On the Use of Asymptotics in Nonlinear Boundary Value Problems (*).

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Summary. – *We consider the solvability of some nonlinear boundary value problems for differential equations where the nonlinearity is bounded. This involves the study of the asymptotic behaviour of certain multivalued functionals.*

In this paper, we show how the study of the asymptotic behaviour of certain integrals when a parameter is large can be used to obtain results on the solvability of some nonlinear boundary value problems of the form

$$Lu = g(u) - f .$$

Here L is a linear differential operator, $g\colon R \to R$ is continuous and either $g(y) \to 0$ as $|y| \to \infty$ or $g(y) = a \sin y$. Our results considerably improve work of a number of authors e.g. FUCIK [14], HESS [17], DANCER [8] and many others. A more complete bibliography can be found in [14]. We do not aim to obtain the best possible results but merely consider some simple special cases which illustrate our techniques. However, we do mention a number of ways in which our results generalize. Indeed, our methods often becomes difficult to apply at the higher eigenvalues of a self-adjoint elliptic operator. (The difficulty is in doing the asymptotics.) The results here are probably the most important of the results announced in [5]. We apologize for the delay in writing them up. This was caused by the author's interest being diverted to other mathematical topics. A good deal of interest has been expressed in the results. For example, they are mentioned in [13]. Our results solve or partially solve some of the problems in [13], [14] and [15].

In § 1, we prove some technical abstract results and, in § 2, we consider the case where $g(y) \to 0$ as $|y| \to \infty$. Finally, in § 3, we consider the case where $g(y) = a \sin y$.

We assume a basic knowledge of the standard results on the solvability of ordinary and elliptic partial differential equations in Sobolev spaces and the standard embedding theorems. They can be found in [14].

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1. – Abstract results.

Assume that E and X are Banach spaces, that $L: E \to X$ is Fredholm of index zero but not invertible and that $H: E \to X$ is completely continuous such that $\|u\|^{-1}\|H(u)\|_1 \to 0$ as $\|u\| \to \infty$. Here $\|\ \|$ and $\|\ \|_1$ denote the norms on E and X respectively. Choose continuous linear projections Q and P on E and X respectively such that $R(P) = R(L)$ and $R(I-Q) = N(L)$. Here $N(L)$ and $R(L)$ denote the kernel and range of L respectively. It is easy to see that L is invertible *when considered as a map* between the Banach spaces $R(Q)$ and $R(P)$. Let K denote its inverse. (Thus K maps $R(P)$ onto $R(Q)$.) Now the equation

$$Lx = H(x) \tag{1}$$

is equivalent to the pair of equations

$$v = KPH(u+v) \tag{2}$$

$$(I-P)H(u+v) = 0\,, \tag{3}$$

where $x = u + v$ with $u \in N(L)$ and $v \in R(Q)$. Note that, by our assumptions, the map $u + v \to KPH(u+v)$ is completely continuous and that equation (2) involves only the *component* $PH(x)$ of $H(x)$. Let $\mathfrak{D} = \{u + v \in E: u + v \text{ is a solution of (2)}\}$. If $N(L)$ is one-dimensional, we write elements of E as $\alpha h + v$ where h spans $N(L)$ and $v \in R(Q)$. Suppose $l \in E^*$ such that $(I-Q)x = l(x)h$ and $l(h) = 1$.

PROPOSITION. – *If $N(L)$ is one-dimensional, there is a closed connected subset T of $\mathfrak{D}$ such that $l(T) = (-\infty, \infty)$.*

PROOF. – We merely sketch the proof since it is a standard application of known techniques (cp. RABINOWITZ [20], TURNER [23] or DANCER [9]).

STEP 1. – *For each $n > 0$ there is a connected subset T_n of $\{x \in \mathfrak{D}: |l(x)| \leqslant n\}$ such that $l(T_n) = [-n, n]$.* To see this, define $A: R(Q) \oplus R \to R(Q)$ by $A(v, \alpha) = KPH(\alpha h + v)$. By our assumptions, A is completely continuous and $\|v\|^{-1}A(v, \alpha) \to 0$ as $\|v\| \to \infty$ uniformly for $\alpha \in [-n, n]$. Choose $K > 0$ such that $\|A(v, \alpha)\| \leqslant \frac{1}{2}\|v\|$ if $\|v\| \geqslant K$ and $\alpha \in [-n, n]$. Thus, by using the homotopy $(v, t) \to v - tA(v, \alpha)$ we see that $\deg\big(I - A(\ , \alpha), B_K\big) = 1$ for each $\alpha \in [-n, n]$. Here B_K is the ball of radius K in $R(Q)$ and deg denotes the Leray-Schauder degree. The existence of T_n now follows from RABINOWITZ [22, Lemma A5].

STEP 2. – *Completion of the proof.* Since the closure of a connected set is connected, we may assume that T_n is closed. By the argument in Step 1, $\|v\| \leqslant K$ if $\alpha h + v \in \mathfrak{D}$ and $\alpha \in [-n, n]$. Thus, since A is completely continuous, $\{x \in \mathfrak{D}: |l(x)| \leqslant n\}$ is compact. Hence we can compactify $\mathfrak{D}$ and obtain a compact metric space $\tilde{\mathfrak{D}}$ by adding two points « ∞ » and « $-\infty$ » corresponding to $\alpha = \infty$ and $\alpha = -\infty$ respectively. (A similar argument appears in § 3 of [9].) Let $\tilde{T} = \limsup_{n\to\infty} T_n$ in the sense of WHYBURN [26, p. 7]. (*Here we are working in the metric space* $\tilde{\mathfrak{D}}$.) By [26, Theorem 9.1], $\tilde{T}$ is connected and compact. (Since T_n contains points with $l(x) = \pm n$, it follows that $\pm\infty \in \liminf_{n\to\infty} T_n$.) Hence, by a wellknown result (cp. ALEXANDER [1, Proposition 5], or WARD [25]), there is a component T of $\tilde{T}\setminus\{\pm\infty\}$ such that $\pm\infty$ are in $\bar{T}$. Now T can be thought of as a subset of $\mathfrak{D}$. With this identification, it is easy to see that T has the required properties. This completes the proof.

It is often convenient to write elements of T as $\alpha h + w(\alpha)$ where $w(\alpha) \in R(Q)$. *Note that, in general, $w(\alpha)$ is multi-valued.* Since T is connected, its image $\{l(H(\alpha h + w(\alpha)): \alpha h + w(\alpha) \in T\}$ is a connected subset of R. Thus, we see that, if there exist $\alpha_0, \alpha_1 \in R$ such that $l(H(\alpha_0 h + w(\alpha_0)) < 0$ for every $\alpha_0 h + w(\alpha_0) \in \mathfrak{D}$ and $l(H(\alpha_1 h + w(\alpha_1)) > 0$ for every $\alpha_1 h + w(\alpha_1) \in \mathfrak{D}$, then there exist α_2 between α_0 and α_1 and $\alpha_2 h + w(\alpha_2) \in T$ such that $l(H(\alpha_2 h + w(\alpha_2)) = 0$. (Otherwise, $\{\alpha h + w(\alpha) \in T$: either (i) $\alpha_0 \leqslant \alpha \leqslant \alpha_1$ and $l(H(\alpha h + w(\alpha)) < 0$, or (ii) $\alpha \leqslant \alpha_0\}$ and $\{\alpha h + w(\alpha) \in T$: either (i) $\alpha_0 \leqslant \alpha \leqslant \alpha_1$ and $l(H(\alpha h + w(\alpha)) > 0$ or (ii) $\alpha \geqslant \alpha_1\}$ are a disconnection of T). Thus we see that $\alpha_2 h + w(\alpha_2)$ is a solution of (1) for some $\alpha_2 h + w(\alpha_2) \in T$. This is the form in which we will use the proposition.

Let us now assume that $\|H(u)\|_1 \leqslant K_1$ on E. Then, by (2), there is an $M > 0$ such that $\|w(\alpha)\| \leqslant M$ whenever $\alpha h + w(\alpha) \in \mathfrak{D}$. Let us further assume that there is an $\tilde{f}$ in X such that $H(\alpha h + w(\alpha)) \to \tilde{f}$ as $\alpha \to \infty$ whenever $\alpha h + w(\alpha) \in \mathfrak{D}$. (For example, this holds if $H(\alpha h + u) \to \tilde{f}$ as $\alpha \to \infty$ uniformly for u in E such that $\|u\| \leqslant M$.) By equation (2), it follows that $w(\alpha) \to KP\tilde{f}$ in E as $\alpha \to \infty$. Note that the limit is singlevalued even though $w(\alpha)$ is multi-valued. This result will be of considerable use in the study of the asymptotic behaviour of $l(H(\alpha h + w(\alpha))$ in § 2 and § 3. Similarly, since continuous linear mapping are weakly continuous, we see that, if $H(\alpha h + w(\alpha)) \rightharpoonup \tilde{f}$ (weakly) as $\alpha \to \infty$, then $w(\alpha) \rightharpoonup KP\tilde{f}$ (weakly) as $\alpha \to \infty$.

We now comment briefly on the case where $N(L)$ is multi-dimensional. Assume that there is a map $t: N(L) \to R(I - P)$, a bounded neighbourhood W of zero in $N(L)$ and a continuous map $h: \mathfrak{D}\times[0, 1] \to R(I - P)$ such that (i) $h(\ , 0) = (I - P)H|_{\mathfrak{D}}$; (ii) $h(\ , 1) = t(I - Q)|_{\mathfrak{D}}$; (iii) $h(u + w(u), s) \neq 0$ if $0 \leqslant s \leqslant 1$, if $u \in \partial W$ and if $u + w(u) \in \mathfrak{D}$ and (iv) $\deg(t, W) \neq 0$. (Here, as before, $w(u)$ is the multi-valued map such that $\mathfrak{D} = \{u + w(u): u \in N(L)\}$.) Then (1) has solution. This is proved by showing that the map $(u, v) \to (Z(I - P)H(u + v), v - KPH(u + v))$ has degree $\pm\deg(t, W)$ on $W\times B$, where B is a ball of large radius in $R(Q)$ and Z is a linear isomorphism of $R(I - P)$ and $N(L)$. Of course, in applications, W is usually something like a large ball. For example, it is easy to see that the above assumptions are satisfied when there is a scalar product $\langle\ ,\ \rangle$ on X such that $\langle g(u + w(u)), Zu\rangle > 0$ (< 0) if $u \in \partial W$. In addition, one can establish the existence of more than one solution if there exist W_i and t_i $(i = 1, 2)$ as above such that $W_1 \subset W_2$ and $\deg(t_1, W_1) \neq \deg(t_2, W_2)$. In practice, this result on the existence of more than one solution are much harder to apply than the corresponding results in the one-dimensional case (especially if $N(L)$ is even-dimensional).

2. – The case of vanishing at infinity.

Let L denote the linear operator $-(d/dt)(py') + qy$ on $L^1[0, \pi]$, where the domain of L is $\{u \in W^{2,1}[0, \pi]: u(0) = u(\pi) = 0\}$. We assume that p and q are smooth on $[0, \pi]$. (This is a much stronger condition than we really need.) It is well known and easy to prove that L is Fredholm of index zero. Moreover, if $h \in N(L)\setminus\{0\}$, $|h(s)| + |h'(s)| > 0$ on $[0, \pi]$. We wish to study the solvability of

$$Lu = g(u) - f, \tag{4}$$

where $g: R \to R$ is continuous, $g(y) \to 0$ as $|y| \to \infty$ and $f \in L^1[0, \pi]$. *We assume these conditions throughout the section unless explicit mention is made to the contrary. Let* $R = \{f \in L^1[0, \pi]:$ (4) has a solution$\}$. If $N(L) = \{0\}$, it is well known that $\mathfrak{R} = L^1[0, \pi]$. *Thus we consider the case where* $N(L) \neq \{0\}$. Since $N(L)$ is one dimensional, we can choose an h which spans $N(L)$. It is convenient to choose h such that $h'(0) > 0$ and $\int_0^\pi h^2\, ds = 1$. Note that $L^1[0, \pi] = R(L) \oplus \operatorname{span}\{h\}$.

We first consider the case where $h(s) > 0$ *on* $[0, \pi]$. (In other words, zero is the smallest eigenvalue of $Lu = \lambda u$.) Since the natural embedding of $W^{2,1}[0, \pi]$ into $C[0, \pi]$ is compact, it is easy to see that the basic assumptions of § 1 hold (where $E = D(L)$ with the usual $W^{2,1}[0, \pi]$ norm, $X = L^1[0, \pi]$ and $H(u) = g(u) - f$). We take l to be the linear functional defined by $l(x) = \int_0^\pi x(s)h(s)\, ds$. Since g is bounded on R, there is an $M > 0$ such that $\|H(u)\|_1 \leq M$ on E, where $\|\ \|_1$ denotes the usual L^1 norm. As in § 1, it follows that, if $\alpha h + w(\alpha)$ is a solution of (2), then $w(\alpha)$ is bounded in $W^{2,1}[0, \pi]$. Hence $w(\alpha)$ is bounded in $C[0, \pi]$. Thus $|\alpha h(s) + w(\alpha)(s)|$ is large if $|\alpha|$ is large except near the ends of the interval. Since $|g(y)| \leq K$ on R and $g(y) \to 0$ as $|y| \to \infty$, it follows by a simple estimation that $g(\alpha h + w(\alpha)) \to 0$ in $L^1[0, \pi]$ as $|\alpha| \to \infty$.

As in § 1, it follows that $w(\alpha) \to Sf_0$ in $W^{2,1}[0, \pi]$ as $\alpha \to \infty$, where Sf_0 is the solution in $R(Q)$ of $Lu = -f_0$ (with $(I - P)x = l(x)h$, $Q = P|_{\mathfrak{D}(L)}$ and $f_0 = Pf$). Note that $w(\alpha)$ really depends upon f_0 rather than f. (Equation (2) can be written as $v = K\big(P(g(\alpha h + v) - f_0)\big)$.) The above argument is still valid with minor modifications even if h has zeros in $(0, \pi)$.

We now want to estimate $F(\alpha) \equiv \int_0^\pi g(\alpha h + w(\alpha))h\, ds$. Note that $F(\alpha)$ may be multi-valued, that $F(\alpha)$ depends on f_0 and that $f_0 + th \in \mathfrak{R}$ if and only if t is in the range of F (since (3) can be written as $l(f) = l\big(g(\alpha h + w(\alpha)\big)$. Thus, we expect $F(\alpha)$ to be the critical term for studying solvability. *Fix* $f_0 \in R(L)$. *We assume that* $yg(y) \geq 0$ *for* y *large and that* $Z(x) \equiv \int_0^x g(y)y\, dy > 0$ *for all* x *large.* The last condition certainly holds if $\int_0^\infty g(y)y\, dy$ diverges. Choose $K > 0$ such that $Z(Kh'(0)) > 0$,

$Z(-Kh'(\pi)) > 0$ and $yg(y) \geq 0$ for $y \geq \inf\{K, Kh'(0), -Kh'(\pi)\}$. Since the natural embedding of $W^{2,1}[0,\pi]$ into $C^1[0,\pi]$ is continuous, $w(\alpha)$ are uniformly bounded in $C^1[0,\pi]$. Since $w(\alpha)(0) = w(\alpha)(\pi) = 0$, $h'(0) > 0$, $h'(\pi) < 0$ and $h(s) > 0$ on $(0,\pi)$, it follows easily that there exist $K_1 \geq K$ and $\tau > 0$ such that $\alpha h(s) + w(\alpha)(s) \geq K$ if $\alpha h + w(\alpha) \in \mathfrak{D}$, if $\alpha \geq \tau$ and if $\alpha^{-1}K_1 \leq s \leq \pi - \alpha^{-1}K_1$. Hence

$$g(\alpha h(s) + w(\alpha)(s))h(s) \geq 0 \quad \text{if} \quad \alpha \geq \tau \quad \text{and} \quad \alpha^{-1}K_1 \leq s \leq \pi - \alpha^{-1}K_1 .$$

Thus we will prove that $\alpha^2 F(\alpha) > 0$ (and hence $F(\alpha) > 0$) for α large positive if we show that

$$\alpha^2 F_1(\alpha) \equiv \int_0^{|\alpha|^{-1}K_1} \alpha^2 g(\alpha h + w(\alpha))h\,ds$$

and

$$\alpha^2 F_2(\alpha) = \int_{\pi - |\alpha|^{-1}K_1}^{\pi} \alpha^2 g(\alpha h + w(\alpha))h\,ds$$

are both positive for large positive α. (When we say that $F(\alpha) > 0$ we mean that every element of the compact set $F(\alpha)$ is strictly positive.)

Now

$$\alpha^2 F_1(\alpha) = \int_0^{K_1} g\big(\alpha h(|\alpha|^{-1}u) + w(\alpha)(|\alpha|^{-1}u)\big)|\alpha| h(|\alpha|^{-1}u)\,du\,,$$

where $s = |\alpha|u$. Since $w(\alpha)$ is bounded in $C^1[0,\pi]$ and $w(\alpha)(0) = 0$, we easily see that

$$\alpha h(|\alpha|^{-1}u) + w(\alpha)(|\alpha|^{-1}u) \to h'(0)u$$

as $\alpha \to \infty$. Thus, by the dominated convergence theorem

$$\alpha^2 F_1(\alpha) \to \int_0^{K_1} g(h'(0)u)h'(0)u\,du = (h'(0))^{-1} Z(K_1 h'(0))$$

as $\alpha \to \infty$. By our assumptions, $Z(K_1 h'(0)) \geq Z(Kh'(0)) > 0$. Thus $\alpha^2 F_1(\alpha) > 0$ for α large positive. (Note that the multivaluedness of $w(\alpha)$ does not affect our arguments. Our use of the dominated convergence theorem shows that, if $\alpha h + \tilde{t}(\alpha) \in \mathfrak{D}$ (i.e. $\tilde{t}(\alpha) \in w(\alpha)$), then

$$\int_0^{|\alpha|^{-1}K} g(\alpha h + \tilde{t}(\alpha))h\,dt > 0 \qquad \text{for } \alpha \geq \alpha_0$$

where α_0 is independent of the choice of $\tilde{t}(\alpha)$.) We can estimate $\alpha^2 F_2(\alpha)$ by using the change of variable $u = |\alpha|(\pi - t)$ and then using a similar argument. Thus, we eventually find that there is an $\alpha_0 > 0$ such that $\alpha^2 F(\alpha) > 0$ (and thus $F(\alpha) > 0$)

for $\alpha \geqslant \alpha_0$. Similarly, if there is a $K > 0$ such that $yg(y) \geqslant 0$ for $y \leqslant -K$ and $\int_{-K}^{0} g(y)y\,dy > 0$, we find that $F(\alpha) < 0$ for α large negative. We can now easily prove our first main result.

THEOREM 1. – *Assume that $h(s) > 0$ on $(0, \pi)$, that $yg(y) \geqslant 0$ for $|y|$ large, that $\int_0^\infty g(y)y\,dy > 0$ and that $\int_{-\infty}^0 g(y)y\,dy > 0$ (where the integrals may diverge to $+\infty$). For each $f_0 \in R(L)$, there is an $\varepsilon > 0$ such that $f_0 + th \in \mathfrak{R}$ if $|t| \leqslant \varepsilon$.*

PROOF. – Note that $f_0 + th \in \mathfrak{R}$ if and only if t is in the range of the multi-valued map F. We proved above that there exist $\alpha_1 > 0$ and $\alpha_2 < 0$ such that $F(\alpha_1) > 0$ and $F(\alpha_2) < 0$. Let T be the connected set of Proposition 1. As in § 1, $T_1 \equiv \{l(g(u)) \colon u \in T\}$ is an *interval*. Since there exist $u_1, u_2 \in T$ such that $l(u_1) = \alpha_1$ and $l(u_2) = \alpha_2$, it follows that T_1 contains positive and negative elements. Thus T_1 contains a neighbourhood of zero and the result follows.

We now consider the case where the signs of the integrals in Theorem 1 are reversed. (Thus $\int_0^\infty g(y)y\,dy < 0$ and $\int_{\infty-}^0 g(y)y\,dy < 0$, where it is assumed that the integrals converge.) We say that g is *regular* if there exists $d\colon R \to R$ such that $d(y) \to 0$ as $|y| \to \infty$, $d(y)\operatorname{sgn} y$ is decreasing for $|y|$ large, $|g(y)| \leqslant |d(y)|$ for $|y|$ large and $\int_{-\infty}^{\infty} d(y)y\,dy$ converges. Note that this slightly generalizes the definition of regular in [5]. We define F_1 and F_2 as before *except that the ranges of integration are from* 0 *to* $\frac{1}{2}\pi$ *and* $\frac{1}{2}\pi$ *to* π *respectively*. By our assumptions and earlier arguments, we easily see that there exist $A, B, C > 0$ such that $As \leqslant h(s) \leqslant Bs$ and $|w(\alpha)(s)| \leqslant Cs$ on $[0, \frac{1}{2}\pi]$. (Remember that $w(\alpha)$ is bounded in $C^1[0, \pi]$ uniformly in α and $w(\alpha)(0) = 0$.) Thus

$$(\alpha A - C)s \leqslant \alpha h(s) + w(\alpha)(s) \leqslant (\alpha B + C)s \tag{5}$$

on $[0, \frac{1}{2}\pi]$ if $\alpha \geqslant 0$. As before, by using the substitution $|\alpha|s = u$, we find that

$$\alpha^2 F_1(\alpha) = \int_0^{\frac{1}{2}\pi|\alpha|} g\big(\alpha h(|\alpha|^{-1}u) + w(\alpha)(|\alpha|^{-1}u)\big)|\alpha|h(|\alpha|^{-1}u)\,du = \int_0^\infty r(\alpha)(u)\,du\,,$$

where $r(\alpha)(u) = g\big(\alpha h(|\alpha|^{-1}u) + w(\alpha)(|\alpha|^{-1}u)\big)|\alpha|h(|\alpha|^{-1}u)$ if $u \leqslant \frac{1}{2}\pi|\alpha|$ and is zero otherwise. As before, we easily see that $r(\alpha)(u) \to g(h'(0)u)h'(0)u$ as $\alpha \to \infty$ for each fixed $u \geqslant 0$. It is easy to see that there is a $K_2 > 0$ such that $|r(\alpha)(u)| \leqslant K_2$ for $u \geqslant 0$ and $\alpha > 0$. Moreover, by (5) and the assumption that g is regular,

$$|r(\alpha)(u)| \leqslant d\big((A - \alpha^{-1}C)u\big)Bu \leqslant Bd\big((A-1)u\big)u$$

if u is large and $\alpha \geqslant C$. (Remember that $d(u)$ is decreasing in u if u is large.) Since $\int_0^\infty d(u)u\,du$ exists, we can apply the dominated convergence theorem, and deduce that

$$\alpha^2 F_1(\alpha) \to \int_0^\infty g\big(h'(0)u\big)h'(0)u\,du = \big(h'(0)\big)^{-1}\int_0^\infty g(v)v\,dv < 0$$

as $\alpha \to \infty$. (Remember that $\int_0^\infty g(y)y\,dy < 0$ and $h'(0) > 0$.) Since we can obtain a similar result for $F_2(\alpha)$ and since $F(\alpha) = F_1(\alpha) + F_2(\alpha)$, it follows that $F(\alpha) < 0$ for large positive α. Similarly, since $\int_{-\infty}^0 g(u)u\,du < 0$, $F(\alpha) > 0$ for α large negative. By the same argument as in the proof of Theorem 1, the theorem below follows.

THEOREM 2. – *Assume that* $h(s) > 0$ on $(0, \pi)$, *that g is regular, that* $\int_0^\infty g(y)y\,dy < 0$ *and that* $\int_{-\infty}^0 g(y)y\,dy < 0$. *Then, for each* $f_0 \in R(L)$, *there is an* $\varepsilon > 0$ *such that* $f_0 + th \in \mathcal{R}$ *if* $|t| \leqslant \varepsilon$.

Note that, in Theorem 2, g may change sign at points where $|y|$ is arbitrarily large. The assumptions on the signs of the integrals could be replaced by the single condition $\int_0^\infty g(y)y\,dy \int_{-\infty}^0 g(y)y\,dy > 0$.

We now wish to consider the *case where h has zeros in* $(0, \pi)$. *Assume that* $t_i \in (0, \pi)$, $h(t_i) = 0$ *and* $y\ g(y) \geqslant 0$ *for* $|y|$ *large.* Thus $h'(t_i) \neq 0$. Suppose $f_0 \in R(L)$. Let

$$\tilde{F}_i(\alpha) \equiv \int_{t_i - |\alpha|^{-1}K}^{t_i + |\alpha|^{-1}K} g\big(\alpha h + w(\alpha)\big)h\,ds\,,$$

where K is large. If we use the change of variable $u = \alpha(s - t_i)$, we find that

$$\alpha^2 \tilde{F}_i(\alpha) = \int_{-K}^{K} g\big(\alpha h(t_i + \alpha^{-1}u) + w(\alpha)(t_i + \alpha^{-1}u)\big)\alpha h(t_i + \alpha^{-1}u)\,du$$

for $\alpha > 0$. Now $w(\alpha) \to Sf_0$ uniformly on $[0, \pi]$ as $\alpha \to \infty$ and $\alpha h(t_i + \alpha^{-1}u) \to$ $\to h'(t_i)u$ as $\alpha \to \infty$. Thus

$$\alpha^2 \tilde{F}_i(\alpha) \to \int_{-K}^{K} g\big(h'(t_i)u + Sf_0(t_i)\big)h'(t_i)u\,du = |h'(t_i)|^{-1}\int_{-K|h'(t_i)|}^{K|h'(t_i)|} g\big(v + Sf_0(t_i)\big)v\,dv\,.$$

Since $w(\alpha)$ is uniformly bounded in $C^1[0, \pi]$, since $h'(t_i) \neq 0$ when $h(t_i) = 0$, and since $yg(y) \geqslant 0$ for $|y|$ large, we easily see that the integrand in the expression for $F(\alpha)$ is non-negative except within $\alpha^{-1}K_1$ of one of the zeros of h, where K_1 is a large constant. Hence, we see that, if K is large, then

$$\alpha^2 F(\alpha) \geqslant (h'(0))^{-1}\int_0^{Kh'(0)} g(v)\,v\,dv + (h'(\pi))^{-1}\int_0^{Kh'(\pi)} g(v)\,v\,dv + \sum_{i=1}^{n} |h'(t_i)|^{-1}\int_{-K|h'(t_i)|}^{K|h'(t_i)|} g\big(v + Sf_0(t_i)\big)\,v\,dv \tag{6}$$

if $\alpha \geqslant \alpha(K)$, where $t_1 < t_2 \ldots < t_n$ are the zeros of h in $(0, \pi)$. (The contributions at 0 and π are calculated by similar arguments to those in the proof of Theorem 1.) Let us consider two cases.

Case (i). – $\int_{-\infty}^{\infty} g(y)y\,dy$ *diverges* (necessarily to $+\infty$). It follows that, for any $v \in R$, $\int_{-\infty}^{\infty} g(y+v)y\,dy$ diverges to $+\infty$. (Otherwise, $\int_{-\infty}^{\infty} g(y)(y-v)\,dy$ would converge. Since $g(y)y$ is dominated by $2g(y)(y-v)$ for $|y|$ large, it would follow that $\int_{-\infty}^{\infty} g(y)y\,dy$ converges). It follows easily that the right hand side of (6) tends to ∞ as $K \to \infty$. (Note that, if $h'(\pi) < 0$, the second term is $|h'(\pi)|^{-1}\int_{-K|h'(\pi)|}^{0} g(y)y\,dy$.) It follows from (6) that $\alpha^2 F(\alpha) > 0$ (and hence $F(\alpha) > 0$) for α large positive.

Case (ii). – $\int_{-\infty}^{\infty} g(y)y\,dy$ *converges*. A simple comparison shows that $\int_{-\infty}^{\infty} g(y+v)y\,dy = \int_{-\infty}^{\infty} g(y)(y-v)\,dy$ converges and that $\int_{-\infty}^{\infty} g(y)\,dy$ converges. Moreover, $\int_{-\infty}^{\infty} g(y+v)y\,dy = \int_{-\infty}^{\infty} g(y)y\,dy - v\int_{-\infty}^{\infty} g(y)\,dy$. Thus, as $K \to \infty$, the right hand side of (6) tends to

$$Z_+(f_0) \equiv (h'(0))^{-1}\int_0^{\infty} g(y)\,y\,dy + \int_{-\infty}^{\infty} g(y)\,y\,dy \sum_{i=1}^{n} |h'(t_i)|^{-1} - \int_{-\infty}^{\infty} g(y)\,dy \sum_{i=1}^{n} Sf_0(t_i)|h'(t_i)|^{-1} + \Big(h'(\pi)^{-1}\int_0^{z\infty} g(y)\,y\,dy\Big),$$

where $z = \operatorname{sgn} h'(\pi)$. Thus, if $Z_+(f_0) > 0$, the right hand side of (6) is positive for large K and hence $F(\alpha) > 0$ for α large positive.

By similar arguments, we find that $F(\alpha) < 0$ for α large negative if either $\int_{-\infty}^{\infty} g(y)y\,dy$ diverges or if $\int_{-\infty}^{\infty} g(y)y\,dy$ converges *and* $Z_-(f_0) > 0$, where

$$Z_-(f_0) = (h'(0))^{-1}\int_{-\infty}^{0} g(y)\,y\,dy + \int_{-\infty}^{\infty} g(y)\,y\,dy \sum_{i=1}^{n} |h'(t_i)|^{-1} - \int_{-\infty}^{\infty} g(y)\,dy \sum_{i=1}^{n} Sf_0(t_i)\,|h'(t_i)|^{-1} - (h'(\pi))^{-1}\int_0^{-z\infty} g(y)\,y\,dy\,.$$

Let us now consider the regular case. We choose s_i such that $0 < s_1 < t_1 \ldots < t_n < s_{n+1} < \pi$. We estimate $\alpha^2 \tilde{F}_i(\alpha)$ where

$$\alpha^2 \tilde{F}_i(\alpha) \equiv \alpha^2 \int_{s_i}^{s_{i+1}} g(\alpha h + w(\alpha)) h \, ds = \int_{\alpha(s_i - t_i)}^{\alpha(s_{i+1}-t_i)} g(\alpha h(t_i + \alpha^{-1}u) + w(\alpha)(t_i + \alpha^{-1}u)) \alpha h(t_i + \alpha^{-1}u) \, du \, .$$

Since $w(\alpha)$ is bounded in $C^1[0, \pi]$,

$$|w(\alpha)(t_i) - w(\alpha)(t_i + \alpha^{-1}u)| \leqslant K\alpha^{-1} u \, .$$

Moreover $w(\alpha)(t_i) \to Sf_0(t_i)$ as $\alpha \to \infty$. Thus, by using the dominated convergence theorem in essentially the same way as in the proof of Theorem 2, we find that

$$\alpha^2 \tilde{F}_i(\alpha) \to \int_{-\infty}^{\infty} g(h'(t_i)u + Sf_0(t_i)) h'(t_i) u \, du = |h'(t_i)|^{-1} \int_{-\infty}^{\infty} g(y + Sf_0(t_i)) y \, dy \, .$$

Since, we could estimate the other terms similarly, we eventually find $\alpha^2 F(\alpha) \to Z_+(f_0)$ as $\alpha \to \infty$. Similarly, $\alpha^2 F(\alpha) \to - Z_-(f_0)$ as $\alpha \to -\infty$. As in the proofs of Theorems 1 and 2, estimates on the sign of $F(\alpha)$ for $|\alpha|$ large yield theorems on the solvability of (4). In this way, we obtain the following theorem.

THEOREM 3. – *Suppose that $f_0 \in R(L)$ and h has a zero in $(0, \pi)$. Assume that (a) $yg(y) \geqslant 0$ for $|y|$ large and $\int_{-\infty}^{\infty} g(y) y \, dy$ diverges or (b) $yg(y) \geqslant 0$ for $|y|$ large, $\int_{-\infty}^{\infty} g(y) y \, dy$ converges, $Z_+(f_0) > 0$ and $Z_-(f_0) > 0$ or (c) g is regular and $Z_+(f_0) Z_-(f_0) > 0$. Then there is an $\varepsilon > 0$ such that $f_0 + th \in \mathfrak{R}$ if $|t| \leqslant \varepsilon$.*

Note that $Z_+(f_0)$ and $Z_-(f_0)$ are independent of f_0 precisely when $\int_{-\infty}^{\infty} g(y) \, dy = 0$. If this integral is non-zero (and $\int_{-\infty}^{\infty} yg(y) \, dy$ converges), there must be an $f_0 \in R(L)$ such that $Z_+(f_0) Z_-(f_0) = 0$.

In some cases, we can use the variational structure of our equation to obtain additional results. Note that some related weaker results appear in [10]. We say that *Assumption U* holds if g is continuous differentiable on R and if $\mu < g'(y) < \nu$ on R, where $\mu = \sup \sigma(L) \cap (-\infty, 0)$ and $\nu = \inf \sigma(L) \cap (0, \infty)$. Here $\sigma(L)$ denotes the spectrum of L. (In fact, our main results below would still hold if this assumption were replaced by appropriate Lipschitz conditions on g.) *We now suppose that Assumption U holds.* In this case, it is easy to prove that $w(\alpha)$ is single-valued and the map $\alpha \to w(\alpha)$ is continuously differentiable. *Assume in addition that $\int_{-\infty}^{\infty} g(y) \, dy$ exists.* Now it is well-known (cp. RABINOWITZ [21] or DANCER [6] for similar arguments) that, in this case, $F(\alpha)$ is the gradient of $\tilde{F}(\alpha) = \frac{1}{2} \langle Lw(\alpha), w(\alpha) \rangle +$

$\int_0^\pi G(\alpha h + w(\alpha))\, dt$, where $G(y) = \int_0^y g(t)\, dt$ and $\langle\, ,\rangle$ is the usual scalar product on $L^2[0, \pi]$. Now it is easy to see that

$$\tilde{F}(\alpha) \to \tilde{F}(\infty) \equiv \tfrac{1}{2}\langle LSf_0, Sf_0\rangle + G(\infty)A_1 + G(-\infty)(\pi - A_1) \qquad \text{as } \alpha \to \infty\,,$$

where A_1 is the measure of $\{x \in [0, \pi]: h(x) > 0\}$. Similarly,

$$\tilde{F}(\alpha) \to \tilde{F}(-\infty) \equiv \tfrac{1}{2}\langle LSf_0, Sf_0\rangle + G(-\infty)A_1 + G(\infty)(\pi - A_1) \qquad \text{as } \alpha \to -\infty\,.$$

Now $\tilde{F}(\infty) = \tilde{F}(-\infty)$ if $A_1 = \frac{1}{2}\pi$ or if $G(\infty) = G(-\infty)$ (that is, $\int_{-\infty}^{\infty} g(y)\, dy = 0$). Since a function on R with equal limits at $\pm\infty$ has a critical point, it follows that, if $\tilde{F}(\infty) = \tilde{F}(-\infty)$, then $F(\alpha)$ has a zero, that is $f_0 \in \mathfrak{R}$. If $\tilde{F}(\infty) = \tilde{F}(-\infty)$ and $\tilde{F}$ is non-constant on R, a simple perturbation argument shows that $f_0 + th \in \mathfrak{R}$ for $|t|$ small (Note that $f_0 + th \in \mathfrak{R}$ if and only if $\tilde{F}(\alpha) - \frac{1}{2}t\alpha^2$ has a critical point.) Since $F(\alpha)$ is the gradient of $\tilde{F}(\alpha)$, our earlier estimates can be used to show that $\tilde{F}$ is non-constant on R. *For example*, if g is regular and *either* $Z_+(f_0) \neq 0$ or $Z_-(f_0) \neq 0$, then $\tilde{F}(\alpha)$ is non-constant on R. In addition, if $\tilde{F}(\infty) > \tilde{F}(-\infty)$ and $F(\alpha) < 0$ for α large negative, then one easily sees that $f_0 + th \in \mathfrak{R}$ for all small t. Once again, our earlier estimates could be used to verify the condition on F. Obviously, one could state a number of variants of this last result. *It would be of interest to prove the results on $\mathfrak{R}$ in this paragraph without requiring Assumption U.* Note that Assumption U is in a sense best possible because, if g is continuously differentiable and if $R(g') \not\subseteq [\mu, \nu]$, then there is an $f_0 \in R(L)$ and an $\alpha_0 \in R$ such that $w(\alpha_0)$ is multi-valued. This is a special case of a much more general abstract result which will appear elsewhere.

Note that, even when variational methods apply, our methods give more information. For example, if Z_+ does not vanish identically but Z_+ has a zero, then there exist $\{u_n\}_{n=1}^\infty$ in $W^{2,1}[0, \pi]$ such that (i) $\|u_n\|_{2,1} \to \infty$ as $n \to \infty$, (ii) $Lu_n - g(u_n) \in R(L)$ for all n and $Lu_n - g(u_n)$ converges in $L^1[0, \pi]$.

Remarks on Theorems 1 to 3

1) In all the above theorems, a simple examination of the proofs shows that, given $f_0 \in R(L)$, there exist $\alpha_0, \varepsilon, \delta > 0$ such that $|F(\alpha)| \geq \delta$ if $|\alpha| = \alpha_0$, if $f \in R(L)$ and if $\|f - f_0\|_1 \leq \varepsilon$. (Under the assumptions of Theorem 3 (b) or (c), we must assume that $Z_+(f_0)Z_-(f_0) \neq 0$.) Since $R(L) \subseteq \mathfrak{R}$, it follows easily as in [8, § 4] that $\mathfrak{R}$ is closed. (Under the assumptions of Theorem 3 (b) or (c), we must assume that $\int_{-\infty}^{\infty} g(y)\, dy = 0$ and $Z_+(0)Z_-(0) > 0$.) In Theorems 1 and 2, one can use the argument in § 4 of [7], to show that there exist functions r_1, r_2: $R(L) \to (0, \infty)$ such that

$$\mathfrak{R} = \{\alpha h + t: -r_1(t) \leq \alpha \leq r_2(t),\ t \in R(L)\}\,.$$

Moreover, if g is continuously differentiable, r_1 and r_2 are continuous and (4) has at least 2 solutions for $f = \alpha h + t$ if $\alpha \neq 0$ and $-r_1(t) < \alpha < r_2(t)$. *Similar results hold if h has zeros in $(0, \pi)$ provided that Assumption U holds but it is not known if this result holds in general.*

2) Our methods can be applied to a great many other problems. For example, they could be used for other boundary conditions, for problems involving higher order differential operators, for problems where $g(y)$ has finite non-zero limits $I^{\pm}$ as $y \to \pm\infty$ and $R(g) \not\subseteq [I^-, I^+]$ and for non-self-adjoint problems. (Note that some results for this last case were announced at the end of § 1 in [5]. The results announced there could be improved by replacing the condition on g' by the corresponding assumption on g. In this case, we no longer know that $v(\alpha)$ is single valued ($w(\alpha)$ in the notation here) for $|\alpha|$ large and we estimate $F(\alpha)$ instead of its derivative.) Our methods of estimation could be combined with the ideas at the end of § 1 to handle a number of problems where $N(L)$ is multi-dimensional. For example, if $Ly = -y'' + 4n^2 y$, our methods apply if (i) $yg(y) \geq 0$ for $|y|$ large and $\int_{-\infty}^{\infty} g(y) y\, dy$ diverges or (ii) $yg(y) \geq 0$ for $|y|$ large, $\int_{-\infty}^{\infty} g(y) y\, dy > 0$ and $\int_{-\infty}^{\infty} g(y)\, dy = 0$ or (iii) g is regular, $\int_{-\infty}^{\infty} g(y) y\, dy \neq 0$ and $\int_{-\infty}^{\infty} g(y)\, dy = 0$. The conclusion is that $f_0 + h \in \mathfrak{R}$ if $h \in N(L)$ and $\|h\|_1$ is small. If $\int_{-\infty}^{\infty} g(y) y\, dy$ converges and $\int_{-\infty}^{\infty} g(y)\, dy \neq 0$, one obtains a similar result if $\tilde{Z}(f_{0\theta}) \neq 0$ for all θ in $[0, \pi]$. Here $\tilde{Z}$ is an affine functional similar to Z_+ and $f_{0\theta}$ is the translate of f_0. The idea in the proof of all the above results is to use our earlier techniques to estimate $\int_0^{\pi} g(\alpha h + w(\alpha)) h\, ds$ as $\alpha \to \infty$ uniformly in h for h in $\{z \in N(L) : \|z\|_1 = 1\}$. With rather more care, the last result could be improved by also estimating

$$\int_0^{\pi} g(\alpha h + w(\alpha)) h'\, ds = -\alpha^{-1} \int_0^{\pi} g(\alpha h + w(\alpha)) w(\alpha)'\, ds$$

if g is regular. If Property U holds, some of the above results could be improved by combining variational methods with our estimates. Finally, the above techniques can be used to solve the open problem on p. 335 of [13]. (The conclusion is the same for $1 < l < \infty$.)

3) If $g(y)$ is eventually decreasing and $\int_{-\infty}^{\infty} g(y) y\, dy$ diverges, our methods can be used to estimate the growth of $F(\alpha)$ as $|\alpha| \to \infty$. (Under reasonable hypotheses, $F(\alpha) \sim \int_0^{K\alpha} g(u) u\, du$.) This idea is useful in fourth order problems where there may be zeros of different orders. Another useful way to do estimates is to split the integral from $[0, \pi]$ into integrals such as $\int_{t_i - \mu(\alpha)}^{t_i + \mu(\alpha)}$ and $\int_{t_i + \mu(\alpha)}^{t_{i+1} - \mu(\alpha)}$, where the t_i are the zeros of h

and $\mu(\alpha)$ tends to zero rather slowly. It can be shown that integrals such as $\int\limits_{t_i+\mu(\alpha)}^{t_{i+1}-\mu(\alpha)}$ are of smaller order by estimating the integrand. This idea is also useful in the partial differential case below.

We now want to discuss in a little more detail the corresponding problem where L is a second order elliptic self-adjoint partial differential operator with smooth coefficients on a smooth bounded domain Ω in R^k.

For simplicity, we only consider Dirichlet boundary conditions. We first consider the case where $h(x) > 0$ in Ω. Theorems 1 and 2 readily generalize to this case if we assume that $f_0 \in L^p(\Omega)$ with $p > k$ and require that $y^2 g(y) \to 0$ as $|y| \to \infty$ in Theorem 2. Let us explain this rather briefly. Now points of Ω near $\partial\Omega$ can be uniquely expressed in the form $z + sn(z)$ where $z \in \partial\Omega$, $n(z)$ is the inward unit normal to $\partial\Omega$ at z and s is small. To prove the analogue Theorem 1, we let $\tilde{\Omega} = \{x \in \Omega: x = z + sn(z),\ z \in \partial\Omega,\ 0 \leqslant s \leqslant \alpha^{-1}K\}$ and $F_K(\alpha) = \int\limits_{\tilde{\Omega}} g(\alpha h + w(\alpha))h\,dx$. We use the coordinates z and $u = \alpha s$ to show that

$$\alpha^2 F_K(\alpha) \to \int\limits_{\partial\Omega}\int\limits_0^K g\left(\frac{\partial h}{\partial n}(z)\,u\right)\frac{\partial h}{\partial n}(z)\,u\,du\,dz\,. \tag{7}$$

(Note that $\alpha h(z + \alpha^{-1}un(z)) \to (\partial h/\partial n)(z)u$ as $\alpha \to \infty$ and that $w(\alpha)$ is bounded in $C^1(\bar{\Omega})$.) If $\int\limits_0^\infty g(u)u\,du > 0$, one easily sees that (7) is positive for K large. (Remember that $(\partial h/\partial n)(z) > 0$ on $\partial\Omega$.) For example, if $\int\limits_0^\infty g(u)u\,du$ converges, then the right hand side of (7) approaches

$$\int\limits_0^\infty g(u)\,u\,du \int\limits_{\partial\Omega}\left(\frac{\partial h}{\partial n}(z)\right)^{-1} dz \quad \text{as } K \to \infty\,.$$

It follows easily as before that $F(\alpha)$ is positive for large positive α. The rest of the proof of Theorem 1 is as before. Theorem 2 is proved similarly. (The extra assumption that $y^2 g(y) \to 0$ as $|y| \to \infty$ is used to ensure that the contribution from the part of Ω not near $\partial\Omega$ is $o(\alpha^{-2})$. This condition can be removed for some simple domains, for example, balls.) If $N(L)$ is one dimensional and if $\nabla h(x) \neq 0$ whenever $x \in \bar{\Omega}$ and $h(x) = 0$, an analogue of Theorem 3 can be easily proved by using the ideas above. The only differences are that we must assume that $y^2 g(y) \to 0$ as $|y| \to \infty$ in Theorem 3 (c) and that $Z_+(f_0)$ and $Z_-(f_0)$ now involve integrals over nodal manifolds. The general case is much more complicated. *For the moment, we always assume that $N(L)$ is one-dimensional.* Under weak assumptions, the dominating term of $F(\alpha)$ comes from near the zero set of h. Let us first assume that $v \in \partial\Omega$, $\nabla h(v) = 0$ and that the Hessian matrix $D^2 h(v)$ of h at v does not vanish. Then it is well known and easy to prove that there exist new coordinates near v such that $\partial\Omega$ is given locally by $x_1 = 0$ and $h(x) = x_1 x_2$ near v. (Note that, since h satisfies an elliptic equation, it can not happen that $(\partial^2 h/\partial x_1^2)(v)$ is the only non-zero second derivative.) If one uses these coordinates, it can be shown that the contribution to $F(\alpha)$ near v is $\sim \alpha^{-2}\ln|\alpha|\int_{-\infty}^{\infty} g(u)u\,du$ as $\alpha\to\infty$. (If $yg(y)\geqslant 0$ for $|y|$ large but g is not regular, we have $\geqslant$ rather than $\sim$.) Now we can use our earlier methods to estimate the contribution to $F(\alpha)$ near a non-critical zero of h. Thus we can obtain a formula for the asymptotic behaviour of $F(\alpha)$ (and further existence theorems) if we assume that the only points in $\bar{\Omega}$ where $|h(x)|^2 + |\nabla h(x)|^2$ vanishes are in $\partial\Omega$ and are of the above type. (To prove an analogue of Theorem 3 (*c*), we need to assume that $(\ln|y|)^{-1}y^2 g(y)\to 0$ as $|y|\to\infty$.) Note that the contribution at these irregular points dominates the contribution near the more regular points and that, generically, the above bad case is the only one to occur. (This follows from the results of UHLENBECK [24].) A similar estimate holds for $k=2$ near a zero v of h in Ω where $\nabla h(v)=0$ and the Hessian matrix is non-singular. (The only difference is that we must replace $\int_{-\infty}^{\infty} g(u)u\,du$ by $\int_{-\infty}^{\infty} g\big(u+Sf_0(v)\big)u\,du$.) The idea of the proof is to use the Morse lemma (cp. [18]) to change coordinates such that $h(x)=x_1x_2$ near v. (If $k>2$, one can use the Morse lemma to show that the contribution to $F(\alpha)$ near such a zero is $o(\alpha^{-2})$, that is, dominated by the contribution near the regular points.) We now restrict ourselves to the case $k=2$. For *simplicity*, we take L to be $-\Delta$. If z_i is a critical zero of h, it is easy to see (and well-known) that there is an integer $n\geqslant 2$ such that $h(x)=r^n\cos(n\theta+\varphi)+o(r^n)$ near z_i. (Here we have chosen axes such that $z_i=0$.) By a theorem of KUIPER [18], there is a C^1 change of coordinates such that $h(x)=r^n\cos n\theta$ near $z_i=0$. We now assume that $n>2$ since we have already covered the case where $n=2$. By using similar ideas to earlier and by using polar coordinates, we find that the contribution to $F(\alpha)$ near 0 is

$$\sim (2n)^{-1}\alpha^{-1-2n^{-1}}\int_{-\infty}^{\infty} g\big(u+Sf(0)\big)|u|^{2n^{-1}}\operatorname{sgn} u\,du\int_0^{2\pi}|\cos n\theta|^{-2n^{-1}}\,d\theta$$

if g is eventually decreasing and $\int_{-\infty}^{\infty} g(u)|u|^{2n^{-1}}\operatorname{sgn} u\,du$ converges. (Without the decreasing condition, we can prove a result with $\sim$ replaced by $\geqslant$ provided that $yg(y)\geqslant 0$ for $|y|$ large.) Once again, one can easily obtain existence results. It is likely that the coarea formula (cp. FEDERER [12, Theorem 3.2.12] can also be used to obtain estimates for $F(\alpha)$. As in Remark 2, our methods can sometimes be used to obtain results in some cases where $N(L)$ is multi-dimensional. However, one extra difficulty often occurs which we have not overcome. It may happen that there is bifurcation in the nodal set of h as h moves on the unit sphere in $N(L)$. This causes difficulties in obtaining estimates which are uniformly valid on the sphere in $N(L)$. For example, a degenerate zero may bifurcate into several less degenerate zeros. Finally, if Assumption U holds, our estimates can sometimes be combined with variational methods to obtain additional results.

3. – The case $g(y) = a \sin y$.

We consider the same equation as in § 2 except that we now assume that $g(y) = a \sin y$. As in § 2, we can obtain results on the solvability of (4) by studying the asymptotic behaviour of $F(\alpha) = \int_0^\pi a \sin(\alpha h + w(\alpha))h\,ds$, where $w(\alpha)$ is defined in § 1. As in § 2, $w(\alpha)$ is multi-valued. *We retain the notation at the beginning of* § 2.

Now

$$F(\alpha) = a \operatorname{Im} \int_0^\pi \exp(i\alpha h + iw(\alpha))h\,ds = a \operatorname{Im} \int_0^\pi \exp(i\alpha h) z(\alpha)\,ds\,, \tag{8}$$

where $z(\alpha)(s) = h(s) \exp iw(\alpha)(s)$. We need the following lemma which is a variant of known results.

LEMMA 1. – *Assume that $y(\alpha) \in W^{2,1}[0, \pi]$ for each α and $\|y(\alpha)\|_{2,1} \leqslant K$ for all α, where $\|\ \|_{2,1}$ denotes the usual norm on $W^{2,1}[0, \pi]$. Suppose that $e \in L^1[0, \pi]$. Then*

(i) $\int_0^\pi \exp(i\alpha h) y(\alpha) e\,ds \to 0$ *as $\alpha \to \infty$ if h' has a finite number of zeros in $[0, \pi]$.*

(ii) *If $[a, b] \subseteq [0, \pi]$ and $h'(s) \neq 0$ on $[a, b]$, then $\left|\int_a^b \exp(i\alpha h) y(\alpha)\,ds\right| \leqslant |\alpha|^{-1} K_1$, where K_1 depends only on K, h, a and b.*

(iii) *Suppose that $t_j \in (0, \pi)$, $h'(t_j) = 0$, $h''(t_j) \neq 0$ and $\varepsilon > 0$ such that $h'(s) \neq 0$ for $|s - t_j| \leqslant \varepsilon$ and $s \neq t_j$. Then*

$$\int_{t_j-\varepsilon}^{t_j+\varepsilon} \exp(i\alpha h) y(\alpha)\,ds$$
$$= \left(\pi/|\alpha h''(t_j)|\right)^{\frac{1}{2}} y(\alpha)(t_j) \exp\left(i\alpha h(t_j) + \tfrac{1}{4}\pi i \operatorname{sgn} h''(t_j)\right) + \alpha^{-\frac{1}{2}} D(\alpha)\,,$$

where $D(\alpha) \to 0$ as $\alpha \to \infty$.

PROOF. – (i) By our assumption, there is a $K_2 > 0$ such that $|y(\alpha)(t)| \leqslant K_2$ on $[0, \pi]$ for all α. Thus $|\exp(i\alpha h(t)) y(\alpha)(t)| \leqslant K_2$ on $[0, \pi]$ for all α. Hence, since we can approximate e in the L^1 norm by smooth functions, it suffices to prove (i) when e is C^∞. Secondly, since our integrand is uniformly bounded (by $K_2 \sup\{|e(s)|: s \in [0, \pi]\}$), its integral over small intervals is small. Thus it suffices to prove the corresponding result for the integral over a closed interval T where $h'(s) \neq 0$. (We simply choose a finite number of small intervals covering the finite number of zeros of h' and note that the remainder of $[0, \pi]$ is a finite union of intervals where h' does not vanish.) Thus (i) will follow from (ii) if we replace $y(\alpha)$ by $y(\alpha) e$.

(ii) If we use the change of variable $u = h(s)$, then the integral becomes

$$\int_{h(a)}^{h(b)} \exp(i\alpha u) y(\alpha)\big(h^{-1}(u)\big)\Big(h'\big(h^{-1}(u)\big)\Big)^{-1} du .$$

(Note that h is monotonic on $[a, b]$.) The result now follows by integration by parts. The uniformity of the error estimate follows by examining the argument in COPSON [4, p. 30].

(iii) This is by the standard stationary phase argument. The only difference is that $y(\alpha)$ depends on α. The easiest way to see the error estimate is to follow the argument on pp. 54-55 of ERDELYI [11] with $l = 2$, $\sigma = 1$, $N = 1$ and $\lambda = \mu = 1$. The key point in estimating the error term (as obtained in [11]) is to note that $y(\alpha)$ and $y(\alpha)'$ are bounded on $[0, \pi]$ uniformly in α.

REMARK. – In generalizations (for example to partial differential equations), it is useful to note that the error estimate in part (iii) is valid if $y(\alpha)$ is bounded in $W^{1,1}[0,\pi]$ uniformly in α and $\{y(\alpha)'(t): \alpha \in R\}$ are equi-integrable on $[0,\pi]$. Moreover, if these hold uniformly for a set of y's, then the corresponding $D(\alpha)$ tend to zero uniformly.

We now use this lemma to estimate the right hand side of (8). *We assume that* $h''(s) \neq 0$ *whenever* $h'(s) = 0$. (Thus h' has only a finite number of zeros on $[0, \pi]$.) Since $|a \sin y| \leqslant a$ on R, it follows easily that the $w(\alpha)$ are bounded in $W^{2,1}[0,\pi]$ uniformly in α. Since the natural embedding of $W^{2,1}[0,\pi]$ into $C^1[0,\pi]$ is continuous, it is easy to show that $\exp iw(\alpha)$ are uniformly bounded in $W^{2,1}[0,\pi]$. Hence, by part (i) of the lemma,

$$\int_0^\pi \exp(i\alpha h) \exp\big(iw(\alpha)\big)\, e\, ds \to 0$$

as $\alpha \to \infty$ for each $e \in L^\infty[0, \pi]$. (Set $y(\alpha) = \exp iw(\alpha)$.) Thus, by taking the imaginary part, $\sin\big(\alpha h + w(\alpha)\big) \rightharpoonup 0$ weakly in $L^1[0,\pi]$ as $\alpha \to \infty$. Hence $\sin\big(\alpha h + w(\alpha)\big) - f_0 \rightharpoonup -f_0$ weakly in $L^1[0,\pi]$ as $\alpha \to \infty$ (where, as in § 2, $f_0 = Pf$). It follows as at the end of § 1 that $w(\alpha) \rightharpoonup -Kf_0 \equiv Sf_0$ weakly in $W^{2,1}[0,\pi]$ as $\alpha \to \infty$. Since the natural embedding of $W^{2,1}[0,\pi]$ into $C[0,\pi]$ is compact, it follows that $w(\alpha) \to Sf_0$ strongly in $C[0,\pi]$. Now parts (ii) and (iii) of the lemma $\big($with $y(\alpha) = z(\alpha) = \exp\big(iw(\alpha)\big)h\big)$ imply that

$$\int_0^\pi a \exp i\alpha h\, z(\alpha)\, ds = a \sum_{j=1}^{n} \pi^{\frac12} |\alpha h''(t_j)|^{-\frac12} z(\alpha)(t_j) \times \exp\big(i\alpha h + \tfrac14 \pi i \operatorname{sgn} h''(t_j)\big) + o(\alpha^{-\frac12}) \tag{9}$$

as $\alpha \to \infty$, where $t_1 < t_2 \ldots < t_n$ are the zeros of h' in $(0, \pi)$. This is simply proved by splitting the interval $[0,\pi]$ into intervals I where either $h'(s) \neq 0$ on I or I contains a single point where h' is zero. Note that, as in § 2, $h'(0) \neq 0$ and $h'(\pi) \neq 0$. Since $w(\alpha) \to Sf_0$ in $C[0, \pi]$ as $\alpha \to \infty$, $z(\alpha) \to \exp(iSf_0)h$ as $\alpha \to \infty$. By using this and equations (8) and (9), we eventually find

$$F(\alpha) = a\alpha^{-\frac{1}{2}}\pi^{\frac{1}{2}} \sum_{j=1}^{n} |h''(t_j)|^{-\frac{1}{2}} h(t_j) \sin\left(\alpha h(t_j) + \tfrac{1}{4}\pi \operatorname{sgn} h''(t_j) + Sf_0(t_j)\right) + o(\alpha^{-\frac{1}{2}})$$

as $\alpha \to \infty$. Hence

$$\alpha^{\frac{1}{2}} F(\alpha)\pi^{-\frac{1}{2}} a^{-1} = \tilde{T}(\alpha) + o(1)\,, \tag{10}$$

as $\alpha \to \infty$, where $\tilde{T}(\alpha) = \sum_{j=1}^{n} \left(\mu_j \cos \alpha h(t_j) + \nu_j \sin \alpha h(t_j)\right)$.

Here $\nu_j + i\mu_j = |h''(t_j)|^{-\frac{1}{2}} h(t_j) \exp i\left(\tfrac{1}{4}\pi \operatorname{sgn} h''(t_j) + Sf_0(t_j)\right)$.

Note that the multivaluedness of $w(\alpha)$ does not affect the above arguments as one can readily see by examining the proofs. (The key point is that we do not need *any* regularity assumptions on the map $\alpha \to w(\alpha)$.) Since $h'(s) \neq 0$ when $h(s) = 0$ (by the uniqueness theorem for ordinary differential equations), $h(t_j) \neq 0$ for $1 \leqslant j \leqslant n$. Thus μ_j and ν_j cannot both vanish. We now discuss the behaviour of $\tilde{T}(\alpha)$ as a function of α. We assume a knowledge of basic properties of almost periodic functions as in BOHR [2]. Note that $\tilde{T}(\alpha)$ is almost periodic. *Assume that there is a k with $1 \leqslant k \leqslant n$ such that $|h(t_j)| \neq |h(t_k)|$ if $j \neq k$.* Then no other terms in the expression for $\tilde{T}(\alpha)$ can cancel the terms with $j = k$. Hence $\tilde{T}(\alpha)$ does not vanish identically. Since $h(t_j) \neq 0$ for $1 \leqslant j \leqslant n$, the integral of $\tilde{T}$ is also almost periodic. (Remember that the sum is finite.) Thus the integral of $\tilde{T}$ is bounded on R. It follows that $\tilde{T}$ must change sign. Remember that $\tilde{T}$ is uniformly continuous and cannot have a limit as $\alpha \to \infty$ (since it is almost periodic). Since an almost periodic function has arbitrarily large translation numbers, it follows that there exist $\delta > 0$ and β_s, γ_s for every positive integer s such that $\beta_s \to \infty$ and $\gamma_s \to \infty$ as $s \to \infty$, $\tilde{T}(\beta_s) \geqslant \delta$ and $\tilde{T}(\gamma_s) \leqslant -\delta$. By choosing subsequences, we can ensure that $\beta_s < \gamma_s < \beta_{s+1}$ for every positive integer s. By (10), it follows that $F(\beta_s) > 0$ and $F(\gamma_s) < 0$ if s is large. It follows as in the proof of Theorem 1 that there exists an $\varepsilon > 0$ such that $f_0 + th \in \mathfrak{R}$ if $|t| \leqslant \varepsilon$. Moreover, by the remarks after the proof of Proposition 1, there exist α in (β_s, γ_s) and $\tilde{w}(\alpha) \in w(\alpha)$ such that $L(\alpha h + \tilde{w}(\alpha)) = a \sin(\alpha h + \tilde{w}(\alpha)) - f_0$ for every large s. Thus (4) has an infinite number of solutions. We have proved the following result.

THEOREM 4. – *Assume that $f_0 \in R(L)$, that $h''(t_j) \neq 0$ for $1 \leqslant j \leqslant n$ where $\{t_j\}_{j=1}^{n}$ are the zeros of h' in $(0, \pi)$. In addition, assume that there is a k such that $1 \leqslant k \leqslant n$ and $|h(t_j)| \neq |h(t_k)|$ if $j \neq k$. Then there is an $\varepsilon > 0$ such that $f_0 + th \in \mathfrak{R}$ if $|t| \leqslant \varepsilon$. Moreover, (4) has an infinite number of solutions for $f = f_0$.*

REMARKS. – 1) Our methods are still valid if $g(y) = a \sin(y + c)$ where $c \neq 0$. The assumption that there is a k such that $|h(t_j)| \neq |h(t_k)|$ for $j \neq k$ is only used to ensure that $\tilde{T}(\alpha)$ does not vanish identically. It is not difficult to show that $Y = \{f_0 \in R(L)\colon \tilde{T}(\alpha)$ vanishes identically$\}$ is a closed nowhere dense subset of $R(L)$ which is non-empty in many cases. Note that the conclusion of Theorem 4 holds for $f_0 \in R(L) \setminus Y$ even if the condition on $|h(t)|$ fails. It can also be shown that Y is empty if there exist $1 \leqslant k < l \leqslant n$, such that $|h(t_j)| \neq |h(t_k)|$ if $j \neq k, l$ and $|h''(t_k)| \neq |h''(t_l)|$. On the other hand, if $Ly = -y'' + 4y$, then Y is non-empty but does not intersect $\{f_0 \in R(L)\colon \sup |Sf_0| < \frac{1}{2}\pi\}$. *If Assumption U of § 2 holds*, then one can obtain other results by variational methods. In particular, if h' has only a finite number of zeros in $(0, \pi)$, then it is easy to use Lemma 1 (i) to prove that $\tilde{F}(\alpha) \to \frac{1}{2}\langle LSf_0, f_0\rangle$ as $|\alpha| \to \infty$ (where, as in § 2, F is the gradient of $\tilde{F}$). By a similar argument to the one there, it follows that $R(L) \subseteq \mathfrak{R}$. Note that the methods used to prove Theorem 4 are also valid for non-self adjoint equations, for some other boundary conditions and for ordinary differential operators of higher order. Our methods can be combined with Erdelyi's techniques to cover cases where h' has degenerate zeros. The degenerate zeros dominate in the study of the asymptotic behaviour of $F(\alpha)$. If Property U holds, our estimates could be combined with variational methods to obtain some results in the cases where $N(L)$ is multidimensional.

2) Once again, it is not difficult to show that $\mathfrak{R}$ is closed under the assumption of Theorem 4. (It is necessary to check that $\alpha^{\frac{1}{2}} G(\alpha) \to 0$ as $\alpha \to \infty$ locally uniformly in f_0.) One can prove further results on the structure of $\mathfrak{R}$ if either $h(t) > 0$ on $(0, \pi)$ or if Assumption U holds. In particular, similar results to the ones in Remark 1 of § 2 hold though a little more care is needed in the proofs. In addition, by a slightly more careful argument, one can prove the following under the assumptions of Theorem 4. For each $f_0 \in R(L)$ there is an $\varepsilon_n > 0$ such that (4) has at least n (but finitely many) solutions for $f = f_0 + th$ if $0 < |t| \leqslant \varepsilon_n$. There is one problem that behaves rather differently. Consider $Ly = -y''$ with periodic (or Neumann) boundary conditions. In this case, h is constant. It is not difficult to use the method of sub- and super-solutions (as in [7]) to deduce that there exist continuous functions $\alpha_1, \alpha_2\colon R(L) \to R$ such that

$$\mathfrak{R} = \{\alpha h + f_0\colon f_0 \in R(L),\ \alpha_1(f_0) \leqslant \alpha \leqslant \alpha_2(f_0)\}\,.$$

Moreover, $R(L) \subseteq \mathfrak{R}$ and thus $\alpha_1(f_0) \leqslant 0 \leqslant \alpha_2(f_0)$. To see that $R(L) \subseteq \mathfrak{R}$, one minimizes

$$\int_0^\pi [x'(t)^2 + a \cos x(t) - f_0(t) x(t)]\, dt$$

over

$$\left\{x \in W^{1,2}[0, \pi];\ x(0) = x(\pi),\ 0 \leqslant \int_0^\pi x(t)\, dt \leqslant 2\pi\right\}$$

(where $f_0 \in R(L) = \{x \in L^1[0, \pi]\colon \int_0^\pi x(t)\, dt = 0\}$. The periodicity of cos then ensures the minimum is also the minimum on $\{x \in W^{1,2}[0, \pi]\colon x(0) = x(\pi)\}$. We do not know of an example where $\alpha_1(f_0)\alpha_2(f_0) = 0$. It can be shown that $\alpha_1(f_0)\alpha_2(f_0) < 0$ if $\sup Sf_0 - \inf Sf_0 < 2\pi$. CASTRO [3] independently obtained similar results to the ones of this paragraph under a restriction on a.

3) The methods of § 2 and § 3 can be combined to consider cases where $g(y) = g_1(y) + a \sin(y + c)$, with $g_1(y) \to 0$ as $|y| \to \infty$. Under natural hypotheses, one can show that this problem behaves like the one in § 2 if $|y|^{\frac{1}{2}} g_1(y) \operatorname{sgn} y \to \infty$ as $|y| \to \infty$ while it behaves like the one in § 3 if $|y|^{\frac{1}{2}} g_1(y) \to 0$ as $y \to \infty$. (This occurs because the magnitude of $g_1(y)$ determines which term dominates in the asymptotics.)

4) Our methods can be generalized to apply to simple eigenvalues of a smooth elliptic operator on a smooth bounded domain Ω in R^k *if* $k = 2$ *or* 3 and $f_0 \in L^p(\Omega)$ where $p > k$. We indicate this briefly. The non-degeneracy assumption becomes that the hessian matrix $D^2h(x)$ is invertible whenever $\nabla h(x) = 0$ and $x \in \bar{\Omega}$. To obtain the estimates in this case, we choose an open cover $\{A_i\}_{i=1}^q$ of $\bar{\Omega}$ such that, in suitable coordinates $h|_{A_i}$ is a linear function or is quadratic with no linear term. (That this can be done follows from the Morse lemma and the implicit function theorem.) We then choose a partition of unity subordinate to the A_i and estimate $F(\alpha)$ by writing it as a sum of integrals over the A_i and where the integrand is zero near ∂A_i: When $h|_{A_i}$ is linear (for some coordinates), it is easy to use two integrations by parts to show that the corresponding contribution to $F(\alpha)$ is $0(\alpha^{-2})$. Near a critical point, one can use Erdelyi's ideas and those in MASLOV [19, pp. 238-239] to obtain a similar estimate to earlier for the contribution to $F(\alpha)$. (If $k = 2$ or 3, the contribution near a critical point x_i is $\sim K\alpha^{-\frac{1}{2}k} \sin\left(\alpha h(x_i) + \frac{1}{4}\pi w + Sf_0(x_i)\right)$, where w is the signature of the symmetric matrix $D^2h(x_i)$ and $K > 0$.) If $k \geqslant 4$, the problem seems to behave a little differently. For example, if $g(y) = -\cos y = \sin(y + \frac{1}{2}\pi)$, integration near $\partial\Omega$ gives a contribution $\sim \tilde{K}\alpha^{-2}$. (On the other hand, if Property U holds, variational methods can be used to show that $R(L) \subseteq \mathfrak{R}$ for all k provided that $\{x \in \bar{\Omega}\colon \nabla h(x) = 0\}$ is finite.) With a little care, our results can be generalized to the generic case (in the sense of UHLENBECK [24]) by allowing slightly more general singularities on $\partial\Omega$ (as at the end of § 2). Finally, by using a theorem of GROMOLL and MEYER [16] to obtain a local canonical form for h, we can generalize our methods to cover all reasonable cases where the Hessian matrix $D^2h(x_i)$ has rank at least $k - 1$ at every critical point x_i in Ω (and $k = 2$ or 3).

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