ON THE USE OF A SPECTRUM–BASED MODEL FOR LINEAR VISCOELASTIC MATERIALS

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Abstract

A new spectrum–based model for describing the behavior of time–dependent materials is presented. In this paper, unlike most prior modeling techniques, the time–dependent response of viscoelastic materials is not expressed through the use of series. Instead, certain criteria have been imposed to select a spectrum function that has the potential of describing a wide range of material behavior. Another consequence of choosing the spectrum function of the type used in this paper is to have a few closed form analytic solutions in the theory of linear viscoelasticity. The Laplace transform technique is used to obtain the necessary formulae for viscoelastic Lame' functions, relaxation and bulk moduli, creep bulk and shear compliance, as well as Poisson's ratio. By using the Elastic-Viscoelastic Correspondence Principle (EVCP), material constants appearing in the proposed model are obtained by comparing the experimental data with the solution of the integral equation for a simple tensile test. The resulting viscoelastic functions describe the material properties which can then be used to express the behavior of a material in other loading configurations. The model's potential is demonstrated and its limitations are discussed.

Keywords: viscoelasticity, spectrum function, constitutive model, Elastic-Viscoelastic Correspondence Principle, Volterra integral equation, epoxy polymer.

1. Introduction

In the discussion about precise formulations in viscoelasticity, Kelvin and Maxwell were the first to give exact solutions to describe the linear viscoelastic phenomena in usable form. Their contributions were important in that their solutions, in the form of an exponential, formed the basis for future, more elaborate models. Thus far, most depictions of viscoelastic behavior use the elements of Kelvin-Voigt, Maxwell, or a combination of these models [1,2,3,4]. These representations are based on simple discrete elastic and viscous elements which are modeled by series. Depending on the complexity of the response, numerous terms are sometimes required to represent the phenomena appropriately. It is also well known that not all materials can be modeled using simple Kelvin-Voigt (K-V) or Maxwell elements. Therefore, the primary focus of this study is to present a methodology that describes general viscoelastic behavior that encompasses both K–V and non K–V materials through the use of a continuous distribution, rather than as a combination of simple mechanical elements. This approach generates closed form solutions for various viscoelastic moduli.

The attraction of using a spectrum function to model viscoelastic behavior is obvious, mainly the possibility of producing closed form solutions from which the material properties can be extracted. Nevertheless, most researchers remain loyal to the series representation due to the inherent difficulties with the use of some functions. Use of distribution functions to model viscoelastic phenomena implies movement between the Laplace and the time domain through the use of operational calculus techniques, and depending on the chosen spectrum function, this may prove to be a difficult task. Even so, the appeal of producing a fully contained model to express all properties of interest deserves another look, for the resulting expression may not be solved analytically, but it can always be determined numerically.

Using distribution functions to model viscoelastic phenomena is not a new practice. In fact, the use of spectral representations to model linear time-dependent behavior has been in existence since the late nineteenth century. An excellent summary and review of efforts made in the area of empirical mathematical modeling of viscoelastic behavior is given by Tschoegl [5]. Much of the efforts from the 1940's to the 1960's resulted in discrete representations [6] or power law representations [5] of time–dependent behavior. Surveying the various functions that were selected, it seems that generally the distribution functions were chosen more for their simplicity rather than their ability to describe realistic viscoelastic behavior. For example, a "box function" or step function was used either singly or in combination with a "wedge" or power law distribution function. By describing these functions at some intermediate time range, these functions effectively avoided the singularity at time $t = 0$. In this paper, a continuous spectrum function that is valid for all times is considered.

In Ref. [7], Eringen suggests the use of memory functions by using a spectrum function to represent viscoelastic behavior. However, he does not suggest the manner in which this distribution function should be chosen. In this study, a spectrum function that satisfies certain criteria is selected to develop the basic viscoelastic functions resulting in a method that allows for the mapping of a continuous distribution to describe the viscoelastic behavior of the material. Though the chosen spectrum function for this study is not a unique function, it does produce results which are consistent with the time–dependent behavior of viscoelastic solids. Other functions were selected for modeling, but these proved to be mathematically cumbersome. The distinctive quality of the selected spectrum function is that it not only represents typical viscoelastic response, but it also produces closed form representations for several important time–dependent properties.

2. Problem Formulation

Typically, the constitutive relationship is described using moduli or compliances. However, for this study, the fundamental Lame' parameters as functions, which are necessary to express all properties of interest, are used to develop the viscoelastic response functions.

From the Boltzman-Volterra theory, the constitutive equation describing a time–dependent linear isotropic viscoelastic solid in Cartesian coordinates is given as

$$T_{kl} = \delta_{kl}\int_{-\infty}^{t}\left[\lambda_e + \lambda_v(t-\tau)\right]\frac{\partial e_{rr}}{\partial\tau}d\tau + 2\int_{-\infty}^{t}\left[\mu_e + \mu_v(t-\tau)\right]\frac{\partial e_{kl}}{\partial\tau}d\tau \tag{2.1}$$

where a repeated index implies a sum and e_{rr} is the trace of the strain tensor **e**. From Ref. [7] the relaxation bulk modulus K(t) and shear modulus G(t) are defined as

$$\begin{aligned} K(t) &= \lambda_e + \frac{2}{3}\mu_e + \lambda_v(t) + \frac{2}{3}\mu_v(t) \\ G(t) &= \mu_e + \mu_v(t) \end{aligned} \tag{2.2}$$

where λ_e and μ_e are the Lame' elastic constants and $\lambda_v(t)$ and $\mu_v(t)$ are the viscoelastic functions of the material. In terms of K(t) and G(t), the stress in Eq. (2.1) is written as

$$T_{kl} = \delta_{kl}\int_{-\infty}^{t}\left[K(t-\tau) - \frac{2}{3}G(t-\tau)\right]\frac{\partial e_{rr}}{\partial\tau}d\tau + 2\int_{-\infty}^{t}G(t-\tau)\frac{\partial e_{kl}}{\partial\tau}d\tau \tag{2.3a}$$

The strain is obtained if the stress in Eq. (2.3a) is inverted as

$$e_{kl} = \frac{1}{3}\delta_{kl}\int_{-\infty}^{t}\left[\frac{1}{3}B(t-\tau) - \frac{1}{2}J(t-\tau)\right]\frac{\partial T_{rr}}{\partial\tau}d\tau + \frac{1}{2}\int_{-\infty}^{t}J(t-\tau)\frac{\partial T_{kl}}{\partial\tau}d\tau \tag{2.3b}$$

where B(t) is the creep bulk compliance and J(t) is the creep shear compliance. If the material is unstressed for $t < 0$, then the integration limit in Eqs. (2.1) and (2.3) will be 0 to t. In this case, taking the Laplace transform of Eqs. (2.3) while using the convolution theorem, results in

$$\overline{T}_{kl} = \delta_{kl}\zeta\left(\overline{K} - \frac{2}{3}\overline{G}\right)\overline{e}_{rr} + 2\zeta\,\overline{G}\,\overline{e}_{kl} \tag{2.4a}$$

$$\bar{e}_{kl} = \frac{1}{3}\delta_{kl}\,\zeta\left(\frac{1}{3}\bar{B} - \frac{1}{2}\bar{J}\right)\bar{T}_{rr} + \frac{1}{2}\zeta\,\bar{J}\,\bar{T}_{kl} \tag{2.4b}$$

where for a function **f** (**r**,t), the Laplace transform is

$$\mathcal{L}\left[\mathbf{f}(\mathbf{r},t)\right] = \bar{\mathbf{f}}(\mathbf{r},\zeta) = \int_0^\infty e^{-\zeta t}\mathbf{f}(\mathbf{r},t)dt \tag{2.5a}$$

and its inverse is

$$\mathbf{f}\,(\mathbf{r},t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\zeta t}\,\bar{\mathbf{f}}(\mathbf{r},t)\,dt \tag{2.5b}$$

If the indices k and l are equated in Eqs. (2.4), i.e., k = l, and a sum is performed, the simple and well known relationship between the Laplace transforms of the relaxation modulus K and the creep bulk compliance B is found to be

$$\bar{K}\,\bar{B} = \frac{1}{\zeta^2} \tag{2.6a}$$

whereas, for $k \neq l$ in Eqs. (2.4), a direct correlation between the Laplace transforms of the shear modulus G and the creep shear compliance J is described by

$$\bar{G}\,\bar{J} = \frac{1}{\zeta^2} \tag{2.6b}$$

It can be seen that if the Laplace transforms of K(t) and G(t), denoted by $\bar{K}(\zeta)$ and $\bar{G}(\zeta)$, respectively, are known, then B(t) and J(t) can be obtained by the inverse Laplace transforms.

3. The Spectrum Function

As noted from the above equations for the relaxation bulk modulus and shear modulus in Eq. (2.2), it is imperative that the Lame functions $\lambda_v(t)$ and $\mu_v(t)$ be determined. The most general form of the Lame functions is obtained from Eringen [7] and is formulated as

$$\lambda_v(t) = -\lambda_e \int_0^\infty \phi_1(\alpha)\left[1 - e^{\frac{-t}{\alpha}}\right] d\alpha \tag{3.1a}$$

$$\mu_v(t) = -\mu_e \int_0^\infty \phi_2(\alpha)\left[1 - e^{\frac{-t}{\alpha}}\right] d\alpha \tag{3.1b}$$

where $\varphi_1(\alpha)$ and $\varphi_2(\alpha)$, called the relaxation spectrums, satisfy two criteria:

$$\phi_{1,2}(\alpha) \geq 0 \quad \phi_{1.2}(\infty) \leq 1 \tag{3.2}$$

To demonstrate the methodology, the case of $\varphi_2(\alpha) = \varphi_1(\alpha) = \varphi(\alpha)$ is considered for this study. As a consequence of this simplification, it must be noted that all relaxation properties are proportional to the same time-function, and that the Poisson behavior is characterized through a constant Poisson ratio, i.e.,

$$\mu_v(t) = C_1 \lambda_v(t), \quad \mu(t) = C_2 \lambda(t) = C_3 K(t) = C_4 E(t) \tag{3.2b}$$

with C_1, C_2 ,C_3 and C_4 denoting appropriate constants, and Poisson's ratio being given by

$$\nu = \frac{\lambda_e}{2(\lambda_e + \mu_e)} = \text{constant} \tag{3.2c}$$

These characteristics do not apply to viscoelastic materials in general. This special case is further discussed below.

At this point, it should be noted that all inter-functional relaxation functions, except for the inverse (creep) functions, can be generated from elastic parameters.

3.1 SELECTION OF $\varphi(\alpha)$

There are many choices for the spectrum function, $\varphi(\alpha)$. A number of known functions were tried, but these yielded expressions for which the Laplace inversion became very difficult. The spectrum function that satisfies the criteria of being non-negative, bounded, monotonic, and able

to produce constant values in a particular limiting condition is selected. The proposed form chosen for this research and quoted in Ref. [8] is

$$\varphi(\alpha) = \frac{r}{\pi\left(k^2 + r^2\alpha^2\right)} \tag{3.3}$$

where r and k are the temperature dependent parameters that are assumed to be material-specific. In Eq. (3.3), k is non-dimensional and r has dimension of 1/sec. Here φ is regarded as a general spectrum function which is continuous and continuously differentiable. For general viscoelastic modeling, from Eq. (3.3), material constants n and n_o are introduced as

$$n = \frac{r}{k^2}, \quad n_o = \frac{r}{k} \tag{3.4}$$

having dimension of [1/sec] and then, φ(α) can be expressed as

$$\varphi(\alpha) = \frac{n}{\pi\left(1 + n_0^2\alpha^2\right)} \tag{3.5}$$

3.1.1. *Similarity in Functional Behavior*

From Eqs. 3.1 with $\varphi_2(\alpha) = \varphi_1(\alpha) = \varphi(\alpha)$ it is obvious that choosing one spectrum function φ(α) will produce similarity in the functional behavior of the viscoelastic moduli. Particularly, the ratio $\frac{\lambda_v}{\lambda_e} = \frac{\mu_v}{\mu_e}$ implies that a constant Poisson's ratio will be obtained. Although Poisson's ratio does not vary dramatically, it is certainly time-dependent for viscoelastic materials [9,10]. For this methodology, the limitation can be remedied by choosing separate functions $\varphi_1(\alpha)$ and $\varphi_2(\alpha)$ for λ_v and μ_v, respectively, as expressed in Eq. 3.1. However, this choice also introduces additional constants to be determined and the closed form solutions are not possible.

To use the current methodology, for cases where Poisson's ratio must be modeled as a time-dependent property, a sort of empirical approach is used. This procedure, demonstrated in Section 4.2, is based on the moduli obtained in this study.

3.1.2. *Power Law Behavior*

As stated before, most viscoelastic phenomena are well represented by various power laws due to their inherent ability in emulating actual time-dependent behavior. The selected spectrum function given in Eq. 3.5 is simply a continuous distribution, valid for all times. But, *for very large values of the argument*, this function *behaves* like a power law with power two, i.e., $\varphi(\alpha) \approx \frac{n}{\pi}(n_0 t)^{-2}$, for $n_0 t \gg 1$. This behavior and the chosen spectrum function's applicability for various time periods are demonstrated in Section 5.

3.2 MODELING VISCOELASTIC LAME' FUNCTIONS USING φ(α)

Having satisfied all the necessary criteria for the spectrum function, the form of φ(α) as given in Eq. (3.5) is used as the general spectrum function. It must be stated here that no limiting procedures are used for the development of any viscoelastic Lame' functions – the complete form of the spectrum function will be used. The validity of this φ and the values of the constants will be established through comparison with the experimental data.

The spectrum function in Eq. (3.5) is substituted in Eqs. (3.1) and the necessary integration is performed to obtain

$$\lambda_v(t) = -\lambda_e\left\{\frac{m}{2} + \frac{m}{\pi}F(t)\right\} \tag{3.6a}$$

$$\mu_v(t) = -\mu_e\left\{\frac{m}{2} + \frac{m}{\pi}F(t)\right\} \tag{3.6b}$$

where

$$m = \frac{n}{n_o} \tag{3.6c}$$

and

$$F(t) = ci\ (n_o t)\ \sin(n_o t) + si\ (n_o t)\cos(n_o t) \tag{3.6d}$$

where si, Si, ci, Ci are the sine and cosine integral functions defined as

$$\begin{aligned} si(x) &= \int_{\infty}^{x} \frac{\sin z}{z}\, dz \\ ci(x) &= -\int_{\infty}^{x} \frac{\cos z}{z}\, dz \\ Si(x) &= \int_{0}^{x} \frac{\sin z}{z}\, dz = \frac{\pi}{2} + si(x) \\ Ci(x) &= \gamma + \ln(x) + \int_{0}^{x} \frac{\cos(z) - 1}{z}\, dz \end{aligned} \tag{3.7}$$

and γ is the Euler's constant = 0.5772. Using the definitions in Eq. (3.7), F(t) can also be written as

$$F(t) = -\int_{n_o t}^{\infty} \frac{\sin(z - n_o t)}{z}\, dz \tag{3.8}$$

The Laplace transform of the integral in Eq. (3.8) is obtained from Erdelyi [11] as

$$\mathcal{L}[F(t)] = \frac{m}{\pi\zeta}\left[\bar{f}\left(\frac{\zeta}{n_o}\right)\right] \tag{3.9}$$

where

$$\bar{f}\left(\frac{\zeta}{n_o}\right) = \frac{m}{\pi}\left[\frac{\frac{\zeta}{n_o}\ln\frac{\zeta}{n_o}}{\frac{\zeta^2}{n_o^2}+1} - \frac{\frac{\pi\zeta^2}{n_o^2}}{2\left(\frac{\zeta^2}{n_o^2}+1\right)}\right] \tag{3.10}$$

Two functions of ζ are defined as $\lambda(\zeta)$ and $\mu(\zeta)$, where

$$\begin{aligned} \lambda(\zeta) &= \lambda_e + \zeta\,\bar{\lambda}_v(\zeta) \\ \mu(\zeta) &= \mu_e + \zeta\,\bar{\mu}_v(\zeta) \end{aligned} \tag{3.11}$$

The form of these functions is consistent with the result of the Laplace transform of the equilibrium equation. Upon taking the Laplace transform of Eqs. (3.6) and using the result in Eqs. (3.11), $\lambda(\zeta)$ and $\mu(\zeta)$ are expressed as

$$\lambda(\zeta) = \lambda_e\left[\left(1-\frac{m}{2}\right)-\bar{f}\left(\frac{\zeta}{n_o}\right)\right]$$
$$\mu(\zeta) = \mu_e\left[\left(1-\frac{m}{2}\right)-\bar{f}\left(\frac{\zeta}{n_o}\right)\right] \qquad (3.12)$$

4. Closed Form Solutions

It must be noted that, based on the choice of the spectrum function $\varphi(\alpha)$, Eqs. (3.1) are the closed form analytical formulae for $\lambda_v(t)$ and $\mu_v(t)$. Since the Lame' functions have now been developed, the viscoelastic response functions, i.e., moduli, compliance, etc., can be obtained. Certainly, as discussed in Section 3.1.1, the functional similarity extends to all properties being expressed through the use of Eqs. 3.12.

4.1 MODELING OF MODULI & COMPLIANCES

Although the bulk modulus may be difficult to measure directly, the analytically closed forms of K(t) and G(t) from Eq. (2.2), under the restriction imposed by Eq. (3.6), is

$$K(t) = \left(\lambda_e + \frac{2}{3}\mu_e\right)\left[\left(1-\frac{m}{2}\right)+\frac{m}{\pi}\int_{n_o t}^{\infty}\frac{\sin(z-n_o t)}{z}\,dz\right] \qquad (4.1a)$$

$$K(t) = \left(\lambda_e + \frac{2}{3}\mu_e\right)\frac{G(t)}{\mu_e} \qquad (4.1b)$$

$$G(t) = \mu_e\left[\left(1-\frac{m}{2}\right)+\frac{m}{\pi}\int_{n_o t}^{\infty}\frac{\sin(z-n_o t)}{z}\,dz\right] \qquad (4.2)$$

where the identical time dependence for the two functions is apparent. The determination of the material constants n and n_o, and hence m will be discussed in Section 4.3.

Unlike K(t) and G(t), there are also properties of interest that are not directly expressed as analytical functions of time. However, since the Lame' functions have been expressed both in the time domain (Eqs. (3.6)) and also in the Laplace domain (Eqs. (3.12)), viscoelastic response functions are obtainable as long as the properties are available in either domain. For example, direct analytic expressions for tensile modulus E(t), bulk compliance B(t), and shear compliance J(t) are commonly expressed only in the form of their Laplace transforms [4,5,7,8].

$$\zeta\overline{E}(\zeta) = \frac{\mu(\zeta)\left[3\lambda(\zeta) + 2\mu(\zeta)\right]}{\lambda(\zeta) + \mu(\zeta)} \tag{4.3a}$$

$$\zeta\overline{B}(\zeta) = \frac{1}{\lambda(\zeta) + \frac{2}{3}\mu(\zeta)} = \frac{1}{\zeta\overline{K}(\zeta)} \tag{4.3b}$$

$$\zeta\overline{J}(\zeta) = \frac{1}{\mu(\zeta)} = \frac{1}{\zeta\overline{G}(\zeta)} \tag{4.3c}$$

With the availability of $\lambda(\zeta)$ and $\mu(\zeta)$ as given in Eqs. (3.12), the above properties can be expressed in the Laplace domain. The full expression for the Laplace transform of E(t) is obtained by substituting Eq. (3.12) into Eq. (4.3a) and performing the Laplace inverse to obtain

$$E(t) = E_e\left[\left(1 - \frac{m}{2}\right) + \frac{m}{\pi}\int_{n_o t}^{\infty} \frac{\sin(z - n_o t)}{z}\, dz\right] \tag{4.4}$$

where

$$E_e = \frac{\mu_e\left(3\lambda_e + 2\mu_e\right)}{\lambda_e + \mu_e} \tag{4.5}$$

To obtain the closed form expression for the bulk compliance B(t), the forms of $\lambda(\zeta)$ and $\mu(\zeta)$ as given in Eqs. (3.12) are used in Eq. (4.3b) to obtain

$$\overline{B}(\zeta) = \frac{3}{3\lambda_e + 2\mu_e}\left[\frac{1}{\zeta\left(1-\frac{m}{2}\right) - \bar{f}\left(\frac{\zeta}{n_o}\right)}\right] \tag{4.6}$$

First, the Laplace inverse of $\bar{f}\left(\frac{\zeta}{n_o}\right)$, defined in Eq. (3.10), must be taken and from [11], it is determined to be

$$\mathcal{L}^{-1}\left[\bar{f}\left(\frac{\zeta}{n_o}\right)\right] = \frac{mn_o}{\pi}\left[\psi(t;n_o) - \frac{\pi}{2}\delta(n_o t)\right] \tag{4.7a}$$

where

$$\psi(t;n_o) = \int_{n_o t}^{\infty} \frac{\cos(z - n_o t)}{z} dz \tag{4.7b}$$

The Laplace inverse of Eq. (4.6) is difficult to obtain. Therefore, a Volterra integral equation for B(t) is formulated by using the convolution theorem. The expression in Eq. (4.6) is first written as

$$\overline{B}(\zeta)\left(1-\frac{m}{2}\right) - \overline{B}(\zeta)\ \bar{f}\left(\frac{\zeta}{n_o}\right) = \frac{3}{3\lambda_e + 2\mu_e}\left(\frac{1}{\zeta}\right)$$

and noting that $\delta(n_o t) = \frac{1}{n_o}\delta(t)$, the following form is obtained.

$$B(t) - \frac{mn_o}{\pi}\int_0^t B(t-\tau)\,\psi(t;n_o)\ d\tau = \frac{3}{3\lambda_e + 2\mu_e} \tag{4.8}$$

Following the same procedure for the shear compliance, J(t) is determined to be

$$J(t) - \frac{mn_o}{\pi}\int_0^t J(t-\tau)\,\psi(t;n_o)\ d\tau = \frac{1}{\mu_e} \tag{4.9}$$

Although the functions B(t) and J(t) as given in Eqs. (4.8) and (4.9) are not in closed analytic forms, the formulated integral equations can be solved by an iterative method.

4.1 MODELING OF POISSON'S RATIO

It must be restated here that the following development does not follow the assumptions stated in Eq. (3.2b) which result in a constant Poisson's Ratio. However, to demonstrate the use of the present methodology, it will be shown how the time-dependent Poisson's Ratio can be formulated by using the spectrum-based moduli E(t) and K(t).

The expression for the transformed Poisson's ratio in terms of the transforms of the bulk and shear moduli is given as [10]

$$\zeta\overline{\nu}(\zeta) = \frac{3\overline{K}(\zeta) - 2\overline{G}(\zeta)}{6\overline{K}(\zeta) + 2\overline{G}(\zeta)} \tag{4.10}$$

The transform of the shear modulus is defined to be:

$$\overline{G}(\zeta) = \frac{3\overline{K}(\zeta)\,\overline{E}(\zeta)}{9\overline{K}(\zeta) - \overline{E}(\zeta)} \tag{4.11}$$

Using Eq. (4.11) in Eq. (4.10), Poisson's ratio is now expressed as

$$\zeta\overline{\nu}(\zeta) = \frac{1}{2} - \frac{\overline{E}(\zeta)}{6\overline{K}(\zeta)} \tag{4.12}$$

For some polymeric materials the bulk modulus does not change dramatically over the time interval considered, and therefore, if K is approximated as a constant [10,12,13], its transform is

$$\overline{K}(\zeta) = \frac{K_e}{\zeta} \tag{4.13}$$

Using Eq. (4.13) in Eq. (4.12) and inverting the final expression results in

$$\nu(t) = \frac{1}{2} - \frac{E(t)}{6K_e} \tag{4.14}$$

For comparison with the above expression for ν(t), an expression obtained from Hilton [10] is considered with $K_0 = 3K_e$:

$$\nu(t) = \frac{0.5 - \left(G(t) \Big/ 3K_e \right)}{1 + \left(G(t) \Big/ 3K_e \right)} \tag{4.15}$$

From Eq. (4.7), it is noted that for $K(t) = K_e$, both Eqs. (4.14) and (4.15) are equal, as expected. The time-dependent behavior of Poisson's ratio, using the same deformation history as that depicted in Figure 1 and expressed by Eqs. (4.14) or (4.15), is shown in Section 5.

4.3 DETERMINATION OF n AND n_O

Material property constants n and n_o, and subsequently m, must be determined to express the properties of interest as given by Eqs. (4.1), (4.2), (4.4), (4.8) and (4.9). The determination of these constants requires experimental data, which is easily obtained from a simple uniaxial tensile creep test. For such a test case, only one stress component exists and that is in the direction of the applied load. Hooke's Law for linearly elastic isotropic materials in uniaxial tension is

$$e_{11} = \frac{T_{11}}{E_e} \tag{4.16}$$

where E_e is the elastic tensile modulus and has been defined in Eq. (4.5). The Principle of Correspondence (EVCP), which is often used in the formulation of viscoelastic functions [14,15], is applied to Eq. (4.16). This mathematical tool simply states that if the linear elastic solution exists, then the corresponding linear viscoelastic solution can be found by replacing the parameters with corresponding substitutions. The Correspondence Principle is applied to Eq. (4.16) and the strain as a function of the Laplace parameter ζ is obtained as

$$\bar{e}_{11} = \frac{\lambda(\zeta) + \mu(\zeta)}{\mu(\zeta)[3\lambda(\zeta) + 2\mu(\zeta)]} \bar{T}_{11} \tag{4.17}$$

Substituting the viscoelastic memory functions, $\lambda(\zeta)$ and $\mu(\zeta)$ from Eqs. (3.12) into Eq. (4.17) results in an expression for the strain as a function of the Laplace parameter ζ which when inverted becomes

$$\left(1-\frac{m}{2}\right)e_{11}(t) - \mathcal{L}^{-1}\left[\bar{e}_{11}(\zeta)\,\bar{f}\left(\frac{\zeta}{n_o}\right)\right] = \frac{T_{11}(t)}{E_e} \tag{4.18}$$

Using the Laplace tables [11] and the convolution theorem, and observing that $T_{11}(t) = \sigma =$ constant for a tensile creep experiment, the strain as a function of time is

$$e_{11}(t) - \frac{n}{\pi}\int_0^t e_{11}(t-\tau)\,\psi(\tau)d\tau = \frac{\sigma}{E_e} \tag{4.19}$$

where

$$\psi(t) = \int_{n_o t}^{\infty} \frac{\cos(z - n_o t)}{z} dz\,. \tag{4.20}$$

Equation (4.19) is a Volterra integral equation and it can be solved by the method of iteration as

$$e_{11}^{(p+1)}(t) = \frac{\sigma}{E_e} + \frac{n}{\pi}\int_0^t e_{11}^{(p)}(t-\tau)\,\psi(\tau)\,d\tau \qquad p = 1, 2, 3, \ldots \tag{4.21}$$

where $e_{11}^{(p)}$ is the $(p)^{th}$ approximation and $e_{11}^{(p+1)}$ is the $(p+1)^{st}$ approximation.

The time dependent strain is calculated by first determining n_o by choosing the initial time, final time, and an intermediate time. The value at the intermediate time which produces the value of n_o that best fits the experimental strain data is selected. The resulting transcendental equation is solved for n_o and this value is then used to calculate n and the remaining properties.

5. Discussion of Results

Experimental tensile and creep data at various temperatures and stress levels were obtained for a vinyl ester polymer matrix material [16]. Figure 1 shows the creep response of the resin samples at 66 °C and at a stress level of 14 MPa, in both real time (inset) and in log-log scales.

As a first approximation, n and n_o were computed based on the iterative solution of the integral equation given in Eq. (4.19). The figure shows that the correct choice of the material constants produces very good agreement between the experimental and analytical data, particularly after about thirty seconds. For larger times, the model begins to behave like a power law with power two. The percent standard deviation, shown in Table 1, is computed based on the experimental and calculated strain and log(strain) values. The largest variation is computed to be approximately 4% around 2000 seconds. Some error also results from the authenticity of the experimental data which is sometimes quite unreliable in the very early stages of testing. A more accurate determination of the material constants could be accomplished if some experimental data had been available for E(t). This is due to the fact that no approximations have to be made in the calculation of the tensile modulus (Eq. 4.4). The extracted viscoelastic material property constants n and $n_{o,}$ are now used in the formulae for the remaining properties of interest.

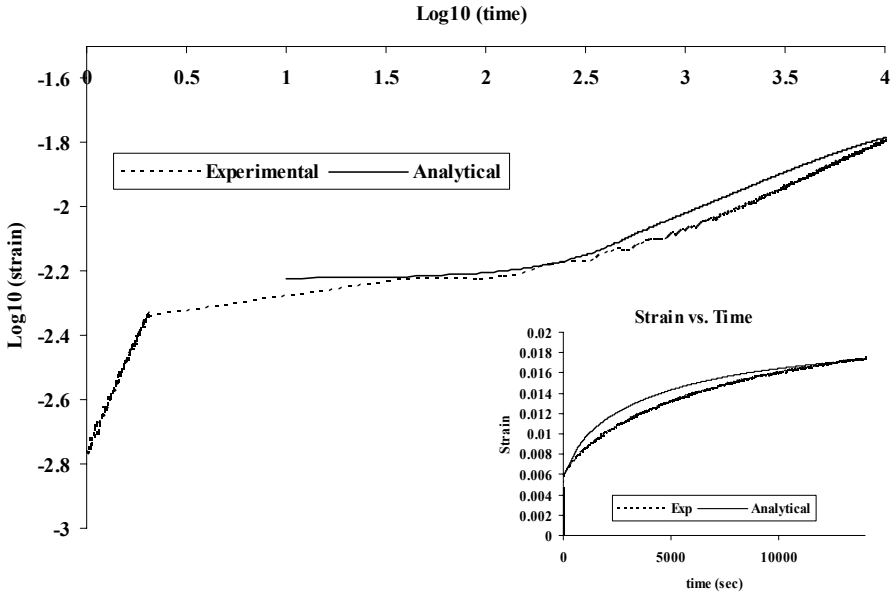

Figure 1.Creep Strain Response: 14 MPa, 66 °C

Table 1. Percent Difference in Experimental and Computed Strain Values (14 MPa, 66 °C)

Time (sec)	Δ_1^*	Δ_2^*
30	1.19E-02	8.78E-01
1000	8.39E-02	4.04E+00
2000	1.02E-01	4.13E+00
4000	9.40E-02	3.15E+00
6000	7.42E-02	2.24E+00
8000	3.74E-02	1.05E+00
10000	2.18E-02	5.82E-01
12000	7.77E-03	2.00E-01
14000	1.60E-06	4.00E-05

Δ_1^* **= Percent standard deviation between experimental and computed strain values.**

Δ_2^* **= Percent standard deviation between experimental and computed log(strain) values.**

Having obtained the values for n and n_0 (hence $m = \frac{n}{n_0}$), for each test case the modulus as a function of time is generated by Eq. (4.4). As mentioned previously, the expression for the time–dependent modulus in Eq. (4.4) is amenable to direct evaluation. The tensile viscoelastic

modulus is one of the properties that is not generally obtained by simply inverting the compliance and, therefore, is not directly obtainable via experiments. Using this methodology, the modulus response can be simulated. Figure 2 shows the degradation in the modulus for the polymer component at 66º C and at a stress level of 14 MPa. This figure also shows the variation of the bulk modulus K(t) and shear modulus G(t) from Eqs. (4.1) and (4.2). Plotting these moduli on a logarithmic ordinate results in the curves being parallel and the multiplication factor being equal to the constants designated in Eq. (3.2b).

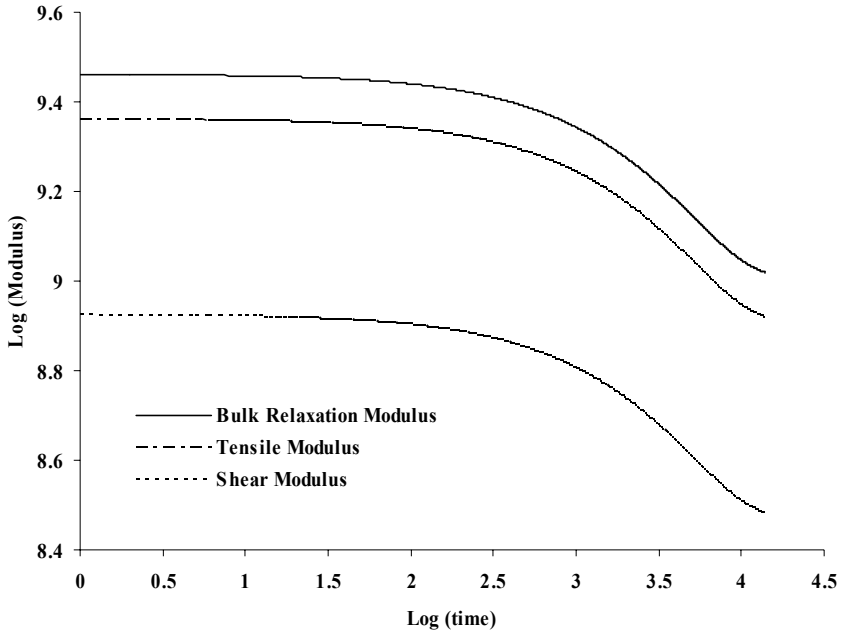

Figure 2. Computed resin modulus response using one distribution function: 14 MPa, 66 °C

Figure 3 is a depiction of the time-dependent behavior of Poisson's ratio using either Eq. (4.14) or (4.15). The reader is again reminded that this depiction of Poisson's ratio is determined solely for the purpose of demonstrating the methodology of using the distribution function. This is generated based on assuming a constant bulk modulus and using the generated tensile modulus based on one distribution function. The reader is referred to Sec. 4.2.

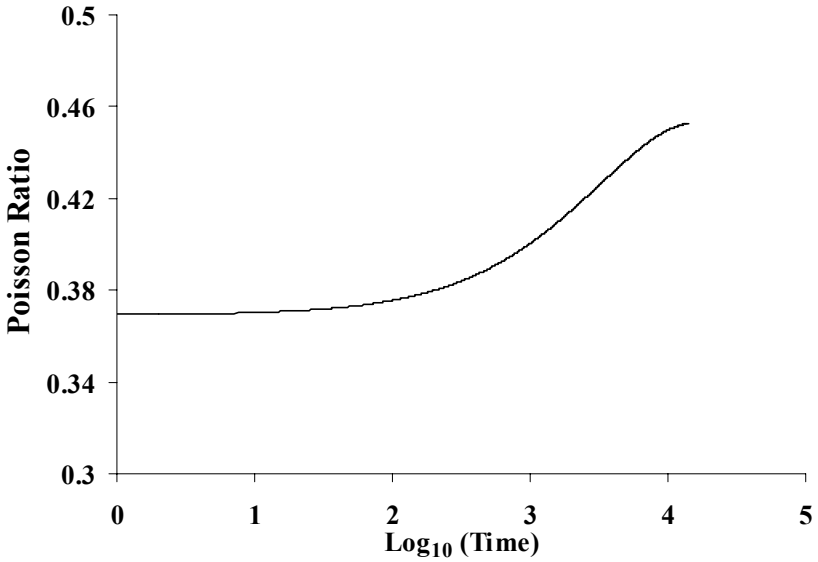

Figure 3. Poisson's Ratio response (14 MPa, 66 °C) for demonstration of the present methodology. (c.f. Section 4.2)

6. Concluding Remarks

A selected spectrum function $\varphi(\alpha)$ is used to develop the fundamental Lame' functions to express mechanical properties of interest of linear viscoelastic materials. It is quite obvious that the chosen spectrum function $\varphi(\alpha)$ is not unique. In fact, other functions may also model the data quite well, but results from the use of these functions may not be easily obtainable and may require further approximations. The Principle of Correspondence is used to express creep strains and moduli in the Laplace domain. The inversion process results in the corresponding numerical solutions in the time domain which are compared to the available experimental data. For time dependent strain, the resulting Volterra integral equation is solved and the computation of the material constants is based on the first approximation of this integral equation.

It was shown that the chosen distribution function has the capability of simulating viscoelastic behavior during the early stages as well as the later time periods. Some discrepancy is seen in the very early time periods and the difficulty in modeling this initial behavior could be due to the

uncertainty in initial experimental data. However, as the time increases, the match between the experimental and analytical data also increases, producing excellent agreement for large times. It is also possible that a more accurate determination of the constants could be made if some tensile modulus data was available. This is due to the fact that the time-dependent modulus is directly computable and approximations would be unnecessary. Because of its availability, the time-dependent strain data was used purely as a demonstrative tool for this methodology.

Functional proportionality is observed due to using the same distribution function for developing both $\lambda_v(t)$ and $\mu_v(t)$. Foremost, the ratio of the Lame' functions produces a constant viscoelastic Poisson's ratio. To demonstrate the versatility of developing the basic Lame' functions by a proper choice of the spectrum function, a time-dependent Poisson's ratio was generated by using a constant bulk modulus and the developed time-dependent tensile modulus. Although the problem becomes more complex, functional similarity becomes non-existent by simply using different spectrum functions for the Lame functions.

This methodology demonstrates the potential of using a realistic distribution function to characterize viscoelastic behavior. Properties that are either not computable from experimental testing (moduli) or that are difficult to measure, such as Poisson's ratio, can be modeled. The development of the memory functions through the use of this spectrum function allows for a compact and complete representation of the pertinent viscoelastic functions.

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