Article:

Bashar, Manijeh, Cumanan, Kanapathippillai orcid.org/0000-0002-9735-7019, Burr, Alister Graham orcid.org/0000-0001-6435-3962 et al. (2 more authors) (2019) On the Uplink Max-Min SINR of Cell-Free Massive MIMO Systems. IEEE Transactions on Wireless Communications. 2021 - 2036. ISSN 1536-1276

https://doi.org/10.1109/TWC.2019.2892463

On the Uplink Max-Min SINR of Cell-Free Massive MIMO Systems

Manijeh Bashar, *Student Member, IEEE*, Kanapathippillai Cumanan, *Member, IEEE*, Alister G. Burr, *Senior Member, IEEE*, Merouane Debbah, *Fellow, IEEE*, and Hien Quoc Ngo, *Member, IEEE*

***Abstract*—A cell-free Massive multiple-input multiple-output (MIMO) system is considered using a max-min approach to maximize the minimum user rate with per-user power constraints. First, an approximated uplink user rate is derived based on channel statistics. Then, the original max-min signal-to-interference-plus-noise ratio (SINR) problem is formulated for optimization of receiver filter coefficients at a central processing unit (CPU), and user power allocation. To solve this max-min non-convex problem, we decouple the original problem into two sub-problems, namely, receiver filter coefficient design and power allocation. The receiver filter coefficient design is formulated as a generalized eigenvalue problem whereas geometric programming (GP) is used to solve the user power allocation problem. Based on these two sub-problems, an iterative algorithm is proposed, in which both problems are alternately solved while one of the design variables is fixed. This iterative algorithm obtains a globally optimum solution, whose optimality is proved through establishing an uplink-downlink duality. Moreover, we present a novel sub-optimal scheme which provides a GP formulation to efficiently and globally maximize the minimum uplink user rate. The numerical results demonstrate that the proposed scheme substantially outperforms existing schemes in the literature.**

***Index Terms*—Cell-free Massive MIMO, max-min resource allocation, geometric programming, uplink-downlink duality, convex optimization, generalized eigenvalue problem.**

I. Introduction

Future fifth generation (5G) wireless communication networks will deliver a wide range of new user services and dramatically increased data rates. Massive multiple-input multiple-output (MIMO) has been recognized as one of the key elements of 5G systems, due to its potential for extremely high spectral efficiency. [1]–[3]. This paper considers cell-free Massive MIMO which has received much attention recently because of its potential to ensure uniformly good service throughput for all users [4]–[9]. Cell-free Massive MIMO is a combination of distributed MIMO and Massive MIMO, and there is no cell boundary [4]. It is a scalable version of network MIMO which is also called coordinated multipoint processing (CoMP) [10], [11]. The distributed access points (APs) are connected to a central processing unit (CPU) via high capacity backhaul links [4]. Cell-free Massive MIMO is thus also a scalable version of the cloud radio access network (CRAN). In CRAN, there are heavy communication burdens on the backhaul, and computation burdens on the CPU, as all signal processing is performed at the CPU [12]. The fog radio access network (FRAN) [13] can overcome some of the problems of CRAN. It moves some signal processing functionalities from the CPU back to the AP, where in this case the APs can also perform part of the signal processing. Hence, the tasks required of the CPU can also be reduced. The more processing is moved to the AP, the less is the burden imposed on the CPU.

In [4], [6], [14] the authors propose that the APs design the linear receivers based on the estimated channels, and that this is carried out locally at the APs. Hence, the CPU exploits only the statistics of the channel for data detection. However, in this paper, we propose to exploit a new receiver filter at the CPU to improve the performance of cell-free Massive MIMO systems. The coefficients of the proposed receiver filter are designed based on only the statistics of the channel, which is different from the linear receiver at the APs. The proposed receiver filter provides more freedom in the design parameters, and hence significantly improves the performance of the uplink of cell-free Massive MIMO. In other words, the receiver filter coefficients are designed after exploiting linear detection at the CPU. Therefore, the uplink problem in the present paper is different from the problem studied in [4], as discussed below.

In this paper, we investigate an uplink max–min signal-to-interference-plus-noise ratio (SINR) problem in a cell-free Massive MIMO system. In particular, we propose a new approach to solve this max-min problem. A similar max–min SINR problem based on SINR known as *SINR balancing* in the literature has been considered in [15]–[20]. In [21], [22], the authors consider MIMO systems and study the problem of max-min user SINR to maximize the smallest user SINR. Note that the same max-min problem is investigated in an uplink cell-free Massive MIMO systems in [4] where user power allocation is utilized by using a bisection search approach.

M. Bashar, K. Cumanan and A. G. Burr are with the Department of Electronic Engineering, University of York, Heslington, York, UK. e-mail: {mb1465, kanapathippillai.cumanan, alister.burr}@york.ac.uk.

M. Debbah is with the Large Networks and Systems Group (LANEAS), CentraleSupelec, Universite Paris-Saclay, Gif-sur-Yvette 91192, France, and also with the Mathematical and Algorithmic Sciences Lab, Huawei Technologies Co., Ltd., Boulogne-Billancourt 92100, France. e-mail: m.debbah@centralesupelec.fr and merouane.debbah@huawei.com.

H. Q. Ngo is with the School of Electronics, Electrical Engineering and Computer Science, Queen's University Belfast, Belfast, UK. hien.ngo@qub.ac.uk.

However, the max-min SINR problem considered in this paper is different from the scheme in [4] due to the design parameters (in terms of receiver filter coefficients and user power allocation) and solution approach. In particular, the receiver filter coefficients and power allocation are optimized in the proposed approach whereas the work in [4] only considered user power allocations. First, we derive the average SINR of the user by incorporating a matched filtering receiver and formulate the corresponding max-min SINR problem. This original max-min problem in terms of receiver filter coefficients and power allocations is not jointly convex. To circumvent this non-convexity issue, we decompose the original problem into two sub-problems, namely, receiver filter coefficient design, and power allocation. It is shown that the receiver filter coefficient design problem can be solved through a generalized eigenvalue problem [23] whereas the user power allocation problems can be formulated using standard geometric programming (GP) [24], [25]. An iterative procedure is proposed whereby at each iteration, one of the sub-problems is solved while the other design variable is fixed. To validate the optimality of the proposed scheme, we show that there exists an equivalent downlink problem to realize the same user rate in the uplink with an equivalent total power constraint and the same receiver filter coefficients. By solving this equivalent problem, the optimality of the proposed scheme in the uplink is proved. The problem of uplink-downlink duality has been investigated in [21], [26]–[29]. Simulation results are provided to demonstrate the performance of the proposed scheme which confirms that the proposed scheme outperforms the scheme in [4] in terms of achieved user rate. In addition, we propose a new sub-optimal max-min SINR scheme using a GP formulation which does not require any iterative approach as in [4]. The contributions and results are as follows:

1. To improve the performance of the system, we propose to use a novel receiver filter, operating at the CPU, which can be designed based only on the statistics of the channel. Note that this is different from the linear matched filtering receiver in [4].
2. The uplink user throughput using the proposed filter is derived based on channel statistics and taking into account the effects of channel estimation errors and the effect of pilot sequences. We propose a novel approach to solve the uplink max-min SINR problem, decoupling the original problem into two sub-problems, which are solved using an iterative algorithm. These sub-problems are formulated as GP and a generalized eigenvalue problem, and both sub-problems are solved at each iteration.
3. We prove that the proposed iterative algorithm provides the globally optimal solution for the original non-convex max-min SINR problem. The optimality of the proposed algorithm is proved through establishing the uplink-downlink duality for cell-free Massive MIMO.
4. We present a sub-optimal max-min SINR scheme by formulating it into a standard GP which does not require an iterative approach and shows the same performance as in [4].
5. We present the complexity analysis of different schemes.

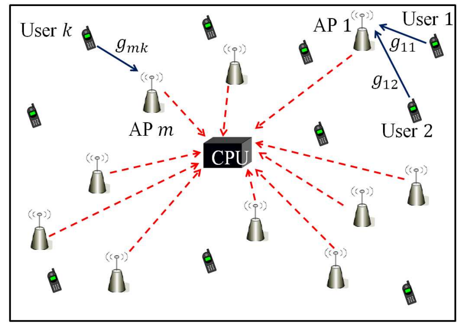

Figure 1. The uplink of a cell-free Massive MIMO system with K single-antenna users and M APs. The solid lines denote the uplink channels and the dashed lines present the backhaul links from the APs to the CPU.

6. We present numerical results supporting the convergence analysis and the theoretical derivations of the optimality of the proposed schemes.

A. Outline

The rest of the paper is organized as follows. Section II describes the system model, and Section III provides performance analysis. The proposed max-min SINR scheme is presented in Section IV and the convergence analysis is provided in Section V. The optimality of the proposed scheme is proved in Section VI. Section VII investigates a sub-optimal max-min SINR scheme. Complexity analysis and a proposed user assignment scheme are presented in Section VIII and Section IX, respectively. Finally, Section X provides numerical results while Section XI concludes the paper.

B. Notation

The following notations are adopted in the rest of the paper. Uppercase and lowercase boldface letters are used for matrices and vectors, respectively. The notation $\mathbb{E}\{\cdot\}$ denotes expectation. $|\cdot|$ stands for absolute value. The conjugate transpose of vector $\mathbf{x}$ is $\mathbf{x}^H$, and $\mathbf{X}^T$ denotes the transpose of matrix $\mathbf{X}$. In addition, $x \sim \mathcal{CN}(0, \sigma^2)$ represents a zero-mean circularly symmetric complex Gaussian random variable with variance σ^2.

II. SYSTEM MODEL

We consider uplink transmission in a cell-free Massive MIMO system with M single-antenna APs and K randomly distributed single-antenna users in the area, as shown in Fig. 1. The channel coefficient between the kth user and the mth AP, g_{mk}, is modeled as [4]

$$g_{mk} = \sqrt{\beta_{mk}} h_{mk}, \tag{1}$$

where β_{mk} denotes the large-scale fading and $h_{mk} \sim \mathcal{CN}(0, 1)$ represents small-scale fading between the kth user and the mth AP.

A. Uplink Channel Estimation

In order to estimate channel coefficients in the uplink, the APs employ an minimum mean-square error (MMSE) estimator. During the training phase, all K users simultaneously transmit their pilot sequences of length τ symbols to the APs. Let $\sqrt{\tau}\boldsymbol{\phi}_k \in C^{\tau\times 1}$, where $\|\boldsymbol{\phi}_k\|^2 = 1$, be the pilot sequence assigned to the kth user. Then, the received signal at the mth AP is given by

$$\mathbf{y}_m^p = \sqrt{\tau p_p}\sum_{k=1}^{K} g_{mk}\boldsymbol{\phi}_k + \mathbf{w}_m^p, \tag{2}$$

where vector $\mathbf{w}_m^p \in \mathbb{C}^{\tau\times 1}$ is the noise whose elements are i.i.d $\mathcal{CN}(0,1)$. Next, the APs exploit the pilot sequence $\boldsymbol{\phi}_k$ to correlate the received signal with the pilot sequence as follows [4]:

$$\check{\mathbf{y}}_{m,k}^p = \boldsymbol{\phi}_k^H \mathbf{y}_m^p = \sqrt{\tau p_p} g_{mk} + \sqrt{\tau p_p}\sum_{k'\neq k}^{K} g_{mk'}\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'} + \dot{w}_{mk}^p,$$

where $\dot{w}_{mk}^p \triangleq \boldsymbol{\phi}_k^H \mathbf{w}_m^p$. The linear MMSE estimate of g_{mk} is

$$\begin{aligned}\hat{g}_{mk} &= \frac{\mathbb{E}\left\{g_{mk}\check{\mathbf{y}}_{m,k}^p\right\}}{\mathbb{E}\left\{\left|\check{\mathbf{y}}_{m,k}^p\right|^2\right\}}\check{\mathbf{y}}_{m,k}^p \\ &= c_{mk}\left(\sqrt{\tau p_p} g_{mk} + \sqrt{\tau p_p}\sum_{k'\neq k}^{K} g_{mk'}\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'} + \dot{w}_{mk}^p\right), \end{aligned}\tag{3}$$

where c_{mk} is obtained as [4]

$$c_{mk} = \frac{\sqrt{\tau p_p}\beta_{mk}}{\tau p_p \sum_{k'=1}^{K}\beta_{mk'}\left|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2 + 1}. \tag{4}$$

Note that, as in [4], we assume that the large-scale fading, β_{mk}, is known. The estimated channels in (3) are used by the APs to design the receiver filter coefficients and determine power allocations at users to maximize the minimum rate of the users. In this paper, we investigate the cases of both random pilot assignment and orthogonal pilots in cell-free Massive MIMO. Here the term “orthogonal pilots” refers to the case where unique orthogonal pilots are assigned to all users, while in “random pilot assignment” each user is randomly assigned a pilot sequence from a set of orthogonal sequences of length τ ($< K$), following the approach of [4], [30].

B. Uplink Transmission

In this subsection, we consider the uplink data transmission, where all users send their signals to the APs. The transmitted signal from the kth user is represented by

$$x_k = \sqrt{\rho\ q_k} s_k, \tag{5}$$

where s_k ($\mathbb{E}\{|s_k|^2\} = 1$) and q_k denote the transmitted symbol and the transmit power from the kth user, respectively. Moreover, ρ refers to the normalized uplink SNR. The received signal at the mth AP from all users is given by

$$y_m = \sqrt{\rho}\sum_{k=1}^{K} g_{mk}\sqrt{q_k} s_k + n_m, \tag{6}$$

where $n_m \sim \mathcal{CN}(0,1)$ is the noise at the mth AP. In addition, a matched filtering approach is employed at the APs, in that the received signal is weighted appropriately. More precisely, the received signal at the mth AP, y_m, is first multiplied by $\hat{g}_{mk}^*$. The resulting $\hat{g}_{mk}^* y_m$ is then forwarded to the CPU for signal detection. In order to improve achievable rate, the forwarded signal is further multiplied by a receiver filter coefficient at the CPU. The aggregated received signal at the CPU can be written as

$$\begin{aligned} r_k &= \sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^* y_m \\ &= \sqrt{\rho}\sum_{k'=1}^{K}\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^* g_{mk'}\sqrt{q_{k'}} s_{k'} + \sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^* n_m. \end{aligned}\tag{7}$$

By collecting all the coefficients $u_{mk},\ \forall\ m$ corresponding to the kth user, we define $\mathbf{u}_k = [u_{1k}, u_{2k}, \cdots, u_{Mk}]^T$ and without loss of generality, it is assumed that $||\mathbf{u}_k|| = 1$. The optimal solution for $\mathbf{u}_k, q_k,\ \forall\ k$ for the considered max-min SINR approach is investigated in Section IV. Similar to [4], [6], [14], we assume that the APs are connected to the CPU via perfect backhaul connections. Such perfect backhaul links might be established through fiber links between the APs and the CPU. Moreover, based on [31], copper-based backhaul links can provide a capacity of 750 Mbits/s for a maximum distance of 1.5 km between the APs and the CPU. In [32]–[36], the authors show that exploiting optimal uniform quantization and wireless microwave links with capacity 100 Mbits/s [37], the performance of limited-backhaul cell-free Massive MIMO system closely approaches the performance of cell-free Massive MIMO with perfect backhaul links.

III. Performance Analysis

In this section, we derive the achievable rate for the considered system model by following a similar approach to [4]. Note that the main difference between the proposed approach and the scheme in [4] is the new set of receiver filter coefficients which are introduced at the CPU to improve the achievable user rate. The benefits of the proposed approach in terms of the achievable uplink rate are demonstrated by the numerical results in Section V. In deriving the achievable rate of each user, it is assumed that the CPU exploits only the knowledge of channel statistics between the users and APs in detecting data from the received signal in (7). Without loss of generality, the aggregate received signal in (7) can be written as

$$\begin{aligned} r_k =& \underbrace{\sqrt{\rho}\mathbb{E}\left\{\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^* g_{mk}\sqrt{q_k}\right\} s_k}_{\mathrm{DS}_k} \\ &+ \underbrace{\sqrt{\rho}\left(\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^* g_{mk}\sqrt{q_k} - \mathbb{E}\left\{\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^* g_{mk}\sqrt{q_k}\right\}\right) s_k}_{\mathrm{BU}_k} \\ &+ \sum_{k'\neq k}^{K}\underbrace{\sqrt{\rho}\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^* g_{mk'}\sqrt{q_{k'}}\, s_{k'}}_{\mathrm{IUI}_{kk'}} + \underbrace{\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^* n_m}_{\mathrm{TN}_k}, \end{aligned}\tag{8}$$

where DS_k and BU_k denote the desired signal (DS) and beamforming uncertainty (BU) for the kth user, respectively,

$$R_k^{\mathrm{UP}} = \log_2\left(1 + \frac{\mathbf{u}_k^H\left(q_k \boldsymbol{\Gamma}_k \boldsymbol{\Gamma}_k^H\right)\mathbf{u}_k}{\mathbf{u}_k^H\left(\sum_{k'\neq k}^K q_{k'}|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}|^2\boldsymbol{\Delta}_{kk'}\boldsymbol{\Delta}_{kk'}^H + \sum_{k'=1}^K q_{k'}\mathbf{D}_{kk'} + \dfrac{1}{\rho}\mathbf{R}_k\right)\mathbf{u}_k}\right). \tag{10}$$

and $\mathrm{IUI}_{kk'}$ represents the inter-user-interference (IUI) caused by the k'th user. In addition, TN_k accounts for the total noise (TN) following the matched filtering. The corresponding SINR of the received signal in (8) can be defined by considering the worst-case of the uncorrelated Gaussian noise as follows [4]:

$$\mathrm{SINR}_k^{\mathrm{UP}} = \frac{|\mathrm{DS}_k|^2}{\mathbb{E}\{|\mathrm{BU}_k|^2\} + \sum_{k'\neq k}^K \mathbb{E}\{|\mathrm{IUI}_{kk'}|^2\} + \mathbb{E}\{|\mathrm{TN}_k|^2\}}. \tag{9}$$

Based on the SINR definition in (9), the achievable uplink rate of the kth user is defined in the following theorem:

Theorem 1. *By employing the matched filtering approach at the APs, the achievable uplink rate of the kth user in the cell-free Massive MIMO system with K randomly distributed single-antenna users and M single-antenna APs is given by (10) (defined at the top of this page). Note that in (10), we have*

$$\boldsymbol{\Gamma}_k = [\gamma_{1k}, \gamma_{2k}, \cdots, \gamma_{Mk}]^T, \tag{11a}$$

$$\boldsymbol{u}_k = [u_{1k}, u_{2k}, \cdots, u_{Mk}]^T, \tag{11b}$$

$$\boldsymbol{\Delta}_{kk'} = [\frac{\gamma_{1k}\beta_{1k'}}{\beta_{1k}}, \frac{\gamma_{2k}\beta_{2k'}}{\beta_{2k}}, \cdots, \frac{\gamma_{Mk}\beta_{Mk'}}{\beta_{Mk}}]^T, \tag{11c}$$

$$\boldsymbol{R}_k = diag\,[\gamma_{1k}, \gamma_{2k}, \cdots, \gamma_{Mk}], \tag{11d}$$

$$\boldsymbol{D}_{kk'} = diag\,[\beta_{1k'}\gamma_{1k}, \beta_{2k'}\gamma_{2k}, \cdots, \beta_{Mk'}\gamma_{Mk}]. \tag{11e}$$

Proof: Please refer to Appendix A. ■

Note that the achievable rate in (10) is a function of only large-scale fading which changes less often than the actual channel. Hence, the rate formula and accordingly the power coefficients only need to be calculated when the large-scale fading changes. Therefore, the APs do not need frequently to update the CPU with the instantaneous channel state and the user rates will change only when the positions of the users change. Moreover, in cell-free massive MIMO, due to the channel hardening property, detection using only the channel statistics is nearly optimal [4].

IV. Proposed Max-Min SINR Scheme

In this section, we formulate the max-min user-fairness problem in the cell-free massive MIMO, where the minimum uplink rates of all users is maximized while satisfying the per-user power constraint. This max-min rate problem can be formulated as the following optimization framework:

$$P_1: \max_{q_k, \mathbf{u}_k} \min_{k=1,\cdots,K} R_k^{\mathrm{UP}}, \tag{12a}$$

$$\text{s.t.} \quad ||\mathbf{u}_k|| = 1, \quad \forall\, k, \tag{12b}$$

$$0 \leq q_k \leq p_{\max}^{(k)}, \quad \forall\, k, \tag{12c}$$

where $p_{\max}^{(k)}$ is the maximum transmit power available at user k. From (10), it can be observed that in the denominator of the expression for the uplink SINR, the power coefficients $q_{k'}, k' \neq k$ are coupled with the receiver filter $\mathbf{u}_k$. Therefore, it is not possible to define a new variable $\mathbf{w}_k = \sqrt{q_k}\mathbf{u}_k$, and solve the problem jointly in terms of $\mathbf{u}_k$ and q_k. As a result, Problem P_1 is not jointly convex in terms of $\mathbf{u}_k$ and power allocation q_k, $\forall\, k$. Therefore, this problem cannot be directly solved through existing convex optimization software. To tackle this non-convexity issue, we decouple the original problem P_1 into two sub-problems: receiver filter coefficient design (i.e., $\mathbf{u}_k$) and the power allocation problem. To obtain a solution for Problem P_1, these sub-problems are alternately solved as explained in the following subsections.

A. Receiver Filter Coefficient Design

In this subsection, we solve the receiver coefficient design problem to maximize the uplink rate of each user for a given set of transmit power allocations at all users. By following the analysis in [21], [26], [27], the receiver filter coefficients (i.e., $\mathbf{u}_k$, $\forall k$) can be obtained by independently maximizing the uplink SINR of each user. Therefore, the optimal receiver filter coefficients for all users for a given set of transmit power allocations can be determined by solving the following optimization problem:

$$P_2: \max_{\mathbf{u}_k} \tag{13a}$$

$$\frac{\mathbf{u}_k^H\left(q_k \boldsymbol{\Gamma}_k \boldsymbol{\Gamma}_k^H\right)\mathbf{u}_k}{\mathbf{u}_k^H\left(\sum_{k'\neq k}^K q_{k'}|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}|^2\boldsymbol{\Delta}_{kk'}\boldsymbol{\Delta}_{kk'}^H + \sum_{k'=1}^K q_{k'}\mathbf{D}_{kk'} + \frac{1}{\rho}\mathbf{R}_k\right)\mathbf{u}_k} \tag{13b}$$

$$\text{s.t.} \quad ||\mathbf{u}_k|| = 1, \quad \forall\, k. \tag{13c}$$

Problem P_2 is a generalized eigenvalue problem [23], where the optimal solutions can be obtained by determining the generalized eigenvalue of the matrix pair $\mathbf{A}_k = q_k\boldsymbol{\Gamma}_k\boldsymbol{\Gamma}_k^H$ and $\mathbf{B}_k = \sum_{k'\neq k}^K q_{k'}|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}|^2\boldsymbol{\Delta}_{kk'}\boldsymbol{\Delta}_{kk'}^H + \sum_{k'=1}^K q_{k'}\mathbf{D}_{kk'} + \frac{1}{\rho}\mathbf{R}_k$ corresponding to the maximum generalized eigenvalue.

B. Power Allocation

In this subsection, we solve the power allocation problem for a given set of fixed receiver filter coefficients which can be formulated as the following max-min problem:

$$P_3: \max_{q_k} \min_{k=1,\cdots,K} \mathrm{SINR}_k^{\mathrm{UP}}, \tag{14a}$$

$$\text{s.t.} \quad 0 \leq q_k \leq p_{\max}^{(k)}. \tag{14b}$$

Without loss of generality, Problem P_3 can be rewritten by introducing a new slack variable as

$$P_4: \max_{t, q_k} \quad t, \tag{15a}$$

$$\text{s.t.} \quad 0 \leq q_k \leq p_{\max}^{(k)}, \quad \forall\, k, \tag{15b}$$

$$\mathrm{SINR}_k^{\mathrm{UP}} \geq t, \quad \forall\, k. \tag{15c}$$

Proposition 1. *Problem P_4 can be formulated into a standard GP.*

Algorithm 1 Proposed algorithm to solve Problem P_1

1. Initialize $\mathbf{q}^{(0)} = [q_1^{(0)}, q_2^{(0)}, \cdots, q_K^{(0)}]$, $i = 1$

2. Repeat

3. $i = i + 1$

4. Set $\mathbf{q}^{(i)} = \mathbf{q}^{(i-1)}$ and determine the optimal receiver coefficients $\mathbf{U}^{(i)} = [\mathbf{u}_1^{(i)}, \mathbf{u}_2^{(i)}, \cdots, \mathbf{u}_K^{(i)}]$ through solving the generalized eigenvalue Problem P_2 in (13)

5. Compute $\mathbf{q}^{(i+1)}$ through solving Problem P_4 in (15)

6. Go back to Step 3 and repeat until required accuracy

Proof: Please refer to Appendix B. ■

Therefore, this problem can be efficiently solved through existing convex optimization software. Based on these two sub-problems, an iterative algorithm is developed by alternately solving each sub-problem at each iteration. The proposed algorithm is summarized in Algorithm 1.

V. Convergence analysis

In this section, the convergence analysis of the proposed Algorithm 1 is provided. Two sub-problems are alternately solved to determine the solution to Problem P_1. At each iteration, one of the design parameters is determined by solving the corresponding sub-problem while other design variable is fixed. Note that each sub-problem provides an optimal solution for the other given design variable. At the ith iteration, the receiver filter coefficients $\mathbf{u}_k^{(i)}$, $\forall k$ are determined for a given power allocation $\mathbf{q}^{(i)}$ and similarly, the power allocation $\mathbf{q}^{(i+1)}$ is updated for a given set of receiver filter coefficients $\mathbf{u}_k^{(i)}$, $\forall k$. The optimal power allocation $\mathbf{q}^{(i+1)}$ obtained for a given $\mathbf{u}_k^{(i)}$ achieves an uplink rate greater than or equal to that of the previous iteration. In addition, the power allocation $\mathbf{q}^{(i)}$ is also a feasible solution in determining $\mathbf{q}^{(i+1)}$ as the receiver filter coefficients $\mathbf{u}_k^{(i+1)}$, $\forall k$ are determined for a given $\mathbf{q}^{(i)}$. This reveals that the achieved uplink rate monotonically increases with each iteration, which can be also observed from the simulation results presented in Figs. 8 and 9. As the achievable uplink max-min rate is upper bounded by a certain value for a given set of per-user power constraints, the proposed algorithm converges to a particular solution. Fortunately, the proposed Algorithm 1 converges to the optimal solution, as we will prove by establishing the uplink-downlink duality in the following section.

VI. Optimality of the proposed max-min SINR algorithm

In this section, we prove the optimality of the proposed max-min SINR scheme in Algorithm 1. In general, converting the original non-convex problem into two sub-problems would remove the global optimality. However, the global optimality of the proposed Algorithm 1 can be proved as follows: first, we show that the solution of the original max-min Problem P_1 can be obtained by solving an uplink problem with an equivalent total power constraint instead of the per-user power constraint. Then, an uplink-downlink duality is established by proving that the same SINRs can be achieved in both the uplink and the downlink with an equivalent total power constraint. In other words, the same SINRs in the uplink Problem P_1 can be realized by solving an equivalent downlink problem. Finally, we present a bisection approach to determine the optimal solution of the equivalent downlink problem. Since both the uplink Problem P_1 and the equivalent downlink problem achieve the same SINRs and the solution of the downlink problem is optimal, it is straightforward to conclude that Algorithm 1 yields the optimal solution for the considered uplink max-min SINR problem in P_1. The details of the proof are provided in the following subsections.

A. Equivalent Max-Min Uplink Problem

In this subsection, we show that both Problem P_1 with per-user power constraint and the uplink max-min fairness problem with the total power constraint achieve the same user rate. In the total power constraint, the maximum available transmit power is defined as the summation of all users' transmit power from the solution of Problem P_1, which can be written as follows:

$$P_{\text{tot}}^c = \sum_{k=1}^{K} q_k^*, \tag{16}$$

where q_k^* is the power allocated to the kth user obtained by solving problem P_1 (Algorithm 1). The equivalent uplink max-min problem with this total power constraint can be formulated as follows:

$$P_5: \max_{q_k, \mathbf{u}_k} \min_{k=1,\cdots,K} \quad R_k^{\text{UP}}, \tag{17a}$$

$$\text{s.t.} \quad ||\mathbf{u}_k|| = 1, \quad \forall\, k, \tag{17b}$$

$$\sum_{k=1}^{K} q_k \leq P_{\text{tot}}^c. \tag{17c}$$

Similar to the original Problem P_1, Problem P_5 is not jointly convex in terms of receiver filter coefficients $\mathbf{u}_k$ and power allocation $q_k, \forall k$. However, we modify Algorithm 1 to incorporate the total power constraint in Problem P_5. Similar to the alternate optimization approach for Problem P_1, Problem P_5 is decoupled into receiver filter coefficient design and power allocation sub problems. The same generalized eigenvalue problem in Problem P_2 is solved to determine the receiver filter coefficients whereas the GP formulation in P_4 is adapted to incorporate the total power constraint (17c). This is a convex constraint (posynomial function in terms of power allocation) and the power allocation problem (GP) with the equivalent total power constraint remains as a convex problem.

Lemma 1. *Both the original Problem P_1 and Problem P_5 yield the same solution with per-user power constraint and equivalent total power constraint.*

Proof: Please refer to Appendix C. ■

$$\mathrm{SINR}_k^{\mathrm{DL}}(\mathbf{U},\mathbf{p}) = \frac{\mathbf{u}_k^H\left(p_k\boldsymbol{\Gamma}_k\boldsymbol{\Gamma}_k^H\right)\mathbf{u}_k}{\sum_{k'\neq k}^K \mathbf{u}_{k'}^H p_{k'}\left|\boldsymbol{\phi}_{k'}^H\boldsymbol{\phi}_k\right|^2\boldsymbol{\Lambda}_{k'k}\boldsymbol{\Lambda}_{k'k}^H\mathbf{u}_{k'}+\sum_{k'=1}^K\mathbf{u}_{k'}^H p_{k'}\boldsymbol{\Upsilon}_{k'k}\mathbf{u}_{k'}+\frac{1}{\rho}}. \tag{18}$$

$$\mathrm{SINR}_k^{\mathrm{UP}}(\mathbf{U},\mathbf{q}) = \frac{\mathbf{u}_k^H\left(q_k\boldsymbol{\Gamma}_k\boldsymbol{\Gamma}_k^H\right)\mathbf{u}_k}{\mathbf{u}_k^H\left(\sum_{k'\neq k}^K q_{k'}\left|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2\boldsymbol{\Delta}_{kk'}\boldsymbol{\Delta}_{kk'}^H+\sum_{k'=1}^K q_{k'}\mathbf{D}_{kk'}+\frac{1}{\rho}\mathbf{R}_{mk}\right)\mathbf{u}_k}. \tag{19}$$

B. Uplink-Downlink Duality for Cell-free Massive MIMO

In this subsection, we establish an uplink-downlink duality for cell-free Massive MIMO systems. In particular, it is shown that the same SINRs (or rate regions) can be realized for all users in the uplink and the downlink with the equivalent total power constraints, respectively [26]–[28], [38]. In other words, the same set of filter coefficients can be utilized in the uplink and the downlink to achieve the same SINRs for all users with different user power allocations. The following theorem defines the achievable downlink rate for cell-free Massive MIMO systems:

Theorem 2. *By employing conjugate beamforming at the APs, the achievable downlink rate of the kth user in the cell-free Massive MIMO system with K randomly distributed single-antenna users and M single-antenna APs is given by (18) (defined at the top of this page).*

Proof: This can be derived by following the same approach as for the uplink in Theorem 1. ■

Note that the symbol $\boldsymbol{\Lambda}_{k'k}$, in (18), is defined as $\boldsymbol{\Lambda}_{k'k} = \left[\frac{\gamma_{1k'}\beta_{1k}}{\beta_{1k'}},\frac{\gamma_{2k'}\beta_{2k}}{\beta_{2k'}},\cdots,\frac{\gamma_{Mk'}\beta_{Mk}}{\beta_{Mk'}}\right]^T$, and $\boldsymbol{\Upsilon}_{k'k}$ denotes the diagonal matrix whose diagonal entries are $[\gamma_{1k'}\beta_{1k},\gamma_{2k'}\beta_{2k},\cdots,\gamma_{Mk'}\beta_{Mk}]$. In addition, p_k, $\forall k$ denotes the downlink power allocation for the kth user. The following Theorem provides the required condition to establish the uplink-downlink duality for cell-free Massive MIMO systems:

Theorem 3. *By employing matched filtering in the uplink and conjugate beamforming in the downlink, to realize the same SINR tuples in both the uplink and the downlink of a cell-free Massive MIMO system, with the same filter coefficients and different transmit power allocations, the following condition should be satisfied:*

$$\sum_{m=1}^M\sum_{k=1}^K\gamma_{mk}\left|w_{mk}\right|^2 = \sum_{k=1}^K q_k^* = P_{tot}^c, \tag{20}$$

where w_{mk} denotes the (m,k)-th entry of matrix $\boldsymbol{W}$ which is defined as follows:

$$\boldsymbol{W} = [\sqrt{p_1}\boldsymbol{u}_1,\sqrt{p_2}\boldsymbol{u}_2,\cdots,\sqrt{p_K}\boldsymbol{u}_K]. \tag{21}$$

Proof: Please refer to Appendix D. ■

C. Equivalent Max-Min Downlink Problem

In this subsection, we present an optimal approach to solve the max-min SINR downlink problem with the equivalent total power constraint. This user-fairness problem can be formulated as follows:

$$P_6:\ \max_{p_k,\mathbf{u}_k}\ \min_{k=1,\cdots,K}\ R_k^{\mathrm{DL}}, \tag{22a}$$
$$\text{s.t.}\quad ||\mathbf{u}_k|| = 1,\quad \forall\ k, \tag{22b}$$
$$\sum_{k=1}^K p_k \le P_{\mathrm{tot}}^c, \tag{22c}$$

where $R_k^{\mathrm{DL}} = \log_2(1+\mathrm{SINR}_k^{\mathrm{DL}})$, and $\mathrm{SINR}_k^{\mathrm{DL}}$ is defined in (18). This problem is difficult to jointly solve in terms of transmit filter coefficients $\mathbf{u}_k$'s and power allocations p_k's. However, similar to [4], it can be reformulated by introducing a new variable by coupling both of these variables as follows:

$$P_7:\ \max_{\mathbf{W}}\ \min_{k=1,\cdots,K}\ R_k^{\mathrm{DL}}, \tag{23a}$$
$$\text{s.t.}\quad \sum_{m=1}^M\sum_{k=1}^K\gamma_{mk}\left|w_{mk}\right|^2 \le P_{\mathrm{tot}}^c. \tag{23b}$$

It can be easily shown that Problem P_7 is quasi-convex, therefore a bisection approach can be exploited to obtain the optimal solution for the original Problem P_7 by sequentially solving the following power minimization problem for a given target SINR t at all users:

$$P_8:\ \min_{\mathbf{W}}\ \sum_{m=1}^M\sum_{k=1}^K\gamma_{mk}\left|w_{mk}\right|^2 \tag{24a}$$
$$\text{s.t.}\ \frac{\mathbf{w}_k^H\left(\boldsymbol{\Gamma}_k\boldsymbol{\Gamma}_k^H\right)\mathbf{w}_k}{\sum_{k'\neq k}^K\mathbf{w}_{k'}^H\left|\boldsymbol{\phi}_{k'}^H\boldsymbol{\phi}_k\right|^2\boldsymbol{\Lambda}_{k'k}\boldsymbol{\Lambda}_{k'k}^H\mathbf{w}_{k'}+\sum_{k'=1}^K\mathbf{w}_{k'}^H\boldsymbol{\Upsilon}_{k'k}\mathbf{w}_{k'}+\frac{1}{\rho}} \ge t, \tag{24b}$$
$$\sum_{m=1}^M\sum_{k=1}^K\gamma_{mk}\left|w_{mk}\right|^2 \le P_{\mathrm{tot}}^c, \tag{24c}$$

where $\mathbf{w}_k$ represents the kth column of the matrix $\mathbf{W}$ defined in (21). Second order cone programming (SOCP) can be exploited to reformulate Problem P_8 as a convex one. More precisely, for a given t, Problem P_8 can be reformulated as follows:

$$P_8^{\mathrm{rewrite}}:\ \min_{\mathbf{W}}\ \sum_{m=1}^M\sum_{k=1}^K\gamma_{mk}\left|w_{mk}\right|^2, \tag{25a}$$

$$s.t.\begin{cases} ||\mathbf{z}_k|| \le \dfrac{\sum_{m=1}^M[\boldsymbol{\Gamma}_k]_m w_{mk}}{\sqrt{t}}, \forall k, & (25b)\\ \sum_{m=1}^M[\boldsymbol{\Lambda}_{k'k}]_m w_{mk'} \le \chi_{k'k}, \forall k'\neq k, & (25c)\\ \sum_{m=1}^M[\boldsymbol{\Upsilon}_{k'k}]_m w_{mk'}^2 \le \psi_{k'k}^2, \forall k, & (25d)\\ \sum_{m=1}^M\sum_{k=1}^K\gamma_{mk}\left|w_{mk}\right|^2 \le P_{\mathrm{tot}}^c, & (25e)\end{cases}$$

where $\chi_{k'k}$ and $\psi_{k'k}^2$ are slack variables, and $[\mathbf{x}]_n$ represents the nth element of vector $\mathbf{x}$. Moreover, we have

$$\mathbf{z}_k \triangleq \Big[\chi_{1k}\boldsymbol{\phi}_1^H\boldsymbol{\phi}_k,\cdots,\chi_{(k-1)k}\boldsymbol{\phi}_{k-1}^H\boldsymbol{\phi}_k,\ \chi_{(k+1)k}\boldsymbol{\phi}_{k+1}^H\boldsymbol{\phi}_k,\cdots,$$
$$\chi_{Kk}\boldsymbol{\phi}_K^H\boldsymbol{\phi}_k,\psi_{1k},\cdots,\psi_{Kk},\frac{1}{\sqrt{\rho}}\Big]. \quad (26)$$

It can be seen that (25b) represents second order cone (SOC) [39]. Hence, Problem P_8^{rewrite} is a SOCP.

Therefore, the optimal solution for Problem P_6 can be derived by extracting the normalized transmit filter coefficients $\mathbf{u}_k$'s and power allocations p_k's as

$$p_k^* = ||\mathbf{w}_k^*||^2,\ \forall k, \quad (27a)$$
$$\mathbf{u}_k^* = \frac{\mathbf{w}_k^*}{||\mathbf{w}_k^*||},\ \forall k, \quad (27b)$$

where $\mathbf{w}_k^*$'s are the optimal solution of Problem P_7. Note that constraint (24b) is an equivalent total power constraint to the per-user power constraint in the original uplink max-min SINR problem in P_1, which is a more relaxed constraint than the per-user power constraint in P_1. However, it is already shown in the previous sub-section that the same SINRs can be realized in both the uplink and the downlink with per-user and the equivalent total power constraints. In addition, the SINRs achieved in the downlink problem in P_7 are optimal and therefore the SINRs achieved in Problem P_1 is optimal. Next, let us again consider the uplink max-min SINR Problems P_1 and P_5. After solving the uplink max-min SINR with total power (with the maximum available power $P_{\text{tot}}^c = \sum_{k=1}^K q_k^*$ defined in Problem P_5), and solving the uplink max-min SINR with per-user power constraints (Problem P_1), we observe that the obtained power allocation for all users ($q_k, \forall k$) after solving Problem P_1 and Problem P_5 are exactly the same. Moreover, after solving Problem P_5 using the proposed Algorithm 1, it is observed that at least one of the users always consumes the maximum power (i.e., there always exists one user with $q_k^* = p_{\max}^{(k)}$). However, it is easy to prove that it is not possible to improve the max-min rate of the system by increasing the power of other users since in this case we would have to decrease the power of user with $q_k^* = p_{\max}^{(k)}$, which decreases the rate of this user, and hence the max-min rate. This validates the optimality of the proposed max-min SINR scheme in Algorithm 1.

VII. Sub-optimal uplink max-min SINR

In this section, we revisit the bisection search based uplink max-min SINR scheme presented in [4]. First, this bisection scheme is summarized and then, we propose another approach to solve this max-min SINR problem by formulating it into a convex optimization framework. This scheme is developed by appropriately allocating transmit powers at each user with an matched filtering technique at the APs. However, no receiver filter coefficient design has been considered at the CPU to enhance the uplink rate as in the previous section. The achievable rate of the kth user is derived in (28) (defined at the bottom of this page), where η_k is the allocated transmit power at user k [4]. For this scenario, the uplink max-min SINR problem can be formulated as the following max-min problem:

$$P_9:\ \max_{\eta_k\geq 0}\ \min_k\ R_k^{\text{UP}}, \quad (29a)$$
$$\text{s.t.}\quad 0 \leq \eta_k \leq p_{max}^{(k)}. \quad (29b)$$

Algorithm 2 Bisection search method to solve Problem P_9

1. Initialize $t_{\min}$, $t_{\max}$ and ϵ
2. Solve Problem P_{10}, defined in (30), with $t = \frac{t_{\max}+t_{\min}}{2}$
3. Repeat
4. If Problem P_{10} is feasible, then $t_{\min} = t$
5. Else, $t_{\max} = t$
6. Repeat until $(t_{\max} - t_{\min}) \leq \epsilon$

A. Bisection Search Method

In this subsection, we present the bisection search method for this quasi-linear problem. As this problem cannot be directly solved in this present form, a series of power minimization problems is solved by setting the same target rate for all users and the corresponding target rate is modified in the next iteration according to the feasibility or infeasibility of the power minimization problem at each iteration. The feasibility of the following power minimization problem is verified for a given target SINR t at all users in each iteration of the bisection search [4]:

$$P_{10}:\min_{\eta_k}\ \sum_{k=1}^K \eta_k, \quad (30a)$$
$$\text{s.t.}\quad 0 \leq \eta_k \leq p_{max}^{(k)},\ \ \forall\ k, \quad (30b)$$
$$\rho\sum_{k'\neq k}^K \eta_{k'}\left(\sum_{m=1}^M \gamma_{mk}\frac{\beta_{mk'}}{\beta_{mk}}\right)^2 \left|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2 t$$
$$+\rho\sum_{k'=1}^K \eta_{k'}\sum_{m=1}^M \gamma_{mk}\beta_{mk'}t + \sum_{m=1}^M \gamma_{mk}t \leq \rho\eta_k\left(\sum_{m=1}^M \gamma_{mk}\right)^2, \forall k. \quad (30c)$$

In this bisection search approach, first an upper and lower bounds of the achievable SINR are set to $t_{\max}$ and $t_{\min}$, respectively and the initial target SINR t is chosen as $(t_{\max}+t_{\min})/2$. If Problem P_{10} is feasible for a given target SINR t, then the lower bound $t_{\min}$ will be set to t and a new target SINR is chosen as $(t_{\max}+t_{\min})/2$ for the next iteration. This procedure is continued until the difference between the upper and the lower bounds is smaller than a predefined threshold ϵ. This bisection search method based uplink max-min SINR scheme is summarized in Algorithm 2. Note that based on the analysis in [21], the bisection search method provides the optimal solution. In the rest of this section, we show that Problem P_9 can be reformulated as a standard GP, which does not require an iterative bisection search to find the optimal solution.

B. Proposed Sub-optimal Scheme

In this subsection, we exploit GP (convex problem) to develop an efficient solution for Problem P_9 defined in (29). As mentioned in previous subsection, Problem P_9 cannot be

Table I
COMPUTATIONAL COMPLEXITY OF DIFFERENT PROBLEMS

Problems	Required arithmetic operations
Problem P_2, given by (13)	$O(KM^3)$
Problem P_4, given by (15)	$O(K^{\frac{7}{2}})$
Problem P_{10}, given by (30)	$\log_2(\frac{t_{\max}-t_{\min}}{\epsilon})\ O(K^4)$
Problem P_{11}, given by (31)	$O(K^{\frac{7}{2}})$

directly solved through the optimization software. Consider the following optimization problem:

$$P_{11}: \quad \max_{t,\eta_k} \quad t, \tag{31a}$$

$$\text{s.t.} \quad 0 \le \eta_k \le p_{\max}^{(k)}, \quad \forall\ k, \tag{31b}$$

$$\text{SINR}_k^{\text{UP}} \ge t, \quad \forall\ k. \tag{31c}$$

Proposition 2. *Problem P_{11} can be reformulated into a GP.*

Proof: The standard form of GP is defined in Appendix B. The SINR constraint in (31c) can be reformulated into the posynomial function. Following a simple transformation, the SINR constraint in (31c) can be represented by the following inequality:

$$\eta_k^{-1}\left(\sum_{k'\neq k}^{K} e_{kk'}\eta_{k'} + \sum_{k'=1}^{K} f_{kk'}\eta_{k'} + r_k\right) < \frac{1}{t}, \tag{32}$$

where

$$e_{kk'} = \frac{\left(\sum_{m=1}^{M}\gamma_{mk}\frac{\beta_{mk'}}{\beta_{mk}}\right)^2 \left|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2}{\left(\sum_{m=1}^{M}\gamma_{mk}\right)^2}, \tag{33a}$$

$$f_{kk'} = \frac{\sum_{m=1}^{M}\gamma_{mk}\beta_{mk'}}{\left(\sum_{m=1}^{M}\gamma_{mk}\right)^2}, \tag{33b}$$

$$r_k = \frac{\sum_{m=1}^{M}\gamma_{mk}}{\rho\left(\sum_{m=1}^{M}\gamma_{mk}\right)^2}. \tag{33c}$$

The transformation in (32) demonstrates that the left-hand side of (32) is a posynomial function. Hence, Problem P_{11} is a standard GP, which completes the proof of Proposition 2. ■
Based on Proposition 2, the objective function and constraints of Problem P_{11} are monomial and posynomials functions in terms of power allocaitons η_ks. Hence, Problem P_{11} is a standard GP, and can be efficiently solved through convex optimization software. Simulation results are provided to show that both bisection and GP based sub-optimal schemes achieve the same user rate for all users.

VIII. COMPLEXITY ANALYSIS

Here, we provide the computational complexity analysis for the proposed Algorithm 1, which solves a generalized eigenvalue problem P_2 and a GP (convex optimization problem) P_4 at each iteration. For the receiver filter coefficient design in P_2, given by (13), an eigenvalue solver requires approximately $O(KM^3)$ flops [40], [41]. Note that the complexity analysis of an eigenvalue solver takes into account the matrix inversion as well. In addition, a standard GP in Problem P_4, defined in (15), can be solved with complexity equivalent to $O(K^{\frac{7}{2}})$ [42, Chapter 10]. The proposed sub-optimal scheme in Section VII solves a GP in Problem P_{11}, defined in (31), which can be solved with $O(K^{\frac{7}{2}})$ [42, Chapter 10]. However, for the scheme in [4], the iterative bisection search method in Algorithm 2 solves a SOCP at each iteration. The complexity of SOCP is $O(K^4)$ in each iteration [43], [44]. Note that the total number of iterations to solve Problem P_9 via a bisection search method is given by $\log_2(\frac{t_{\max}-t_{\min}}{\epsilon})$, where ϵ refers to a predetermined threshold [39]. The number of arithmetic operations required for Algorithm 1, Algorithm 2, and the proposed sub-optimal scheme are provided in Table I.

IX. PROPOSED USER ASSIGNMENT SCHEME

In practice, the total backhaul capacity required between the mth AP and the CPU increases linearly with the total number of users served by the mth AP, which motivates the need to pick a proper set of active users for each AP [32]. In [32], we proposed a user assignment algorithm which can reduce the required capacity of backhaul link by assigning a limited number of users to each AP, however, this paper assumes perfect backhaul links. Hence, for simplicity we assume here that only th_m% of the total number of users can be supported by the mth AP. Hence, we have

$$K_m \le \left(\frac{\text{th}_m}{100} \times K\right), \tag{34}$$

where K_m denotes the size of the set of active users for the mth AP. First, we find an upper bound on the size of the set of active users for each AP. In the next step, we propose for all APs that the users are sorted according to β_{mk}, $\forall k$, and find the K_m users which have the highest values of β_{mk} among all users. If a user is not selected by any AP, we propose to find the AP which has the best link to this user. Then, we add the user to the set of active users for this AP and drop the user which has the lowest β_{mk}, $\forall k$, among active users for that AP which have links to other APs as well. We next solve the original max-min SINR problem with $\tilde{\gamma}_{mk} \leftarrow \gamma_{mk}$, where $\tilde{\gamma}_{mk}$ is given by

$$\tilde{\gamma}_{mk} = \begin{cases} \gamma_{mk}, & m \in \mathcal{S}_k \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

$$R_k^{\text{UP}} = \left(1 + \frac{\rho\eta_k\left(\sum_{m=1}^{M}\gamma_{mk}\right)^2}{\rho\sum_{k'\neq k}^{K}\eta_{k'}\left(\sum_{m=1}^{M}\gamma_{mk}\frac{\beta_{mk'}}{\beta_{mk}}\right)^2\left|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2 + \rho\sum_{k'=1}^{K}\eta_{k'}\sum_{m=1}^{M}\gamma_{mk}\beta_{mk'} + \sum_{m=1}^{M}\gamma_{mk}}\right). \tag{28}$$

where $\mathcal{S}_k$ refers to the set of active APs for the kth user. Note that optimum user assignment scheme can be considered in future work.

X. Numerical Results and Discussion

In this section, we provide numerical simulation results to validate the performance of the proposed max-min SINR scheme with different parameters. A cell-free Massive MIMO system with M APs and K single-antenna users is considered in a $D \times D$ simulation area, where both APs and users are uniformly located at random. In the following subsections, we define the simulation parameters and then present the corresponding simulation results.

A. Simulation Parameters

The channel coefficients between users and APs are modeled in (1) where the coefficient β_{mk} is given by [4]

$$\beta_{mk} = \text{PL}_{mk} 10^{\frac{\sigma_{sh}\ z_{mk}}{10}}, \tag{36}$$

where PL_{mk} is the path loss from the kth user to the mth AP and the second term in (36), $10^{\frac{\sigma_{sh} z_{mk}}{10}}$, denotes the shadow fading with standard deviation $\sigma_{sh} = 8$ dB, and $z_{mk} \sim \mathcal{N}(0, 1)$. In the simulation, an uncorrelated shadowing model is considered and a three-slope model for the path loss is given by [4], [45]

$$\text{PL}_{mk} = \begin{cases} -L - 35\log_{10}(d_{mk}), & d_{mk} > d_1, \\ -L - 15\log_{10}(d_1) - 20\log_{10}(d_{mk}), & d_0 < d_{mk} \leq d_1, \\ -L - 15\log_{10}(d_1) - 20\log_{10}(d_0), & d_{mk} \leq d_0, \end{cases} \tag{37}$$

and $L = 46.3 + 33.9\log_{10}(f) - 13.82\log_{10}(h_{AP}) - \left(1.1\log_{10}(f) - 0.7\right) h_k + \left(1.56\log_{10}(f) - 0.8\right)$, where f denotes the carrier frequency (in MHz), h_{AP} and h_k represent the AP antenna height (in m) and user height (in m), respectively. The noise power is given by $p_n = \text{BW} \times k_B \times T_0 \times W$, where $\text{BW} = 20$ MHz denotes the bandwidth, $k_B = 1.381 \times 10^{-23}$ represents the Boltzmann constant, and $T_0 = 290$ (Kelvin) denotes the noise temperature. Moreover, $W = 9$ dB, and denotes the noise figure. It is assumed that that $\bar{\rho}_p$ and $\bar{\rho}$ denote the pilot sequence and the uplink data powers, respectively, where $\rho_p = \frac{\bar{\rho}_p}{p_n}$ and $\rho = \frac{\bar{\rho}}{p_n}$. In simulations, we set $\bar{\rho}_p = 200$ mW and $\bar{\rho} = 200$ mW. Similar to [4], we assume that the simulation area is wrapped around at the edges which can simulate an area without boundaries. Hence, the square simulation area has eight neighbours. We evaluate the average rate of the system over 300 random realizations of the locations of APs, users and shadow fading. Furthermore, to consider the channel estimation overhead in our comparison, we exploit the net throughput of the system which is defined as [4] $R_{\text{net,k}} = \text{BW}\frac{1 - \frac{\tau}{\tau_c}}{2} R_k$, where τ_c represents the coherence interval in samples.

B. Simulation Results

1) Performance of the Proposed Max-Min SINR Algorithm: In this subsection, we evaluate the performance of the proposed uplink max-min SINR scheme. To assess the performance, a cell-free Massive MIMO system is considered with

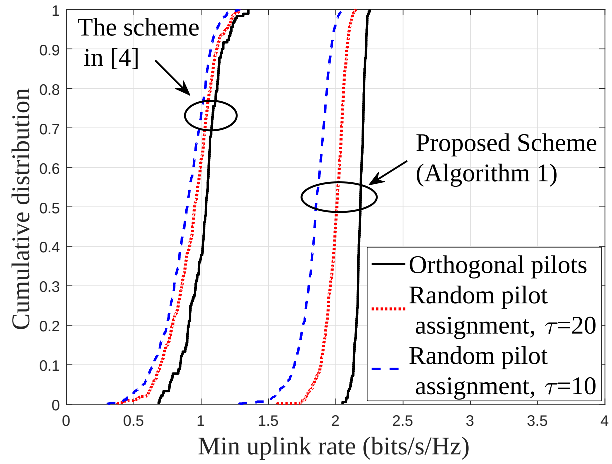

Figure 2. The cumulative distribution of the min uplink rate, with orthogonal and random pilots for $M = 120$, $K = 30$ and $D = 1$ km^2.

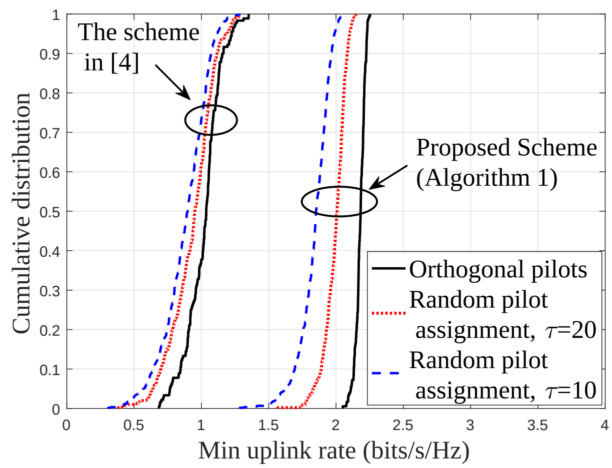

Figure 3. The cumulative distribution of the min uplink rate, with orthogonal and random pilots for $M = 120$, $K = 30$ and $D = 1$ km^2.

120 APs ($M = 120$) and 30 users ($K = 30$) who are randomly distributed over the simulation area of size 1×1 km^2. Fig. 3 presents the cumulative distribution of the achievable uplink rates for the proposed Algorithm 1 and the scheme in [4], for the cases of orthogonal and random pilots. As seen in Fig. 3, the performance of the proposed scheme is almost three times than that of the scheme in [4]. Next, the performance of the algorithm is evaluated for a system with 150 APs ($M = 150$) and 50 users ($K = 50$)[1]. Fig. 4 similarly compares the rate of the proposed algorithm with the scheme in [4]. The simulation results in Figs. 3 and 4 show that the proposed Algorithm 1 achieves more than double the 10% outage capacity compared to the scheme in [4]. Moreover, Figs. 3 and 4 demonstrate that the rate of the proposed max-min SINR approach is more concentrated around the median value.

[1]The analysis in [46] demonstrates that in the limit of Massive MIMO ($M, K \to \infty$ and $\alpha = \frac{M}{K}$), when $\alpha \geq 4$, linear precoding is "virtually optimal", and can be used instead of dirty paper coding (DPC). In this paper, we consider the two cases $\alpha = \frac{120}{30} = 4$ and $\alpha = \frac{150}{50} = 3$.

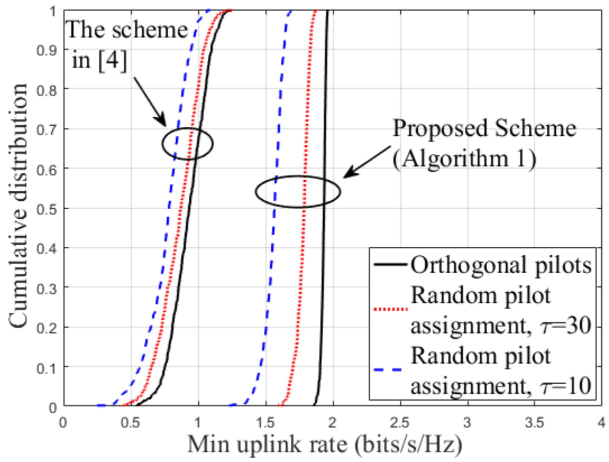

Figure 4. The cumulative distribution of the min uplink rate, with random pilots for M = 150, K = 50 and D = 1 km^2.

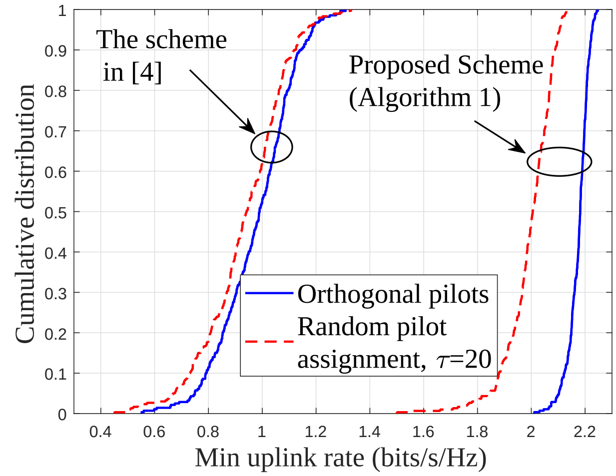

Figure 5. The cumulative distribution of the min uplink rate with proposed user assignment scheme in Section IX, with orthogonal and random pilots for M = 120, K = 30, K_m = 20, $\forall m$, and D = 1 km^2.

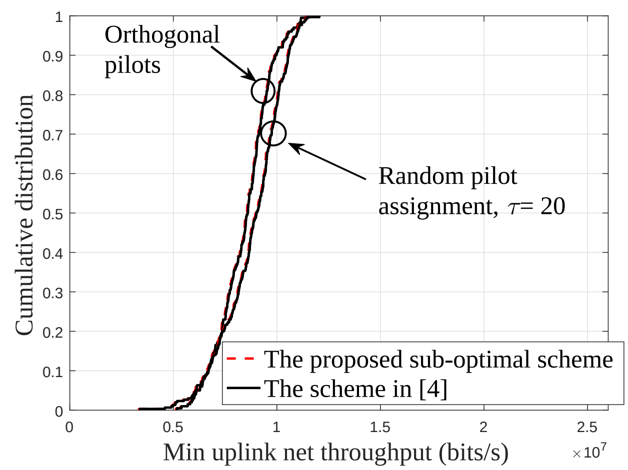

Figure 6. The cumulative distribution of the min uplink net throughput, with orthogonal and random pilots for M = 120, K = 30, D = 1 km^2 and τ_c = 200.

2) User Assignment: In this subsection, the performance of the proposed uplink max-min SINR scheme with the proposed user assignment scheme in Section IX is investigated. We set 120 APs (M = 120) and 30 users (K = 30), and assume 66.66% of the total number of users can be supported by each AP. Based on the analysis in Section IX, this results in a total number of users supported users by each AP of $K_m = 20, \forall m$. Fig. 5 presents the cumulative distribution of the achievable uplink rates for the proposed Algorithm 1 and the scheme in [4] with the proposed user assignment algorithm in Section IX, for the cases of orthogonal and random pilots. As seen in Fig. 5, the performance of the proposed scheme is significantly better than that of the scheme in [4]. In addition, it can be observed from figure that the rate of the proposed Algorithm 1 is more concentrated around the median. Interestingly, by comparing the results in Figs. 3 and 5, the performance degradation is negligible exploiting the proposed user assignment scheme whereas based on the analysis in [32], the backhaul rate is significantly reduced.

3) Performance of the Proposed Sub-optimal Scheme: In this subsection, we study the effect of the proposed sub-optimal scheme on the system performance. Fig. 6 compares the cumulative distribution of the achievable uplink net throughput for our proposed sub-optimal scheme with scheme in [4]. In order to generate the numerical results for the scheme in [4], the iterative bisection search method in Algorithm 2 is used whereas the proposed sub-optimal scheme solves the standard GP with polynomial time complexity. In Fig. 6, the same cell-free Massive MIMO system is considered with 120 APs (M = 120) and 30 users (K = 30). Figs. 6 and 7 compare the performance of the proposed sub-optimal approach with the scheme in [4] for different system parameters. As evidenced from these numerical results, both proposed GP approach and the bisection search scheme in [4] shows the same performance in terms of the achieved user rate. However, the scheme in [4] is developed through iterative bisection search in which a SOCP is solved at each iteration, whereas the proposed GP approach does not require any iterative methods and solves the problem with polynomial time complexity.

4) Convergence: Next, we provide simulation results to validate the convergence of the proposed algorithm for a set of different channel realizations. These results are generated over the simulation area of size 1×1 km^2 with random and orthogonal pilot sequences. Fig. 8 investigates the convergence of the proposed Algorithm 1 with 120 APs (M = 120) and 30 users (K = 30) and orthogonal pilot sequences, whereas Fig. 9 demonstrates the convergence of the proposed Algorithm 1 for the case of M = 150 APs and K = 50. The figures confirm that the proposed algorithm converges after a few iterations, while the minimum rate of the users increases with the iteration number.

5) Uplink-Downlink Duality in Cell-Free Massive MIMO System: Here, the simulation results are provided to support the theoretical derivations of the uplink-downlink duality and

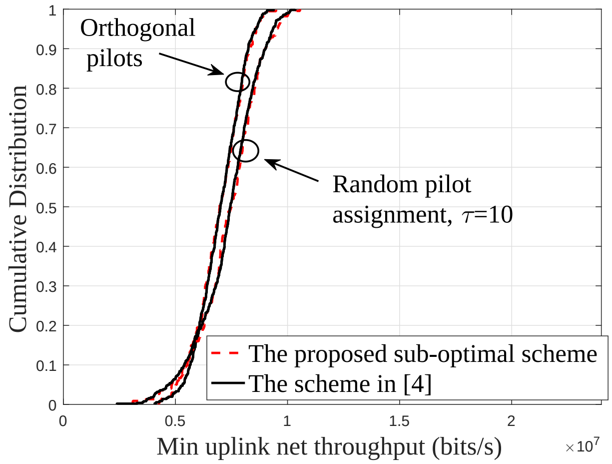

Figure 7. The cumulative distribution of the min uplink net throughput, with orthogonal and random pilots for $M = 150$, $K = 50$, $D = 1$ km^2 and $\tau_c = 200$.

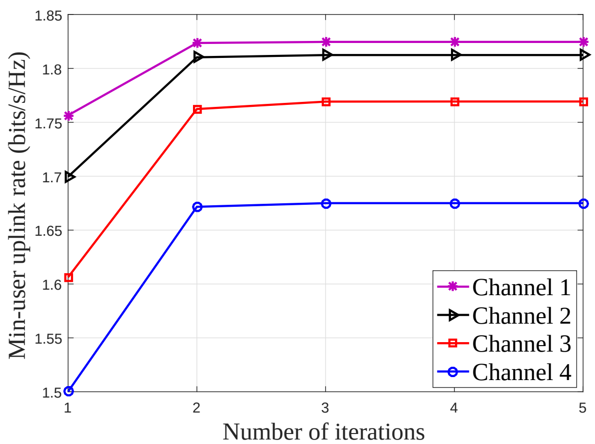

Figure 8. The convergence of the proposed max-min SINR approach (Algorithm 1) for $M = 120$, $K = 30$ and $D = 1$ km with orthogonal pilots.

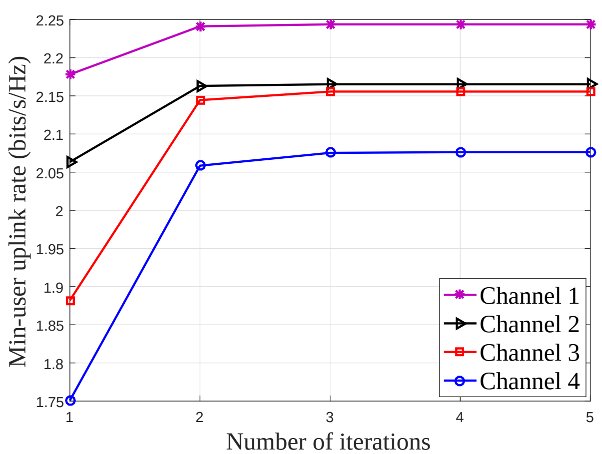

Figure 9. The convergence of the proposed max-min SINR approach (Algorithm 1) for $M = 150$, $K = 50$, $D = 1$ km, and the length of the pilot sequences is set to 30 ($\tau = 30$).

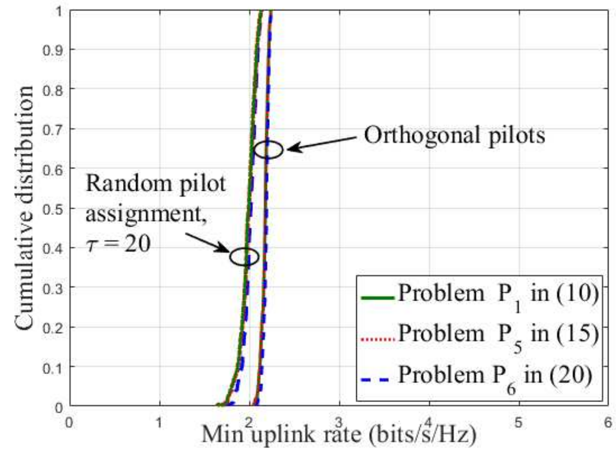

Figure 10. The cumulative distribution of the min uplink rate for the original problem with per-user power constraint (Problem P_1), the equivalent uplink problem with total power constraint (Problem P_5), and the equivalent downlink problem (Problem P_6), with orthogonal and random pilots for $M = 120$, $K = 30$ and $D = 1$ km.

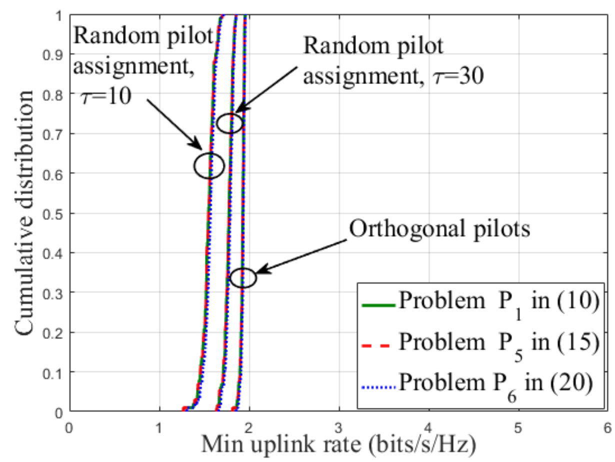

Figure 11. The cumulative distribution of the min uplink rate for the original problem with per-user power constraint (Problem P_1), the equivalent uplink problem with total power constraint (Problem P_5), and the equivalent downlink problem (Problem P_6), with orthogonal and random pilots for $M = 150$, $K = 50$ and $D = 1$ km.

the optimality of Algorithm 1. It is assumed that users are randomly distributed through the simulation area of size 1×1 km^2. Figs. 10 and 11 compare the cumulative distribution of the achievable uplink rates between the original uplink max-min problem (Problem P_1), the equivalent uplink problem (Problem P_5) and the equivalent downlink problem (Problem P_6). In Fig. 10, the minimum uplink rate is obtained for a system with 120 APs ($M = 120$) and 30 users ($K = 30$) whereas Fig. 11 presents the same results for 150 APs ($M = 150$) and 50 users ($K = 50$). The simulation results provided in Figs. 10 and 11 validate our result that the problem formulations P_1, P_5 and P_6 are equivalent and achieve the same minimum user rate. In addition, these results support our result on the uplink-downlink duality for cell-free Massive MIMO in Section VI and the proof of optimality of Algorithm 1.

XI. Conclusions

We have considered cell-free Massive MIMO which has the potential to meet the capacity requirements of 5G. Compared to the collocated massive MIMO, the distributed version brings the APs much closer to the "cell edge" users, which leads to a uniformly good service for all users. We have investigated the uplink max-min SINR problem in cell-free Massive MIMO systems and proposed an optimal solution to maximize the minimum uplink user rate. To realize the solution, the original max-min problem was divided into two sub-problems which were iteratively solved by formulating them respectively as a generalized eigenvalue problem and as GP. The optimality of the proposed solution has been validated by establishing the uplink-downlink duality for cell-free Massive MIMO systems. Next, a novel sub-optimal scheme was developed through formulating the max-min power allocation problem as a standard GP, which efficiently and globally solves the max-min SINR problem. Simulation results have been provided to demonstrate the effectiveness and the optimality of the proposed schemes in comparison with the existing schemes. In addition, these results confirm that the proposed max-min SINR algorithm can significantly improve the uplink user rate, compared to existing algorithms.

Appendix A: Proof of Theorem 1

The desired signal for user k is given by

$$\mathrm{DS}_k = \sqrt{\rho}\mathbb{E}\left\{\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^{*}g_{mk}\sqrt{q_k}\right\} = \sqrt{\rho q_k}\sum_{m=1}^{M} u_{mk}\gamma_{mk}. \tag{38}$$

Hence,

$$|\mathrm{DS}_k|^2 = \rho q_k\left(\sum_{m=1}^{M} u_{mk}\gamma_{mk}\right)^2. \tag{39}$$

Moreover, the term $\mathbb{E}\{|\mathrm{BU}_k|^2\}$ can be obtained as

$$\begin{aligned}
&\mathbb{E}\left\{|\mathrm{BU}_k|^2\right\}\\
&= \rho\mathbb{E}\left\{\left|\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^{*}g_{mk}\sqrt{q_k} - \rho\mathbb{E}\left\{\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^{*}g_{mk}\sqrt{q_k}\right\}\right|^2\right\}\\
&= \rho\sum_{m=1}^{M} q_k u_{mk}^2\left(\mathbb{E}\left\{\left|\hat{g}_{mk}^{*}g_{mk} - \mathbb{E}\left\{\hat{g}_{mk}^{*}g_{mk}\right\}\right|^2\right\}\right)\\
&= \rho q_k\sum_{m=1}^{M} u_{mk}^2\gamma_{mk}\beta_{mk},
\end{aligned} \tag{40}$$

where the last equality comes from the analysis in [4, Appendix A], and using the following fact that; $\gamma_{mk} = \mathbb{E}\{|\hat{g}_{mk}|^2\} = \sqrt{\tau p_p}\beta_{mk}c_{mk}$. The term $\mathbb{E}\{|\mathrm{IUI}_{kk'}|^2\}$ is derived as

$$\begin{aligned}
\mathbb{E}\left\{|\,\mathrm{IUI}_{kk'}|^2\right\} &= \rho\mathbb{E}\left\{\left|\sum_{m=1}^{M} u_{mk}\hat{g}_{mk}^{*}g_{mk'}\sqrt{q_{k'}}\right|^2\right\}\\
&= p\mathbb{E}\left\{\left|\sum_{m=1}^{M} c_{mk}u_{mk}g_{mk'}\sqrt{q_{k'}}\left(\sqrt{\tau p_p}\sum_{i=1}^{K} g_{mi}\boldsymbol{\phi}_k^H\boldsymbol{\phi}_i + \boldsymbol{\phi}_k^H\mathbf{n}_{p,m}\right)^{*}\right|^2\right\}\\
&= \underbrace{\rho\, q_{k'}\mathbb{E}\left\{\left|\sum_{m=1}^{M} c_{mk}u_{mk}g_{mk'}\tilde{n}_{mk}^{*}\right|^2\right\}}_{A}\\
&\quad + \underbrace{\rho\tau p_p\mathbb{E}\left\{q_{k'}\left|\sum_{m=1}^{M} c_{mk}u_{mk}g_{mk'}\left(\sum_{i=1}^{K} g_{mi}\boldsymbol{\phi}_k^H\boldsymbol{\phi}_i\right)^{*}\right|^2\right\}}_{B},
\end{aligned} \tag{41}$$

where the third equality in (41) is due to the fact that for two independent random variables X and Y and $\mathbb{E}\{X\} = 0$, we have $\mathbb{E}\{|X+Y|^2\} = \mathbb{E}\{|X|^2\} + \mathbb{E}\{|Y|^2\}$ [4]. Since $\tilde{n}_{mk} = \boldsymbol{\phi}_k^H\mathbf{n}_{p,m} \sim \mathcal{CN}(0,1)$ is independent of the term $g_{mk'}$, the term A in (41) is given immediately by

$$A = q_{k'}\sum_{m=1}^{M} c_{mk}^2 u_{mk}^2\beta_{mk'}. \tag{42}$$

The term B in (41) can be obtained as

$$\begin{aligned}
B =& \underbrace{\tau p_p q_{k'}\mathbb{E}\left\{\left|\sum_{m=1}^{M} c_{mk}u_{mk}\,|g_{mk'}|^2\,\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2\right\}}_{C}\\
&+ \underbrace{\tau p_p q_{k'}\mathbb{E}\left\{\left|\sum_{m=1}^{M} c_{mk}u_{mk}g_{mk'}\left(\sum_{i\neq k'}^{K} g_{mi}\boldsymbol{\phi}_k^H\boldsymbol{\phi}_i\right)^{*}\right|^2\right\}}_{D}.
\end{aligned} \tag{43}$$

The first term in (43) is given by

$$\begin{aligned}
C &= \tau p_p q_{k'}\mathbb{E}\left\{\left|\sum_{m=1}^{M} c_{mk}u_{mk}\,|g_{mk'}|^2\,\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2\right\}\\
&= 2\tau p_p q_{k'}\left|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2\sum_{m=1}^{M} c_{mk}^2u_{mk}^2\beta_{mk'}^2 + \tau p_p q_{k'}\\
&\quad\mathbb{E}\left\{\left|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2\sum_{m=1}^{M}\sum_{n\neq m}^{M} c_{mk}c_{nk}u_{mk}u_{nk}\,|g_{mk'}|^2|g_{nk'}|^2\right\}\\
&= \tau p_p q_{k'}\left|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2\sum_{m=1}^{M} c_{mk}^2u_{mk}^2\beta_{mk'}^2\\
&\quad + q_{k'}\left|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}\right|^2\left(\sum_{m=1}^{M} u_{mk}\gamma_{mk}\frac{\beta_{mk'}}{\beta_{mk}}\right)^2,
\end{aligned} \tag{44}$$

where the last equality is derived based on the fact $\gamma_{mk} = \sqrt{\tau p_p}\beta_{mk}c_{mk}$. The second term in (43) can be obtained as

$$D = \tau p_p q_{k'} \mathbb{E}\left\{\left|\sum_{m=1}^{M} c_{mk} u_{mk} g_{mk'} \left(\sum_{i\neq k'}^{K} g_{mi} \boldsymbol{\phi}_k^H \boldsymbol{\phi}_i\right)^*\right|^2\right\}$$
$$= \tau p_p \sum_{m=1}^{M} \sum_{i\neq k'}^{K} q_{k'} c_{mk}^2 u_{mk}^2 \beta_{mk'} \beta_{mi} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_i\right|^2. \quad (45)$$

Hence, (41) can be written as

$$\mathbb{E}\left\{|\mathrm{IUI}_{kk'}|^2\right\} = \underbrace{q_{k'} \sum_{m=1}^{M} c_{mk}^2 u_{mk}^2 \beta_{mk'}}_{C_1}$$
$$+ \tau p_p q_{k'} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}\right|^2 \sum_{m=1}^{M} c_{mk}^2 u_{mk}^2 \beta_{mk'}^2$$
$$+ \tau p_p q_{k'} \sum_{m=1}^{M} \sum_{i\neq k'}^{K} c_{mk}^2 u_{mk}^2 \beta_{mk'} \beta_{mi} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_i\right|^2$$
$$q_{k'} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}\right|^2 \left(\sum_{m=1}^{M} u_{mk} \gamma_{mk} \frac{\beta_{mk'}}{\beta_{mk}}\right)^2, \quad (46)$$

and

$$C_2 = \tau p_p q_{k'} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}\right|^2 \sum_{m=1}^{M} c_{mk}^2 u_{mk}^2 \beta_{mk'}^2$$
$$+ \underbrace{\tau p_p q_{k'} \sum_{m=1}^{M} \sum_{i\neq k'}^{M} c_{mk}^2 u_{mk}^2 \beta_{mk'} \beta_{mi} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_i\right|^2}_{C_3}. \quad (47)$$

For the last term of (47), we have

$$C_3 = \tau p_p q_{k'} \sum_{m=1}^{M} \sum_{i\neq k'}^{K} c_{mk}^2 u_{mk}^2 \beta_{mk'} \beta_{mi} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_i\right|^2$$
$$= \tau p_p q_{k'} \left(\sum_{m=1}^{M} u_{mk}^2 c_{mk} \beta_{mk'} \sum_{i=1}^{K} c_{mk} \beta_{mi} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_i\right|^2 \right.$$
$$\left. \sum_{m=1}^{M} u_{mk}^2 c_{mk}^2 \beta_{mk'} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}\right|^2 \right)$$
$$= \sqrt{\tau p_p} q_{k'} \sum_{m=1}^{M} u_{mk}^2 c_{mk} \beta_{mk'} \beta_{mk} - q_{k'} \sum_{m=1}^{M} u_{mk}^2 c_{mk}^2 \beta_{mk'}$$
$$- \tau p_p q_{k'} \sum_{m=1}^{M} u_{mk}^2 c_{mk}^2 \beta_{mk'} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}\right|^2, \quad (48)$$

where in the last step, we used equation (4). As a result, $C_1 + C_2 = \sqrt{\tau p_p} q_{k'} \sum_{m=1}^{M} u_{mk}^2 c_{mk} \beta_{mk'} \beta_{mk}$. Then finally we have

$$\mathbb{E}\left\{|\mathrm{IUI}_{kk'}|^2\right\} = \rho q_{k'} \left(\sum_{m=1}^{M} u_{mk}^2 \beta_{mk'} \gamma_{mk}\right)$$
$$+ \rho q_{k'} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}\right|^2 \left(\sum_{m=1}^{M} u_{mk} \gamma_{mk} \frac{\beta_{mk'}}{\beta_{mk}}\right)^2. \quad (49)$$

The total noise for user k is given by

$$\mathbb{E}\left\{|\mathrm{TN}_k|^2\right\} = \mathbb{E}\left\{\left|\sum_{m=1}^{M} u_{mk} \hat{g}_{mk}^* n_m\right|^2\right\} = \sum_{m=1}^{M} u_{mk}^2 \gamma_{mk}, \quad (50)$$

where the last equality is due to the fact that the terms $\hat{g}_{mk}$ and n_m are uncorrelated. Finally, by substituting (39), (40), (49) and (50) into (9), SINR of kth user is obtained by (10). which completes the proof of Theorem 1. ■

Appendix B: Proof of Proposition 1

The standard form of GP is defined as follows [39]:

$$P_{12}: \min \quad f_0(\mathbf{x}), \quad (51a)$$
$$\text{s.t.} \quad f_i(\mathbf{x}) \leq 1, \quad i = 1, \cdots, m, \quad g_i(\mathbf{x}) = 1, \quad i = 1, \cdots, p, \quad (51b)$$

where f_0 and f_i are posynomial and g_i are monomial functions. Moreover, $\mathbf{x} = \{x_1, \cdots, x_n\}$ represent the optimization variables. The SINR constraint in (15c) is not a posynomial function in its form, however it can be rewritten into the following posynomial function:

$$\frac{\mathbf{u}_k^H \left(\sum_{k'\neq k}^{K} q_{k'} \left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}\right|^2 \boldsymbol{\Delta}_{kk'} \boldsymbol{\Delta}_{kk'}^H + \sum_{k'=1}^{K} q_{k'} \mathbf{D}_{kk'} + \frac{1}{\rho} \mathbf{R}_k\right) \mathbf{u}_k}{\mathbf{u}_k^H \left(q_k \boldsymbol{\Gamma}_k \boldsymbol{\Gamma}_k^H\right) \mathbf{u}_k}$$
$$< \frac{1}{t}, \ \forall k. \quad (52)$$

By applying a simple transformation, (52) is equivalent to the following inequality:

$$q_k^{-1} \left(\sum_{k'\neq k}^{K} a_{kk'} q_{k'} + \sum_{k'=1}^{K} b_{kk'} q_{k'} + c_k\right) < \frac{1}{t}, \quad (53)$$

where

$$a_{kk'} = \frac{\mathbf{u}_k^H \left(\left|\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}\right|^2 \boldsymbol{\Delta}_{kk'} \boldsymbol{\Delta}_{kk'}^H\right) \mathbf{u}_k}{\mathbf{u}_k^H \left(\boldsymbol{\Gamma}_k \boldsymbol{\Gamma}_k^H\right) \mathbf{u}_k}, \quad (54a)$$
$$b_{kk'} = \frac{\mathbf{u}_k^H \mathbf{D}_{kk'} \mathbf{u}_k}{\mathbf{u}_k^H \left(\boldsymbol{\Gamma}_k \boldsymbol{\Gamma}_k^H\right) \mathbf{u}_k}, \quad (54b)$$
$$c_k = \frac{\mathbf{u}_k^H \mathbf{R}_k \mathbf{u}_k}{\rho \mathbf{u}_k^H \left(\boldsymbol{\Gamma}_k \boldsymbol{\Gamma}_k^H\right) \mathbf{u}_k}. \quad (54c)$$

The transformation in (53) shows that the left-hand side of (52) is a posynomial function. Therefore, the power allocation problem P_4 is a standard GP (convex problem), where the objective function and constraints are monomial and posynomial, respectively, which completes the proof of Proposition 1. ■

Appendix C: Proof of Lemma 1

This lemma is proven by exploiting the unique optimal solution of uplink max-min SINR problem with total power through an eigensystem [26]. This problem is iteratively solved and the optimal receiver filter coefficient $\tilde{\mathbf{U}}$ is determined by solving Problem P_3. Next, we scale the power allocation at each user such that the per-user power constraints are satisfied. Let us consider the following optimization problem for a given receiver filter coefficient $\tilde{\mathbf{U}}$:

$$P_{11}: C_k^{\mathrm{UP}}\left(\tilde{\mathbf{U}}, P_{\mathrm{tot}}\right) = \max_{q_k} \quad \min_{k=1,\cdots,K} \quad \mathrm{SINR}_k^{\mathrm{UP}}\left(\tilde{\mathbf{U}}, \mathbf{q}\right), \quad (55a)$$

$$\text{subject to} \quad \sum_{k=1}^{K} q_k \le P_{\mathrm{tot}}, \quad (55b)$$

The optimal solution of Problem P_{11} can be determined by finding the unique eigenvector of an eigensystem and the power allocation $\tilde{\mathbf{q}}$ satisfies the condition $\sum_{k=1}^{K}\tilde{q}_k = P_{\mathrm{tot}}$ [26]. The SINRs of all users defined in (10), can be collectively written as

$$\tilde{\mathbf{q}}\frac{1}{C_k^{\mathrm{UP}}\left(\tilde{\mathbf{U}}, P_{\mathrm{tot}}\right)} = \mathbf{D}\boldsymbol{\Psi}\left(\tilde{\mathbf{U}}\right)\tilde{\mathbf{q}} + \mathbf{D}\boldsymbol{\sigma}\left(\tilde{\mathbf{U}}\right), \quad (56)$$

where $\boldsymbol{\sigma}\left(\tilde{\mathbf{U}}\right) \in \mathbb{C}^{K\times 1}$, $\sigma_k\left(\mathbf{u}_k\right) = \frac{1}{\rho}\sum_{m=1}^{M}\tilde{u}_{mk}\gamma_{mk}$ and $\mathbf{D}$ and $\boldsymbol{\Psi}\left(\tilde{\mathbf{U}}\right)$ are defined as

$$\mathbf{D} = \mathrm{diag}\left[\frac{1}{\tilde{\mathbf{u}}_1^H\tilde{\mathbf{D}}_1\tilde{\mathbf{u}}_1}, \cdots, \frac{1}{\tilde{\mathbf{u}}_K^H\tilde{\mathbf{D}}_K\tilde{\mathbf{u}}_K}\right],$$

$$\left[\boldsymbol{\Psi}\left(\tilde{\mathbf{U}}\right)\right]_{kk'} = \begin{cases} \tilde{\mathbf{u}}_k^H\tilde{\tilde{\mathbf{R}}}_{kk}\tilde{\mathbf{u}}_k, & k = k', \\ \tilde{\mathbf{u}}_k^H\tilde{\mathbf{R}}_{kk'}\tilde{\mathbf{u}}_k + \tilde{\mathbf{u}}_k^H\tilde{\tilde{\mathbf{R}}}_{kk'}\tilde{\mathbf{u}}_k, & k \neq k', \end{cases} \quad (57)$$

where using (10), $\tilde{\mathbf{D}}_k$ $\tilde{\mathbf{R}}_{kk'}$ and $\tilde{\tilde{\mathbf{R}}}_{kk'}$ are defined as

$$\mathrm{SINR}_k^{\mathrm{UP}} = \frac{q_k\mathbf{u}_k^H\overbrace{\left(\boldsymbol{\Gamma}_k\boldsymbol{\Gamma}_k^H\right)}^{\tilde{\mathbf{D}}_k}\mathbf{u}_k}{\mathbf{u}_k^H\left(\sum_{k'\neq k}^{K} q_{k'}\underbrace{|\boldsymbol{\phi}_k^H\boldsymbol{\phi}_{k'}|^2\boldsymbol{\Delta}_{kk'}\boldsymbol{\Delta}_{kk'}^H}_{\tilde{\mathbf{R}}_{kk'}} + \sum_{k'=1}^{K} q_{k'}\underbrace{\mathbf{D}_{kk'}}_{\tilde{\tilde{\mathbf{R}}}_{kk'}} + \frac{1}{\rho}\mathbf{R}_k\right)\mathbf{u}_k}. \quad (58)$$

Having both sides of (56) multiplied by $\mathbf{1}^T = [1, \cdots, 1]^T$, we obtain $\frac{1}{C_k^{\mathrm{UP}}(\tilde{\mathbf{U}}, P_{\mathrm{tot}})} = \frac{1}{P_{\mathrm{tot}}}\mathbf{1}^T\tilde{\mathbf{D}}\boldsymbol{\Psi}\left(\tilde{\mathbf{U}}\right)\tilde{\mathbf{q}} + \frac{1}{P_{\mathrm{tot}}}\mathbf{1}^T\mathbf{D}\boldsymbol{\sigma}\left(\tilde{\mathbf{U}}\right)$, which can be combined with (56) to define the following eigensystem:

$$\boldsymbol{\Lambda}\left(\tilde{\mathbf{U}}, P_{\mathrm{tot}}\right)\tilde{\mathbf{q}}_{\mathrm{ext}} = \frac{1}{C_k^{\mathrm{UP}}\left(\tilde{\mathbf{U}}, P_{\mathrm{tot}}\right)}\tilde{\mathbf{q}}_{\mathrm{ext}}, \ \left[\tilde{\mathbf{q}}_{\mathrm{ext}}\right]_{K+1} = 1, \quad (59)$$

where the extended coupling matrix $\boldsymbol{\Lambda}\left(\tilde{\mathbf{D}}, P_{\mathrm{tot}}\right)$ is given by

$$\boldsymbol{\Lambda}\left(\tilde{\mathbf{D}}, P_{\mathrm{tot}}\right) = \begin{bmatrix} \mathbf{D}\boldsymbol{\Psi}^T\left(\tilde{\mathbf{U}}\right) & \mathbf{D}\boldsymbol{\sigma}\left(\tilde{\mathbf{U}}\right) \\ \frac{1}{P_{\mathrm{tot}}}\mathbf{1}^T\mathbf{D}\boldsymbol{\Psi}^T\left(\tilde{\mathbf{U}}\right) & \frac{1}{P_{\mathrm{tot}}}\mathbf{1}^T\mathbf{D}\boldsymbol{\sigma}\left(\tilde{\mathbf{U}}\right) \end{bmatrix}. \quad (60)$$

The optimal power allocation $\tilde{\mathbf{q}}$ is obtained by determining the eigenvector corresponding to the maximum eigenvalue of $\boldsymbol{\Lambda}\left(\tilde{\mathbf{U}}, P_{\mathrm{tot}}\right)$ and scaling the last element to one as follows:

$$\tilde{\mathbf{q}}_{\mathrm{ext}} = \begin{bmatrix}\tilde{\mathbf{q}} \\ 1\end{bmatrix}, \ \boldsymbol{\Lambda}\left(\tilde{\mathbf{U}}, P_{\mathrm{tot}}\right)\tilde{\mathbf{q}}_{\mathrm{ext}} = \lambda_{\max}\left(\boldsymbol{\Lambda}\left(\tilde{\mathbf{U}}, P_{\mathrm{tot}}\right)\right)\tilde{\mathbf{q}}_{\mathrm{ext}}. \quad (61)$$

Note that the dominant eigenvector can be scaled by any positive value to satisfy a particular condition. As such, we further scale $\tilde{\mathbf{q}}$ to satisfy the per-user power constraints as follows:

$$\tilde{\mathbf{q}} = \begin{bmatrix}\frac{\hat{q}_1}{\max(\hat{\mathbf{q}})} \\ \vdots \\ \frac{\hat{q}_K}{\max(\hat{\mathbf{q}})}\end{bmatrix}, \text{where } \hat{\mathbf{q}} = \begin{bmatrix}\frac{\tilde{q}_1}{p_{\max}^{(1)}} \\ \vdots \\ \frac{\tilde{q}_K}{p_{\max}^{(K)}}\end{bmatrix}, \quad (62)$$

where first the ratios between each component of the allocated power, $\tilde{q}_k, \forall k$, and the maximum available power, $p_{\max}^{(k)}, \forall k$, are calculated. Then the power allocation $\tilde{\mathbf{q}}$ is obtained by dividing all components of $\tilde{\mathbf{q}}$ by the maximum value among the components of $\hat{\mathbf{q}}$, i.e., $\max(\hat{\mathbf{q}})$. In the next iteration, the same max-min problem is solved with a new total power constraint obtained by summing up the allocated power to all users in the previous iteration, i.e., $P_{\mathrm{tot}} = \sum_{k=1}^{K}\tilde{q}_k$. At the convergence, the per-user power constraints are satisfied with achieving the same uplink SINR for all users. Interestingly, if this max-min problem is solved with the corresponding total power constraint, then it will converge to the same optimal solution of max-min problem with per-user power constraints. This is due to the property that the eigensystem exploited to obtain the power allocation in (59) has a unique positive eigenvalue and a corresponding unique eigenvector. Therefore, Problems P_1 and P_5 are equivalent and have the same optimal solution. ■

Appendix D: Proof of Theorem 3

To achieve the same SINR tuples in both the uplink and the downlink, the following condition should be satisfied:

$$\mathrm{SINR}_k^{\mathrm{DL}}(\mathbf{U}, \mathbf{p}) = \mathrm{SINR}_k^{\mathrm{UP}}(\mathbf{U}, \mathbf{q}), \forall k. \quad (63)$$

By substituting uplink and downlink SINRs, in (19) and (18), respectively, in equation (63) and summing all equations by both sides, we have

$$p_1\sum_{m=1}^{M}u_{m1}^2\gamma_{m1} + \cdots + p_K\sum_{m=1}^{M}u_{mK}^2\gamma_{mK} = \sum_{k=1}^{K}q_k. \quad (64)$$

Therefore, this condition between the total transmit power on the uplink and the equivalent total transmit power on the downlink should be satisfied to realize the same SINRs for all set of users, which completes the proof of Theorem 3. ■

References

[1] F. Boccardi, R. W. Heath, A. Lozano, T. L. Marzetta, and P. Popovski, "Five disruptive technology directions for 5G," *IEEE Commun. Mag.*, vol. 52, no. 2, pp. 74–80, Feb. 2014.

[2] A. Zappone, L. Sanguinetti, G. Bacci, E. A. Jorswieck, and M. Debbah, "Energy-efficient power control: a look at 5G wireless technologies," *IEEE Trans. Signal Process.*, vol. 64, no. 7, pp. 1668–1683, Apr. 2016.

[3] E. Bjornson, E. A. Jorswieck, M. Debbah, and B. Ottersten, "Multiobjective signal processing optimization: the way to balance conflicting metrics in 5G systems," *IEEE Signal Process. Mag.*, vol. 31, no. 6, pp. 14–23, Oct. 2014.

[4] H. Q. Ngo, A. Ashikhmin, H. Yang, E. G. Larsson, and T. L. Marzetta, "Cell-free Massive MIMO versus small cells," *IEEE Trans. Wireless Commun.*, vol. 16, no. 3, pp. 1834–1850, Mar. 2017.

[5] M. Bashar, K. Cumanan, A. G. Burr, M. Debbah, and H. Q. Ngo, "Enhanced max-min SINR for uplink cell-free Massive MIMO systems," in *Proc. IEEE ICC*, May 2018, pp. 1–7.

[6] E. Nayebi, A. Ashikhmin, T. L. Marzetta, H. Yang, and B. D. Rao, "Precoding and power optimization in cell-free Massive MIMO systems," *IEEE Trans. Wireless Commun.*, vol. 16, no. 22, pp. 1–15, May 2017.

[7] G. Interdonato, H. Q. Ngo, E. G. Larsson, and P. Frenger, "On the performance of cell-free Massive MIMO with short-term power constraints," in *Proc. IEEE CAMAD*, Oct. 2016, pp. 1–6.

[8] S. Buzzi and C. DAndrea, “Cell-free Massive MIMO: user-centric approach,” *IEEE Wireless Commun. Lett.*, vol. 6, no. 6, pp. 1–4, Aug. 2017.

[9] A. Burr, M. Bashar, and D. Maryopi, “Ultra-dense radio access networks for smart cities: Cloud-RAN, fog-RAN and cell-free Massive MIMO,” in *Proc. IEEE PIMRC*, Sep. 2018, pp. 1–6.

[10] M. Karakayali, G. Foschini, and R. Valenzuela, “Network coordination for spectrally efficient communications in cellular systems,” *IEEE Trans. Wireless Commun. Mag.*, vol. 13, no. 4, pp. 56–61, Aug. 2006.

[11] E. Bjornson, R. Zakhour, D. Gesbert, and B. Ottersten, “Cooperative multicell precoding: rate region characterization and distributed strategies with instantaneous and statistical CSI,” *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4298–4310, Aug. 2010.

[12] S. H. Park, O. Simeone, O. Sahin, and S. S. (Shitz), “Fronthaul compression for cloud radio access networks: Signal processing advances inspired by network information theory,” *IEEE Signal Process. Mag.*, vol. 31, no. 6, pp. 69–79, Nov. 2014.

[13] S. H. Park, O. Simeone, and S. S. (Shitz), “Joint optimization of cloud and edge processing for fog radio access networks,” *IEEE Trans. Wireless Commun.*, vol. 15, no. 11, pp. 7621–7632, Nov. 2016.

[14] H. Q. Ngo, L. Tran, T. Q. Duong, M. Matthaiou, and E. G. Larsson, “On the total energy efficiency of cell-free Massive MIMO,” *IEEE Trans. Green Commun. and Net.*, vol. 2, no. 1, pp. 25–39, Mar. 2017.

[15] K. Cumanan, R. Krishna, L. Musavian, and S. Lambotharan, “Joint beamforming and user maximization techniques for techniques for cognitive radio networks based on branch and bound method,” *IEEE Trans. Wireless Commun.*, vol. 9, no. 10, pp. 3082–3092, Oct. 2010.

[16] K. Cumanan, L. Musavian, S. Lambotharan, and A. B. Gershman, “SINR balancing technique for downlink beamforming in cognitive radio networks,” *IEEE Signal Process. Lett.*, vol. 17, no. 2, pp. 133–136, Feb. 2010.

[17] K. Cumanan, J. Tang, and S. Lambotharan, “Rate balancing based linear transceiver design for multiuser MIMO system with multiple linear transmit covariance constraints,” in *Proc. IEEE ICC*, Jun. 2011, pp. 1–5.

[18] K. Cumanan, R. Zhang, and S. Lambotharan, “A new design paradigm for mimo cognitive radio with primary user rate constraint,” *IEEE Commun. Lett.*, vol. 16, no. 5, pp. 706–709, Aug. 2012.

[19] K. Cumanan, R. Krishna, Z. Xiong, and S. Lambotharan, “SINR balancing technique and its comparison to semidefinite programming based QoS provision for cognitive radios,” in *Proc. IEEE VTC*, Apr. 2009, pp. 1–5.

[20] K. Cumanan, G. C. Alexandropoulos, Z. Ding, and G. K. Karagiannidis, “Secure communications with cooperative jamming: Optimal power allocation and secrecy outage analysis,” *IEEE Trans. Veh. Technol.*, vol. 66, no. 8, pp. 7495–7505, Aug. 2017.

[21] A. Wiesel, Y. C. Eldar, and S. Shamai, “Linear precoding via conic optimization for fixed MIMO receivers,” *IEEE Trans. Signal Process.*, vol. 54, no. 1, pp. 161 – 176, Jan. 2006.

[22] D. W. H. Cai, T. Q. S. Quek, and C. W. Tan, “A unified analysis of max-min weighted SINR for MIMO downlink system,” *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3850–3862, Aug. 2011.

[23] G. Golub and C. V. Loan, *Matrix Computations*, 2nd ed. Baltimore, MD: The Johns Hopkins Univ. Press, 1996.

[24] S. P. Boyd, S. J. Kim, A. Hassibi, and L. Vandenbarghe, “A tutorial on geometric programming,” *Optim. Eng.*, vol. 8, no. 1, pp. 67–128, Apr. 2007.

[25] M. Chiang, C. W. Tan, D. P. Palomar, D. O’Neill, and D. Julian, *Power Control by Geometric Programming, in Resource Allocation in Next Generation Wireless Networks*. W. Li, Y. Pan, Editors, Nova Sciences Publishers, 2006.

[26] M. Schubert and H. Boche, “Solution of the multiuser downlink beamforming problem with individual SINR constraints,” *IEEE Trans. Veh. Technol.*, vol. 53, no. 1, pp. 18 – 28, Jan. 2004.

[27] ——, “Iterative multiuser uplink and downlink beamforming under sinr constraints,” *IEEE Trans. Signal Process.*, vol. 53, no. 7, pp. 2324–2334, Jul. 2005.

[28] D. N. C. Tse and P. Viswanath, “Downlink-uplink duality and effective bandwidths,” in *Proc. IEEE ISlT*, Jul. 2002.

[29] M. Chiang, P. Hande, T. Lan, and C. W. Tan, *Power Control in Wireless Cellular Networks, Foundation and Trends in Networking*. Now Publishers, 2007.

[30] H. Ahmadi, A. Farhang, N. Marchetti, and A. MacKenzie, “A game theoretic approach for pilot contamination avoidance in Massive MIMO,” *IEEE Wireless Commun. Lett.*, vol. 5, no. 1, pp. 12 – 15, Feb. 2016.

[31] M. A. Imran, S. A. R. Zaidi, and Z. Shakir, *Access, Fronthaul and Backhaul Networks for 5G and Beyond*. Institution of Engineering and Technology, 2017.

[32] M. Bashar, K. Cumanan, A. G. Burr, H. Q. Ngo, and M. Debbah, “Cell-free Massive MIMO with limited backhaul,” in *Proc. IEEE ICC*, May 2018, pp. 1–7.

[33] ——, “Max-min SINR of cell-free Massive MIMO uplink with optimal uniform quantization,” *IEEE Trans. Commun.*, submitted.

[34] A. G. Burr, M. Bashar, and D. Maryopi, “Cooperative access networks: Optimum fronthaul quantization in distributed Massive MIMO and cloud RAN,” in *Proc. IEEE VTC*, Jun. 2018, pp. 1–7.

[35] M. Bashar, H. Q. Ngo, A. Burr, D. Maryopi, K. Cumanan, and E. G. Larsson, “On the performance of backhaul constrained cell-free Massive MIMO with linear receivers,” in *Proc. IEEE Asilomar*, Nov. 2018, pp. 1–7.

[36] D. Maryopi, M. Bashar, and A. G. Burr, “On the uplink throughput of zero-forcing in cell-free Massive MIMO with coarse quantization,” *IEEE Trans. Veh. Technol.*, submitted.

[37] A. J. Fehske, P. Marsch, and G. P. Fettweis, “Bit per joule efficiency of cooperating base stations in cellular networks,” in *Proc. IEEE Globecom Workshops (GC Wkshps)*, Dec. 2010, pp. 1406–1411.

[38] F. Rashid-Farrokhia, K. J. R. Liu, and L. Tassiulas, “Transmit beamforming and power control for cellular wireless systems,” *IEEE J. Sel. Areas Commun.*, vol. 16, no. 8, pp. 1437–1450, Oct. 1998.

[39] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, UK: Cambridge University Press, 2004.

[40] C. Chien, H. Su, and H. Li, “Joint beamforming and power allocation for MIMO relay broadcast channel with individual SINR constraints,” *IEEE Trans. Veh. Technol.*, vol. 63, no. 4, pp. 1660–1677, May 2004.

[41] G. H. Golub and C. F. V. Loan, *Matrix Computations*, 3rd ed. Baltimore, MD, USA: Johns Hopkins Univ. Press, 1996.

[42] Y. Nesterov and A. Nemirovsky, *Interior-point polynomial methods in convex programming*. Studies in Applied Mathematics, SIAM, Philadelphia, 1994.

[43] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, “Applications of second-order cone programming,” *Linear Algebra Applicat*, pp. 193–228, Nov. 1998.

[44] J. F. Sturm, “Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones,” *Optim. Meth. Softw.*, vol. 11, pp. 625 – 653, Aug. 1999.

[45] A. Tang, J. Sun, and K. Gong, “Mobile propagation loss with a low base station antenna for NLOS street microcells in urban area,” in *Proc. IEEE VTC*, May 2001, pp. 333–336.

[46] F. Rusek, D. Persson, B. K. Lau, E. G. Larsson, T. L. Marzetta, O. Edfors, and F. Tufvesson, “Scaling up MIMO: Opportunities and challenges with very large arrays,” *IEEE Signal Process. Mag.*, vol. 30, no. 5, pp. 40–60, Jan. 2013.