p_1<\delta\phi\right]} h(X_2,s)\big(1-\gamma_2\big)\phi_t
\end{aligned}$$

Being the function under the last integral bounded, we get:

$$\lim_{\delta\to 0}|K| = 0 \tag{3.14}$$

Now, let us study the integral:

$$J = \int_{Q_1\times Q_2}\Big[h^3(z{+}\sigma,\tau{+}\theta)\nabla_z p_1(z{+}\sigma,\tau{+}\theta) - h^3(z{-}\sigma,\tau{-}\theta)\nabla_z p_2(z{-}\sigma,\tau{-}\theta)\Big]\nabla_z\Big(min\big(\frac{(p_1-p_2))^+}{\delta},\phi\big)\Big)$$

By separating the two regions of integration, we write:

$$\begin{aligned}
J &= \int_{A^\delta}\Big[h^3(z+\sigma,\tau+\theta)\nabla_z p_1(z+\sigma,\tau+\theta) - h^3(z-\sigma,\tau-\theta)\nabla_z p_2(z-\sigma,\tau-\theta)\Big]\nabla_z(\xi\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}) \\
&+ \int_{B^\delta}\Big[h^3(z+\sigma,\tau+\theta)\nabla_z p_1(z+\sigma,\tau+\theta) - h^3(z-\sigma,\tau-\theta)\nabla_z p_2(z-\sigma,\tau-\theta)\Big]\nabla_z\Big(\frac{P_1-p_2}{\delta}\Big) \\
&= J_1 + J_2
\end{aligned}$$

Going back to the old variables in J_2, we have:

$$\begin{aligned}
J_2 &= \int_{B^\delta}\Big[h^3(X_1,t)\big|\nabla_1\frac{p_1}{\delta}\big|^2 + h^3(X_2,s)\big|\nabla_2\frac{p_2}{\delta}\big|^2\Big] \\
&\quad - \int_{B^\delta} h^3(X_1,t)\nabla_1 p_1\nabla_2\big(\frac{p_2}{\delta}\big) - \int_{B^\delta} h^3(X_1,t)\nabla_2 p_2\nabla_1\big(\frac{p_1}{\delta}\big)
\end{aligned}$$

The first integral in the right hand side is positive, and the two others integrals satisfy:

$$\begin{aligned}
&-\int_{B^\delta} h^3(X_1,t)\nabla_1 p_1\nabla_2\big(\frac{p_1-p_2}{\delta}\big) = -\int_{Q_1\times Q_2} h^3(X_1,t)\nabla_1 p_1\nabla_2\Big(min\big(\frac{(p_1-p_2))^+}{\delta},\phi\big)\Big) \\
&+\int_{A^\delta} h^3(X_1,t)\nabla_1 p_1\nabla_2\big(\xi\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}\big) \\
= &-\int_{Q_1\times Q_2} h^3(X_1,t)\nabla_1 p_1\nabla_2\Big(min\big(\frac{(p_1-p_2))^+}{\delta},\phi\big)\Big) + \int_{Q_1\times Q_2} h^3(X_1,t)\nabla_1 p_1\nabla_2\big(\xi\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}\big) \\
&-\int_{B^\delta} h^3(X_1,t)\nabla_1 p_1\nabla_2\big(\xi\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}\big) \\
= &-\int_{B^\delta} h^3(X_1,t)\nabla_1 p_1\nabla_2\big(\xi\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}\big)
\end{aligned}$$

since $h^3(X_1,t)\nabla_1 p_1$ does not depend on X_2 and using the boundary conditions of $min\big(\dfrac{(p_1-p_2))^+}{\delta},\phi\big)$. Now, by Holder inequality and since $\lim_{\delta\to 0}|B^\delta|=0$ we conclude:

$$\lim_{\delta\to 0}\Big|-\int_{B^\delta} h^3(X_1,t)\nabla_1 p_1\nabla_2\big(\frac{p_2}{\delta}\big)\Big| \le \lim_{\delta\to 0}|B^\delta|^{1/2}\Big[\int_B h^6(X_1,t)\big|\nabla_1 p_1\big|^2\big|\nabla_2\big(\xi\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}\big)\big|^2\Big]^{1/2}=0.$$

In a similar way, we prove:

$$\lim_{\delta\to 0}\Big|-\int_{B^\delta} h^3(X_2,s)\nabla_2 p_2\nabla_1\big(\frac{p_1}{\delta}\big)\Big|=0$$

Now, by taking into account relations (3.12), (3.13) and (3.14), as well as the equality

$$I+J_1+J_2=K+L$$

we get

$$\lim_{\epsilon,\epsilon',\epsilon''\to 0}\big(\lim_{\delta\to 0}J_1\big)\le 0$$

By Lebesgue theorem, we have

$$\lim_{\delta\to 0}J_1=\int_{Q_1\times Q_2}\Big[h^3(z+\sigma,\tau+\theta)\frac{\partial p_1}{\partial z_2}-h^3(z-\sigma,\tau-\theta)\frac{\partial p_2}{\partial z_2}\Big]\chi[p_1>p_2]\xi'(z_2)\rho_\epsilon(\sigma_2)\hat{\rho}_{\epsilon'}(\sigma_1)\tilde{\rho}_{\epsilon''}(\theta)$$

which, can be decomposed in

$$\begin{aligned}&\int_{Q_1\times Q_2} h^3(z+\sigma,\tau+\theta)\frac{\partial(p_1-p_2)}{\partial z_2}\chi[p_1>p_2]\xi'(z_2)\rho_\epsilon(\sigma_2)\hat{\rho}_{\epsilon'}(\sigma_1)\tilde{\rho}_{\epsilon''}(\theta)\\ &\qquad\int_{Q_1\times Q_2}\Big[h^3(z+\sigma,\tau+\theta)-h^3(z-\sigma,\tau-\theta)\Big]\frac{\partial p_2}{\partial z_2}\chi[p_1>p_2]\xi'(z_2)\rho_\epsilon(\sigma_2)\hat{\rho}_{\epsilon'}(\sigma_1)\tilde{\rho}_{\epsilon''}(\theta)\\ &= J_1^1+J_1^2\end{aligned}$$

The integral J_1^2 satisfies

$$\big|J_1^2\big| \le C\int_{Q_1\times Q_2}\big|\frac{\partial p_2}{\partial z_2}\big|\big|(h^3(z+\sigma,\tau+\theta)-h^3(z-\sigma,\tau-\theta)\big|\rho_\epsilon\hat{\rho}_{\epsilon'}\tilde{\rho}_{\epsilon''}$$

The constant C does not depend on parameters ϵ, ϵ' and ϵ''. By Holder inequality we get:

$$\begin{aligned}\big|J_1^2\big| &\le C\big\|\frac{\partial p_2}{\partial z_2}\big\|\int_{Q_1\times Q_2}\big|(h^3(z+\sigma,\tau+\theta)-h^3(z-\sigma,\tau-\theta)\big|^2\\ &\le C_2(\epsilon+\epsilon'+\epsilon'')^{1/2}\end{aligned}$$

since The function h^3 is Lipschitz continuous. We conclude

$$\lim_{\delta,\epsilon,\epsilon',\epsilon''\to 0}J_1^2=0$$

Moreover, by letting $\epsilon'\to 0$, $\epsilon''\to 0$ and $\epsilon\to 0$ (see[1]) in the integral J_1^1, we get

$$\int_Q h^3\frac{\partial}{\partial z_2}\big(p_1(z,\tau)-p_2(z,\tau)\big)^+\xi'\le 0$$

Now, by denoting $g=(p_1-p_2)^+$, we have:

$$\int_Q h^3\frac{\partial g}{\partial y}\xi'\le 0 \tag{3.15}$$

Let be $\epsilon_0 = d(supp(\xi), \partial[0,1])$, and let us define

$$A = \{(x,t) \in \mathbb{R}^2 \times (0,T) \setminus \Sigma_0 \cup \Sigma_a\}$$

and

$$A_{\epsilon_0} = \{(x,t) \in \mathbb{R}^2 \times (0,T) / d((x,t), \Sigma_0 \cup \Sigma_a) > \epsilon_0/2\}.$$

Let us still denote by g the function

$$g(x,t) = \begin{cases} g(x,t) & a.e \quad in \quad Q \\ 0 \quad in & A \setminus Q \end{cases} \tag{3.16}$$

Now, for $\epsilon < \epsilon_0/2$, let $\rho_\epsilon \in \mathcal{D}(\mathbb{R}^2 \times (0,T))$ be a regularizing sequence with $supp(\rho_\epsilon) \subset B(0,\epsilon)$ and let $g_\epsilon = g * \rho_\epsilon$. Then from (3.15) we deduce:

$$\int_{A_{\epsilon_0}} h^3 \frac{\partial g_\epsilon}{\partial y} \xi' \leq 0$$

Integrating by parts, we get

$$\int_{A_{\epsilon_0}} g_\epsilon \frac{\partial (h^3 \xi')}{\partial y} \geq 0$$

and letting $\epsilon \to 0$, we conclude

$$\int_Q g \frac{\partial (h^3 \xi')}{\partial y} \geq 0$$

Which achieves the proof □

Theorem 3.2. *Let $(p_i, \gamma_i)(i = 1,2)$ be two pairs satisfying (3.1)-(3.5) then $p_1 \leq p_2$ a.e in Q.*

Proof. From theorem (3.1) we have

$$\int_Q (p_1(x,y,t) - p_2(x,y,t))^+ \frac{\partial (h^3 \xi')}{\partial y} dxdydt \geq 0$$

for any $\xi \in \mathcal{D}^+(0,1)$. Equivalently

$$\int_Q (p_1(x,y,t) - p_2(x,y,t))^+ h^3 \xi'' + \int_Q (p_1(x,y,t) - p_2(x,y,t))^+ \frac{\partial h^3}{\partial y} \xi' \geq 0, \quad \forall \xi \in \mathcal{D}^+(0,1)$$

By separating the integration variables and denoting

$$a(y) = \int_0^T \int_0^{2\pi} (p_1 - p_2)^+ h^3 \quad and \quad b(y) = \int_0^T \int_0^{2\pi} (p_1 - p_2)^+ \frac{\partial h^3}{\partial y}$$

we obtain:

$$\int_0^1 \Big(a(y)\xi'' + b(y)\xi' \Big) dy \geq 0, \quad \forall \xi \in \mathcal{D}^+(0,1) \tag{3.17}$$

Now, let us suppose that there exists an interval $[y_0, y_1] \subset [0,1]$ where a is continuous and it satisfies $a(y) > 0$, $\forall y \in (y_0, y_1)$. We distinct two cases:
First case: We suppose that $a(y_0) = a(y_1) = 0$.
Let be $\psi \in C^\infty[y_0, y_1]$ satisfying $\psi''(y) < 0$ $\forall y \in [y_0, y_1]$ (for example $\psi(y) = \sin\big(\frac{y - y_0}{y_1 - y_0}\pi\big)$); and let us consider the following two points boundary problem in the interval $[y_0, y_1]$:

$$\begin{cases} a(y)\xi'' + b(y)\xi' = a(y)\psi'' \\ \xi(y_0) = \xi(y_1) = 0 \end{cases} \tag{3.18}$$

It is not difficult to see that there exists a solution ξ_0 of the problem with bounded derivatives. Moreover, by means of the minimum principle $\xi_0(y) \geq 0$, $\forall y \in [y_0, y_1]$.
Now, let $\delta > 0$ be a positive parameter. We consider the function g defined in the interval $[y_0, y_1]$:

$$g(y) = \begin{cases} 2(\frac{y-y_0}{\delta})^2 & y \in (y_0, y_0 + \frac{\delta}{2}) \\ 1 - 2(1 - \frac{y-y_0}{\delta})^2 & y \in (y_0 + \frac{\delta}{2}, y_0 + \delta) \\ 1 & y \in (y_0 + \delta, y_1 - \delta) \\ 1 - 2(1 - \frac{y_1 - y}{\delta})^2 & y \in (y_1 - \delta, y_1 - \frac{\delta}{2}) \\ 2(\frac{y_1 - y}{\delta})^2 & y \in (y_1 - \frac{\delta}{2}, y_1) \end{cases} \tag{3.19}$$

and we denote by $\tilde{\xi}(y) = g(y)\xi_0(y)$, $\forall y \in [y_0, y_1]$. We have $\tilde{\xi} \in C^2(y_0, y_1)$, $\tilde{\xi}(y_0) = \tilde{\xi}(y_1) = 0$ and $\tilde{\xi}'(y_0) = \tilde{\xi}'(y_1) = 0$. Therefore, we can take $\xi = \tilde{\xi}$ in (3.17) and get:

$$\int_{y_0}^{y_1} \left(a(y)\tilde{\xi}'' + b(y)\tilde{\xi}'\right) dy \geq 0$$

By separating the integration intervals, we decompose this integral in the form:

$$\int_{y_0}^{y_0+\delta} \left(a(y)(g\xi_0)'' + b(y)(g\xi_0)'\right) dy \; + \; \int_{y_0+\delta}^{y_1-\delta} \left(a(y)\xi_0'' + b(y)\xi_0'\right) dy$$
$$+ \int_{y_1-\delta}^{y_1} \left(a(y)(g\xi_0)'' + b(y)(g\xi_0)'\right) dy \geq 0 \tag{3.20}$$

From (3.18) the second integral is strictly negative, and for the two others integrals we have:

$$\int_{y_0}^{y_0+\delta} \left(a(y)(g\xi_0)'' + b(y)(g\xi_0)'\right) dy = \int_{y_0}^{y_0+\delta} a(y)\left(g''\xi_0 + 2g'\xi_0' + g\xi_0''\right) + b(y)\left(g'\xi_0 + g\xi_0'\right) dy$$
$$= \int_{y_0}^{y_0+\delta} \left(a(y)g''\xi_0 + 2a(y)g'\xi_0' + a(y)g\xi_0'' + b(y)g'\xi_0 + b(y)g\xi_0'\right) dy \tag{3.21}$$

Since $|g'(y)| \approx \frac{1}{\delta}$, $|g''(y)| \approx \frac{1}{\delta^2}$ and being the functions $a(y)$ and $\xi_0(y)$ continuous in the interval $(y_0, y_0 + \delta)$, the terms under the last integral in (3.21) are bounded and we obtain:

$$\int_{y_0}^{y_0+\delta} \left(a(y)(g\xi_0)'' + b(y)(g\xi_0)'\right) dy \approx \delta$$

In the same way we obtain:

$$\int_{y_1-\delta}^{y_1} \left(a(y)(g\xi_0)'' + b(y)(g\xi_0)'\right) dy \approx \delta$$

Now, letting $\delta \to 0$ in (3.20) we get

$$\int_{y_0}^{y_1} [a(y)\xi_0'' + b(y)\xi_0'] dy \geq 0.$$

but

$$\int_{y_0}^{y_1} [a(y)\xi_0'' + b(y)\xi_0'] dy = \int_{y_0}^{y_1} a(y)\psi'' < 0$$

Second case: We suppose that $a(y_0) \neq 0$ or $a(y_1) \neq 0$.
Let us define a sequence of functions $(g_n)_{n\geq 1}$ in the interval (y_0, y_1) by:

$$g_n(y) = \begin{cases} (\frac{y-y_0}{\delta})^{\frac{1}{n}} & y \in (y_0, y_0+\delta) \\ 1 & y \in (y_0+\delta, y_1-\delta) \\ (\frac{y_1-y}{\delta})^{\frac{1}{n}} & y \in (y_1-\delta, y_1) \end{cases} \tag{3.22}$$

and consider $a_n(y) = a(y)g_n(y)$ and $b_n(y) = b(y)g_n(y)$. We have $\lim\limits_{n\to\infty} g_n(y) = 1$, $\forall y \in (y_0, y_1)$; and
for all $\xi \in \mathcal{D}^+(0,1)$:

$$\begin{aligned} \int_{y_0}^{y_1} a_n(y)\xi''(y) + b_n(y)\xi'(y) &= \int_{y_0}^{y_1} a(y)\xi''(y) + b(y)\xi'(y) \\ &+ \int_{y_0}^{y_0+\delta} a(y)\big(g_n(y)-1\big)\xi'' + b(y)\big(g_n(y)-1\big)\xi' \\ &+ \int_{y_1-\delta}^{y_1} a(y)\big(g_n(y)-1\big)\xi'' + b(y)\big(g_n(y)-1\big)\xi' \end{aligned} \tag{3.23}$$

Now we take $\xi(y) = g(y)\xi_0(y)$, where g and ξ_0 are the functions introduced in first case. From (3.20)

$$\begin{aligned} \int_{y_0}^{y_1} a_n(y)(g\xi_0)''(y) + b_n(y)(g\xi_0)'(y) &= \int_{y_0+\delta}^{y_1-\delta} a_n(y)\xi_0''(y) + b_n(y)\xi_0'(y) + O(\delta) \qquad (3.24) \\ &= \int_{y_0+\delta}^{y_1-\delta} a(y)\xi_0''(y) + b(y)\xi_0'(y) + O(\delta) \end{aligned}$$

By other hand we have

$$\begin{aligned} &\int_{y_0}^{y_0+\delta} a(y)\big(g_n(y)-1\big)(g\xi_0)'' + b(y)\big(g_n(y)-1\big)(g\xi_0)' \\ &= \int_{y_0}^{y_0+\delta} (a(y)\big(g_n(y)-1\big)g''\xi_0 + 2a(y)\big(g_n(y)-1\big)g'\xi_0' + a(y)\big(g_n(y)-1\big)g\xi_0'' \\ &+ \int_{y_0}^{y_0+\delta} b(y)\big(g_n(y)-1\big)g'\xi_0 + b(y)\big(g_n(y)-1\big)g\xi_0')dy \\ &\approx O(\frac{1}{n}) \end{aligned} \tag{3.25}$$

since $g' \approx \dfrac{1}{\delta}$, $g'' \approx \dfrac{1}{\delta^2}$, $(g_n - 1) \approx \dfrac{1}{n}$ in $(y_0, y_0+\delta)$ and $\xi_0(y_0) = g_0(y_0) = 0$.
Analogously we have

$$\int_{y_1-\delta}^{y_1} a(y)\big(g_n(y)-1\big)\xi'' + b(y)\big(g_n(y)-1\big)\xi' = O(\frac{1}{n}) \tag{3.26}$$

Taking into account estimations (3.24), (3.25) and (3.26)in (3.23), we get

$$\int_{y_0+\delta}^{y_1-\delta} a(y)\xi_0''(y) + b(y)\xi_0'(y) + O(\delta) = \int_{y_0}^{y_1} a(y)(g\xi_0)''(y) + b(y)(g\xi_0)'(y) + O(\frac{1}{n})$$

for all $\delta > 0$ and $n \geq 2$. Letting $\delta \to 0$ and then $n \to \infty$ we get

$$0 > \int_{y_0}^{y_1} a(y)\xi_0''(y) + b(y)\xi_0'(y) \geq 0$$

Finally, we deduce

$$a(y) = \int_0^T \int_0^{2\pi} (p_1 - p_2)^+ h^3 \leq 0 \quad a.e \quad in \quad (0,1)$$

and then $p_1 \leq p_2$ a.e in Q. □

Theorem 3.3. *The problem $\mathcal{P}$ has a unique solution p*

Proof. Is a consequence of theorem 3.2 □

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