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On the Univalence of Two General Integral Operators

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Abstract. In this paper, we define two new general integral operators in the unit disc $\mathbb{U}$. The main object of this paper is to give sufficient conditions for these integral operators, which are defined here by means of the normalized form of univalent functions and belongs to some special subclasses of its, to be univalent in the open unit disk $\mathbb{U}$.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk

$\mathbb{U} = \{z : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We show by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in $\mathbb{U}$ and $\mathcal{S}^{(2)}$ be the class of all odd functions in $\mathcal{S}$. A function $f \in \mathcal{A}$ is said to be starlike if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*$, the class of all such functions. On the other hand, a function $f \in \mathcal{A}$ is said to be convex if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

We denote by C, the class of all such functions.

Let α, β, γ, μ and λ be any complex numbers. Let us denote by $\mathcal{J}_{\alpha,\beta,\gamma}$ and $\mathcal{I}_{\lambda,\mu}$ the analytic functions in $\mathbb{U}$ defined by the formula:

$$\mathcal{J}_{\alpha,\beta,\gamma}(z) = \left(\gamma \int_0^z (g(t))^{\gamma-1}(f'(t))^{\alpha}(t^{-1}h(t))^{\beta}dt\right)^{1/\gamma} \qquad (z \in \mathbb{U}), \tag{1}$$

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$$\mathcal{I}_{\lambda,\mu}(z)=\left(\mu\int\limits_0^z (g(t))^{\mu-1}\left(e^{f(t)}\right)^{\lambda}dt\right)^{1/\mu}\qquad (z\in\mathbb{U}) \tag{2}$$

where f, g and h are functions of the class $\mathcal{S}$ or one of its subclasses. The problem of univalence of the integral operators $\mathcal{J}_{\alpha,\beta,\gamma}$ and $\mathcal{I}_{\lambda,\mu}$ in $\mathbb{U}$ for special cases of parameters α, β, γ, μ and λ, and functions f, g and h were discussed by many authors. For example; Royster [31] for $\gamma=1$, $\beta=0$ showed that $\mathcal{J}_{\alpha,0,1}\in\mathcal{S}$ if $|\alpha|>\frac{1}{3}$, $\alpha\neq 1$. Duren, Shapiro and Shields [12] for $\gamma=1$, $\beta=0$ proved that $\mathcal{J}_{\alpha,0,1}\in\mathcal{S}$ if $|\alpha|\leqslant\frac{\sqrt{5}-2}{3}=0.078...$. Becker [1] showed that $\mathcal{J}_{\alpha,0,1}$ belongs to $\mathcal{S}$ if $\gamma=1$, $\beta=0$ and $|\alpha|\leqslant\frac{1}{6}=0.166...$. Pfaltzgraff [30] in the case $\gamma=1$, $\beta=0$ proved that $\mathcal{J}_{\alpha,0,1}\in\mathcal{S}$ if $|\alpha|\leqslant\frac{1}{4}=0.25$. He showed that the bound $\frac{1}{4}$ is sharp. On the other hand, Causey [2] for $\gamma=1$, $\alpha=0$ showed that $\mathcal{J}_{0,\beta,1}\in\mathcal{S}$ if $0\leqslant\beta\leqslant\frac{\sqrt{5}-2}{4}=0.05...$ and $\mathcal{J}_{0,\beta,1}\notin\mathcal{S}$ if $\beta\geqslant\frac{1}{2}$ $(\beta\neq 1)$. Some authors improved the result of Causey [2]. Nunokawa [25] for $\gamma=1$, $\alpha=0$ proved that $\mathcal{J}_{0,\beta,1}\in\mathcal{S}$ if $0\leqslant\beta\leqslant\left(\sqrt{1025}-25\right)/100=0.07...$. The result was again improved by Causey [3], who showed that $\left|\beta\right|\leqslant\left(\sqrt{2}-1\right)/4=0.102...$. Kim and Merkes [19] for $\gamma=1$, $\alpha=0$ showed that $\mathcal{J}_{0,\beta,1}\in\mathcal{S}$ if $\left|\beta\right|\leqslant\frac{1}{4}=0.25...$. Godula [13] proved that $\mathcal{J}_{\alpha,\beta,1}\in\mathcal{S}$ if $\gamma=1$ and $4|\alpha|+4\left|\beta\right|\leqslant 1$. Also, Miazga and Wesolowski [22] improved the result of Godula [13], they showed that $\mathcal{J}_{\alpha,\beta,1}\in\mathcal{S}$ if $4|\alpha|+3\left|\beta\right|\leqslant 1$. Moldoevanu and Pascu [23] for $\alpha=\beta=0$ proved that $\mathcal{J}_{0,0,\gamma}\in\mathcal{S}$ if $4\left|\gamma-1\right|\leqslant 1$.

Besides, recently many mathematician have engaged in research of univalence criteria of the integral operators which preserve the class $\mathcal{S}$ by aid of well-known lemmas already used in papers on univalence criteria (see, for example, [4–8, 14–17, 27–29, 32, 33]). In terms of different methods, Kanas and Srivastava [18], and Deniz and Orhan [9–11] studied univalence criteria for analytic functions defined in $\mathbb{U}$ by using the Loewner chains method. Kiryakova, Saigo and Srivastava [20] obtained some univalence criteria for certain generalized fractional integral and derivatives, ancompassing all the linear integro-differential operators. Nunokawa et al [26] gave an interesting new condition for univalence of $f\in\mathcal{A}$ by using the same way in proof of the Noshiro-Warshawski theorem (see [24],[34]).

In our paper, we are mainly interested in two integral operators of the types (1) and (2). More precisely, we would like to show that by using some equalities for the functions belonging to the class $\mathcal{S}$, $\mathcal{S}^*$ and C, the univalence of these integral operators which contains functions belongs to the class of univalent functions and some special subclasses of its can be derived easily via a well-known univalence criterion. In particular, we obtain simple sufficient conditions for some integral operators which involve special cases of the functions f, g and h, and complex numbers α, β, γ, λ and μ.

In the proofs of our main results we need the following interesting univalence criterion given by Deniz and Orhan [9].

Lemma 1.1. *(see Deniz and Orhan [9]). Let γ and c be complex numbers such that*

$$\left|\gamma-1\right|<1 \text{ and } |c|\leqslant 1\ (c\neq -1).$$

If the functions $f,g\in\mathcal{A}$ satisfies the following inequality

$$\left|c\,|z|^2+(1-|z|^2)\left[(\gamma-1)\frac{zg'(z)}{g(z)}+\frac{zf''(z)}{f'(z)}\right]\right|\leqslant 1\quad (z\in\mathbb{U})$$

then the function $\mathcal{F}_\gamma$ defined by

$$\mathcal{F}_\gamma(z)=\left[\gamma\int\limits_0^z g^{\gamma-1}(u)f'(u)du\right]^{1/\gamma}$$

is in the class $\mathcal{S}$.

Note that $\mathcal{J}_{1,0,\gamma}(z) = \mathcal{F}_{\gamma}(z)$.

The following lemmas are of fundamental importance in our investigation.

Lemma 1.2. *(Miazga and Wesolowski [22]). For each function $f \in \mathcal{S}$ and a fixed z, $z \in \mathbb{U}$, the inequality*

$$\left|\frac{z}{f(z)} - 1\right| \leq 2|z| + |z|^2$$

holds.

Lemma 1.3. *(Libera ve Ziegler [21]) If $\phi \in \mathcal{S}^*$ and z_0 is a fixed point from the unit disk $\mathbb{U}$, then the function ϕ_*,*

$$\phi_*(z) = \frac{zz_0\phi\left(\frac{z+z_0}{1+z\bar{z}_0}\right)}{\phi(z_0)(z+z_0)(1+z\bar{z}_0)}$$

is a function of the class $\mathcal{S}^*$.

Lemma 1.4. *(Libera ve Ziegler [21]) If $\varphi \in C$ and z_0 is a fixed point from the unit disk $\mathbb{U}$, then the function ψ_*,*

$$\psi_*(z) = \frac{\psi'\left(\frac{z+z_0}{1+z\bar{z}_0}\right)}{\psi'(z_0)(1+z\bar{z}_0)^2}$$

is a function of the class C.

2. Univalence Conditions Associated with the Integral Operators (1) and (2)

Our first main result is a application of Lemma 1.1 and contains sufficient conditions for an general integral operator $\mathcal{J}_{\alpha,\beta,\gamma}$ of the type (1).

Theorem 2.1. *Let α, β and γ any complex numbers. Also let f, g and h are functions of the class $\mathcal{S}$. Moreover, suppose that the following inequality*

$$4|\gamma - 1| + 4|\alpha| + 3|\beta| \leqslant 1 \tag{3}$$

is satisfied. Then, the function $\mathcal{J}_{\alpha,\beta,\gamma}(z)$ defined by (1) is in the class $\mathcal{S}$.

Proof. From (1) we begin by setting

$$\mathcal{J}_{\alpha,\beta,1}(z) = \int_0^z (f'(t))^{\alpha}(t^{-1}h(t))^{\beta}dt \tag{4}$$

so that, obviously,

$$\frac{z\mathcal{J}''_{\alpha,\beta,1}(z)}{\mathcal{J}'_{\alpha,\beta,1}(z)} = \alpha\left(\frac{zf''(z)}{f'(z)}\right) + \beta\left(\frac{zh'(z)}{h(z)} - 1\right). \tag{5}$$

From a well-known transformation of Bieberbach preserving the class of univalent functions $f \in \mathcal{S}$

$$f(z) = \frac{k\left(\frac{z+z_0}{1+z\bar{z}_0}\right) - k(z_0)}{k'(z_0)(1-|z_0|^2)}, \quad z \in \mathbb{U},\ k \in \mathcal{S},$$

z_0−is a fixed point of the unit disk $\mathbb{U}$, we obtain the value of the functional at the point $z = -z_0$

$$\frac{-z_0 f'(-z_0)}{f(-z_0)} = \frac{z_0}{k(z_0)(1-|z_0|^2)} \text{ and } \frac{-z_0 f''(-z_0)}{f'(-z_0)} = \frac{2|z_0|^2 - 2a_2 z_0}{1-|z_0|^2} \tag{6}$$

where a_2 is the second coefficient in Maclaurin expansion of the function k.
From (5) and (6) putting $z_0 = -z$ we have

$$\begin{aligned} &\left| c|z|^2 + \left(1-|z|^2\right)\left((\gamma-1)\frac{zg'(z)}{g(z)} + \frac{z\mathcal{J}''_{\alpha,\beta,1}(z)}{\mathcal{J}'_{\alpha,\beta,1}(z)}\right)\right| \qquad (7) \\ = &\left|(c+2\alpha+\beta)|z|^2 + (\gamma-1)\left[\frac{z}{-l(-z)}\right] + 2\alpha a_2 z + \beta\left[\frac{z}{-q(-z)} - 1\right]\right| \end{aligned}$$

where $q, l \in \mathcal{S}$. Putting $c = -2\alpha - \beta$ and using Lemma 1.2 and the fact that $f \in \mathcal{S}$ from (7) we obtain

$$\begin{aligned} &\left| c|z|^2 + \left(1-|z|^2\right)\left((\gamma-1)\frac{zg'(z)}{g(z)} + \frac{z\mathcal{J}''_{\alpha,\beta,1}(z)}{\mathcal{J}'_{\alpha,\beta,1}(z)}\right)\right| \\ \leqslant\ &\left|\gamma-1\right|\left|\frac{z}{-l(-z)}\right| + 2|\alpha||a_2||z| + \left|\beta\right|\left|\frac{z}{-q(-z)} - 1\right| \\ \leqslant\ &4\left|\gamma-1\right| + 4|\alpha| + 3\left|\beta\right| \end{aligned}$$

Finally, by applying Lemma 1.1, we conclude that the function $\mathcal{J}_{\alpha,\beta,\gamma}(z)$ defined by (1) is in the univalent function class $\mathcal{S}$. This evidently completes the proof of Theorem 2.1. □

Using the method given in the proof of Theorem 2.1, we can prove the following results.

Theorem 2.2. *Let α, β and γ any complex numbers. Also let f, g and h are functions of the class $\mathcal{S}$. Moreover, suppose that the following inequality*

$$\left|\gamma-1\right| + 4|\alpha| + 3\left|\beta\right| \leqslant 1$$

is satisfied. Then, the function defined by

$$\mathcal{J}^*_{\alpha,\beta,\gamma}(z) = \left(\gamma\int_0^z t^{\gamma-1}(f'(t))^{\alpha}(t^{-1}h(t))^{\beta}dt\right)^{1/\gamma} \qquad (z \in \mathbb{U}) \tag{8}$$

is in the class $\mathcal{S}$.

Theorem 2.3. *Let μ and λ any complex numbers. Also let $g \in \mathcal{S}$ and $f \in \mathcal{A}$ such that $z\,f'(z) \in \mathcal{S}^{(2)}$. Moreover, suppose that the following inequality*

$$4\left|\mu-1\right| + |\lambda| \leqslant 1$$

is satisfied. Then, the function $\mathcal{I}_{\lambda,\mu}(z)$ defined by (2) is in the class $\mathcal{S}$.

Theorem 2.4. *Let μ, λ any complex numbers and $f \in \mathcal{A}$ such that $z\,f'(z) \in \mathcal{S}^{(2)}$. Moreover, suppose that the following inequality*

$$\left|\mu-1\right| + |\lambda| \leqslant 1$$

is satisfied. Then, the function defined by

$$\mathcal{I}^*_{\lambda,\mu}(z) = \left(\mu\int_0^z t^{\mu-1}\left(e^{f(t)}\right)^{\lambda}dt\right)^{1/\mu} \qquad (z \in \mathbb{U}) \tag{9}$$

is in the class $\mathcal{S}$.

Putting $\mu = 1$ in Theorem 2.4, we immediately arrive at the following application of Theorem 2.4.

Corollary 2.5. *Let $\lambda \in \mathbb{C}$ and let $f \in \mathcal{A}$ such that $z f'(z) \in \mathcal{S}^{(2)}$. If the inequality $|\lambda| \leqslant 1$ is satisfy, then* $\mathcal{I}_{\lambda}^{*}(z) = \int_{0}^{z} \left(e^{f(t)}\right)^{\lambda} dt \in \mathcal{S}$.

For example, since the function $h(z) = z f'(z) = \sin z \in \mathcal{S}^{(2)}$, from Corollary 2.5 the integral operator $\mathcal{I}_{1}^{*}(z) = \int_{0}^{z} e^{\int_0^t \frac{\sin u}{u} du} dt \in \mathcal{S}$.

Another example, $F(z) = \int_{0}^{z} e^{t + \frac{t^{2n+1}}{(2n+1)^2}} dt \in \mathcal{S}$ for $n = 1, 2, \ldots$.

Theorem 2.6. *Let α, β and γ any complex numbers. Also let $g, h \in \mathcal{S}^*$ and $f \in C$. Moreover, suppose that the following inequality*

$$3\left|\gamma - 1\right| + 2\left|\alpha\right| + 2\left|\beta\right| \leqslant 1$$

is satisfied. Then, the function $\mathcal{J}_{\alpha,\beta,\gamma}(z)$ defined by (1) is in the class $\mathcal{S}$.

Proof. In the proof of Theorem 2.1, we obtained the equalities (4) and (5) where $g, h \in \mathcal{S}^*$ and $f \in C$.

Also, in Lemma 1.3 for $z_0 = -z$ and $\phi_*(z) \in \mathcal{S}^*$ we obtain

$$\frac{z\phi'(z)}{\phi(z)} = \frac{1 + \phi_2 z + |z|^2}{1 - |z|^2} \tag{10}$$

and from Lemma 1.4 for $\psi_*(z) \in C$

$$\frac{z\psi''(z)}{\psi'(z)} = \frac{2\psi_2 z + 2|z|^2}{1 - |z|^2} \tag{11}$$

where ϕ_2 and ψ_2 are the second coefficients in Maclaurin expansion of the functions ϕ_* and ψ_*, respectively. In the equality (5) if we write (10) and (11) we have

$$\frac{z\mathcal{J}''_{\alpha,\beta,1}(z)}{\mathcal{J}'_{\alpha,\beta,1}(z)} = \frac{1}{1 - |z|^2}\left\{(2\alpha b_2 + \beta a_2)z + (2\alpha + 2\beta)|z|^2\right\}$$

and

$$\begin{aligned} &\left| c|z|^2 + \left(1 - |z|^2\right)\left((\gamma - 1)\frac{zg'(z)}{g(z)} + \frac{z\mathcal{J}''_{\alpha,\beta,1}(z)}{\mathcal{J}'_{\alpha,\beta,1}(z)}\right)\right| \\ = \; & \left|(c + \gamma - 1 + 2\alpha + 2\beta)|z|^2 + (\gamma - 1) + ((\gamma - 1)c_2 + 2\alpha b_2 + \beta a_2)z\right| \end{aligned}$$

where a_2, b_2 and c_2 are the second coefficients in Maclaurin expansion of the functions $h \in \mathcal{S}^*$, $f \in C$ and $g \in \mathcal{S}^*$, respectively. Putting $c = 1 - \gamma - 2\alpha - 2\beta$ and using Lemma 1.2 we obtain

$$\begin{aligned} &\left| c|z|^2 + \left(1 - |z|^2\right)\left((\gamma - 1)\frac{zg'(z)}{g(z)} + \frac{z\mathcal{J}''_{\alpha,\beta,1}(z)}{\mathcal{J}'_{\alpha,\beta,1}(z)}\right)\right| \\ \leqslant \; & 3\left|\gamma - 1\right| + 2\left|\alpha\right| + 2\left|\beta\right| \end{aligned}$$

Finally, by applying Lemma 1.1, we conclude that the function $\mathcal{J}_{\alpha,\beta,\gamma}(z)$ defined by (1) is in the univalent function class $\mathcal{S}$. This evidently completes the proof of Theorem 2.6. □

Corollary 2.7. *Let α, β and γ any complex numbers. Also let $g, h \in \mathcal{S}^*$ and $f \in C$. Moreover, suppose that the following inequality*

$$\left|\gamma - 1\right| + 2\left|\alpha\right| + 2\left|\beta\right| \leqslant 1$$

is satisfied. Then, the function $\mathcal{J}^{*}_{\alpha,\beta,\gamma}(z)$ defined by (8) is in the class $\mathcal{S}$.

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