On the Use of Hexagonal Constellation for Peak-to-Average Power Ratio Reduction of an OFDM Signal

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***Abstract*— One of the main drawbacks of orthogonal frequency division multiplexing (OFDM) is the high peak-to-average power ratio (PAPR) of the OFDM signal. In this paper, we propose the use of hexagonal constellation for PAPR reduction of an OFDM signal. Because hexagonal constellation is the densest packing of regularly spaced points in two dimensions, we can have more signal points in a given area with hexagonal constellation than with quadrature amplitude modulation (QAM) constellation. We can exploit these extra degrees of freedom provided by the hexagonal constellation for PAPR reduction of an OFDM signal. We will apply the proposed technique to eliminate data rate loss due to the side information in partial transmit sequence (PTS) technique and selected mapping (SLM) technique.**

***Index Terms*— OFDM, PAPR, hexagonal constellation, partial transmit sequence (PTS), selected mapping (SLM).**

I. Introduction

Orthogonal frequency division multiplexing (OFDM) is a multicarrier modulation technique that has recently found wide adoption in a variety of high data-rate communication systems, including digital subscriber lines, wireless LANs, digital video broadcasting, and other emerging wireless broadband systems. One of the major problems of OFDM is that OFDM signals have higher peak-to-average power ratio (PAPR) than single carrier signals because an OFDM signal is the sum of many narrowband signals in the time domain [1]. The high PAPR necessitates using larger and expensive linear power amplifiers. Since high peaks occur irregularly and infrequently, this means that power amplifiers will be operating inefficiently.

In this paper, we propose a novel PAPR reduction technique based on hexagonal constellation. By using a hexagonal constellation instead of quadrature amplitude modulation (QAM) constellation, it is possible to have more signal points in a given area and these extra degrees of freedom can be utilized for PAPR reduction. The application of the hexagonal constellation to the tone injection technique was proposed in [2] by the authors. In this paper, we will use partial transmit sequence (PTS) technique and selected mapping (SLM) technique as applications of the proposed hexagonal constellation based PAPR reduction technique. It is possible to eliminate data rate loss due to the side information in PTS technique and SLM technique by applying the proposed technique to them. Similar idea of using hexagonal constellation for PAPR reduction was independently proposed in [3]. But, this paper generalizes the application of the hexagonal constellation also to the SLM technique, presents efficient decoding methods for hexagonal constellation, provides detailed computational complexity comparison with other technique, and considers realistic nonlinear amplifier model in evaluating the symbol error rate performance.

Manuscript received February 9, 2006; revised September 12, 2006, July 26, 2007, and September 13, 2007; accepted September 13, 2007. The associate editor coordinating the review of this paper and approving it for publication was C. Tellambura. This work was supported in part by the Information and Telecommunication National Scholarship Program of the Ministry of Information and Communication, Korea.

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Digital Object Identifier 10.1109/TWC.2007.06104.

II. System Model

Let us denote the data block of length N as a vector $\mathbf{X} = [X_0, X_1, \cdots, X_{N-1}]^T$ where N is the number of subcarriers. Each symbol in $\mathbf{X}$ modulates one of a set of subcarriers, $\{f_n, n = 0, 1, \cdots, N-1\}$. The N subcarriers are chosen to be orthogonal, that is, $f_n = n\Delta f$, where $\Delta f = 1/NT$ and T is the duration of the data symbol X_n. The duration of an OFDM data block is NT. The complex envelope of the transmitted OFDM signal is given by

$$x(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X_n e^{j2\pi f_n t}, 0 \leq t < NT. \tag{1}$$

The PAPR of the transmitted signal is defined as

$$\text{PAPR} = \frac{\max\limits_{0 \leq t < NT} |x(t)|^2}{1/NT \cdot \int_0^{NT} |x(t)|^2 dt}. \tag{2}$$

Since most systems employ discrete-time signals, the amplitude of samples of $x(t)$ is used in many of the PAPR reduction techniques instead of continuous-time signal $x(t)$ itself [1]. Since symbol-spaced sampling of (1) sometimes misses some of the signal peaks and results in optimistic results for the PAPR, the signal samples can be obtained by oversampling (1) by a factor of L to approximate the true PAPR better. The oversampled time-domain samples can be obtained by an LN-point inverse discrete Fourier transform (IDFT) of the data block with $(L-1)N$ zero-padding [1]. It was shown in [4] that $L = 4$ is sufficient to capture the peaks. The PAPR computed from the L-times oversampled time-domain signal samples is given by

$$\text{PAPR} = \frac{\max\limits_{0 \leq k \leq NL-1} |x_k|^2}{E\left[|x_k|^2\right]}. \tag{3}$$

where $x_k = x(k \cdot T/L)$ and $E[\cdot]$ denotes expectation.

III. Hexagonal Constellation and PAPR Reduction

The densest packing of regularly spaced points in two dimensions is the hexagonal lattice shown in Fig. 1(b) [5]. The volume or area of the decision region for each point is

$$\nu_H = d^2 \frac{\sqrt{3}}{2} \tag{4}$$

where d is the minimum distance between points. When compared with square QAM constellation shown in Fig. 1(a), we can pack more signal points with hexagonal constellation in a given area with hexagonal constellation. Specifically, the ratio between the numbers of signal points is

$$(1/\nu_H)/(1/\nu_S) = \frac{d^2}{d^2 \frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \tag{5}$$

where $\nu_S = d^2$ is the volume or area of the decision region for each point in square QAM constellation.

We can have extra degrees of freedom by using hexagonal constellation with appropriate number of points instead of QAM constellation. A rough idea can be illustrated with the hexagonal constellation with 7 signal points (7-HEX) and 4-QAM shown in Fig. 1. Assume that 7-HEX is used instead of 4-QAM which has 4 symbols '1', '2', '3', and '4'. In this case, we have 3 excess signal points out of 7 signal points. So some of the points in 4-QAM may be associated with more than one point in 7-HEX. In Fig. 1, symbol '1' in 4-QAM has 1 representation in 7-HEX and symbols '2', '3', and '4' in 4-QAM have 2 representations in 7-HEX. We can freely choose the points with more than one representation so that the PAPR is reduced in the transmitted signal. At the receiver, we use demodulation for 7-HEX and either 'pA' or 'pB' can be considered as symbol 'p', $p = 2, 3, 4$. Note that it is not necessary for the receiver to know the representation selected at the transmitter side.

This idea can directly be related to the elimination of the side information in the PTS and SLM techniques because we can take advantage of these extra degrees of freedom in some of the symbols that have more than one representation. In the following, the number of representations for the symbols that have more than one representation will be denoted R.

IV. Applications to PTS and SLM Techniques

Ordinary PTS and SLM techniques require the side information to be transmitted from the transmitter to the receiver in order to let the receiver know what has been done in the transmitter. In addition, the entire OFDM data block may be lost if the side information is received in error. This may increase the symbol error at the receiver. To protect the side information against all channel impairments, a channel coding technique can be used. However, this increases the amount of side information and results in a further data rate loss. In the following, we will apply the proposed technique to eliminate data rate loss due to the side information in PTS and SLM techniques.

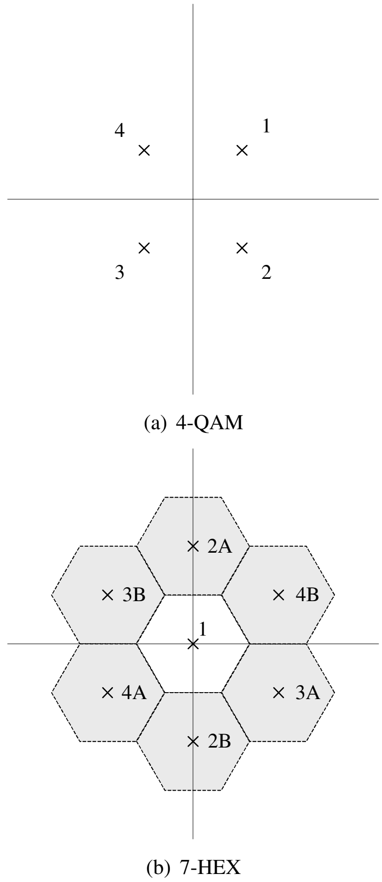

Fig. 1. 4-QAM and 7-HEX constellations.

A. PTS Technique with Hexagonal Constellation

In the PTS technique, an input data block of N symbols is partitioned into disjoint subblocks. The subcarriers in each subblock are weighted by a phase factor for that subblock. The phase factors are selected such that the PAPR of the combined signal is minimized. In the ordinary PTS technique [6], the input data block $\mathbf{X}$ is partitioned into M disjoint subblocks $\mathbf{X}_m = [X_{m,0}, X_{m,1}, \cdots, X_{m,N-1}]^T$, $m = 1, 2, \cdots, M$, such that $\sum_{m=1}^{M} \mathbf{X}_m = \mathbf{X}$ and the subblocks are combined to minimize the PAPR in the time-domain. The L-times oversampled time-domain signal of $\mathbf{X}_m$, $m = 1, 2, \cdots, M$, is denoted as $\mathbf{x}_m = [x_{m,0}, x_{m,1}, \cdots, x_{m,NL-1}]^T$. $\mathbf{x}_m$, $m = 1, 2, \cdots, M$, is obtained by taking an IDFT of length NL on $\mathbf{X}_m$ concatenated with $(L-1)N$ zeros. These are called the partial transmit sequences. Complex phase factors, $b_m = e^{j\phi_m}$, $m = 1, 2, \cdots, M$, are introduced to combine the partial transmit sequences. The set of phase factors is denoted as a vector $\mathbf{b} = [b_1, b_2, \cdots, b_M]^T$. The time-domain signal after combining is given by

$$\mathbf{x}'(\mathbf{b}) = \sum_{m=1}^{M} b_m \cdot \mathbf{x}_m \tag{6}$$

where $\mathbf{x}'(\mathbf{b}) = [x'_0(\mathbf{b}), x'_1(\mathbf{b}), \cdots, x'_{NL-1}(\mathbf{b})]^T$. The objective is to find the set of phase factors which minimizes the PAPR. Minimization of PAPR is related to the minimization of $\max_{0 \le k \le NL-1} |x'_k(\mathbf{b})|$. In general, the selection of the phase factors is limited to a set with a finite number of elements to reduce the search complexity. The set of allowed phase factors is written as $P = \{e^{j2\pi l/W} | l = 0, 1, \cdots, W-1\}$ where W is the number of allowed phase factors. In addition, we can set $b_1 = 1$ without any loss of performance. So, the number of required side information bits is $\lfloor \log_2 W^{M-1} \rfloor$ where $\lfloor z \rfloor$ denotes the smallest integer which does not exceed z.

We can use the hexagonal constellation in the PTS technique to eliminate the exchange of side information between the transmitter and the receiver. As explained earlier, some signal points in the square QAM constellation can have more than one associated points in the hexagonal constellation. It is possible to modify ordinary PTS technique for OFDM system with hexagonal constellation. Details of this PTS technique with hexagonal constellation are as follows:

1) Data block is divided into M subblocks as in the ordinary PTS technique.
2) Subcarriers in each subblock are classified into two categories: subcarriers with one representation and subcarriers with R representations.
3) Subcarriers with one representation in *all subblocks* are collected into a new (i.e., $(M+1)$th) subblock and these subcarriers are nulled in the corresponding subblocks. Now we have $(M+1)$ subblocks and the first M subblocks have less than or equal to N/M nonzero subcarriers.
4) Apply ordinary PTS technique for the $(M+1)$ subblocks with the phase factor for the $(M+1)$th subblock being fixed to 1.

In the ordinary PTS technique, different versions of each subblock are obtained by multiplying the subblock by different phase factors. In the PTS technique with hexagonal constellation, rth modified version of each subblock is obtained by selecting rth representations for all symbols with more than one representation in the subblock. We use demodulation for hexagonal constellation. Note that the side information is not necessary for the PTS technique with hexagonal constellation.

B. SLM Technique with Hexagonal Constellation

In the SLM technique, the transmitter generates a set of modified data blocks, all representing the same information as the original data block, and selects the most favorable for transmission [7]. Specifically, each data block is multiplied component-wise by U different phase sequences, each of length N, $\mathbf{B}^{(u)} = [b_{u,0}, b_{u,1}, \cdots, b_{u,N-1}]^T$, $u = 1, 2, \cdots, U$. To include the unmodified data block in the set of the phase rotated data blocks, we set $\mathbf{B}^{(1)}$ as the all-one vector of length N. Let us denote the phase rotated data block for the uth phase sequence as $\mathbf{X}^{(u)} = [X_0 b_{u,0}, X_1 b_{u,1}, \cdots, X_{N-1} b_{u,N-1}]^T$, $u = 1, 2, \cdots, U$. After applying the SLM to $\mathbf{X}$, the time-domain signal for the uth phase sequence, $u = 1, 2, \cdots, U$, becomes

$$x^{(u)}(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X_n b_{u,n} \cdot e^{j2\pi n \Delta f t}, 0 \le t < NT. \quad (7)$$

Among the phase rotated data blocks, the one with the lowest PAPR is selected and transmitted. The information about the selected phase sequence should be transmitted to the receiver as side information. At the receiver, the reverse operation is performed to recover the unmodified data block. The number of required side information bits is $\lfloor \log_2 U \rfloor$.

We can also use the hexagonal constellation in the SLM technique to eliminate the exchange of side information. In the SLM technique with hexagonal constellation, slight modifications are made for the construction and interpretation of the phase sequences. Details of the SLM technique with hexagonal constellation are as follows:

1) For each data block, denote the set of subcarrier indices where the data symbols have more than one representation as $I = \{i_1, i_2, \cdots, i_J\}$ where J is the size of I and $0 \le i_1, i_2, \cdots, i_J \le N-1$.
2) Also re-define the phase sequences as the sequences of integers between 0 and R. Each position of the phase sequences has randomly selected integer value between 1 and R for the indices in I while it has value of 0 for the rest of subcarriers.
3) Construct uth modified data block according to the uth phase sequence. When the nth element of the uth phase sequence is 0, the value of the nth position of the uth modified data block will be the same as that in the unmodified one. Otherwise, choose a specific representation according to the value of the phase sequence.
4) Choose the modified data block with the lowest PAPR.

C. Design and Construction of Hexagonal Constellation

We can design the hexagonal constellation such that the amplitude of the equivalent points (the points in hexagonal constellation associated with one point in QAM) is the same and the average power of the signal points is less than or equal to that of the square QAM constellation with same data rate. Since the amplitude of all equivalent signal points is the same, there is no power increase due to PTS or SLM. For example, hexagonal constellation with 91 signal points (91-HEX) is shown in Fig. 2. When 91-HEX is used instead of 64-QAM, symbol '1', '2', $\cdots$, '37' in 64-QAM have 1 representation in 91-HEX and symbol '38', '39', $\cdots$, '64' in 64-QAM have 2 representations in 91-HEX. In this case, 2 representations of a single point have same amplitude and opposite signs. So we can find another representation by simply multiplying -1 to one representation.

It is possible to generalize the proposed scheme to general QAM constellations. We can think the hexagonal constellation as a layered structure. For example, the layer 2 is made up of points '2', '3', '4', '5', '6', and '7' in Fig. 2. For general QAM, we can choose the number of layers in hexagonal constellation so that the number of points in those layers exceeds that of QAM constellation under consideration. For instance, there are 7 points in layers up to 2 and 19 points in layers up to 3. So it is required to use more than 3 layers for 16-QAM. If we choose to use layers up to 3, 13 points in 16-QAM have 1 representation in hexagonal constellation with 19 points (19-HEX) and 3 points in 16-QAM have 2 representations in 19-HEX. If we use more layers than minimum required, it is possible to increase the number of points in QAM that have more than one representation in hexagonal constellation. It is also possible to partially use the outermost layer.

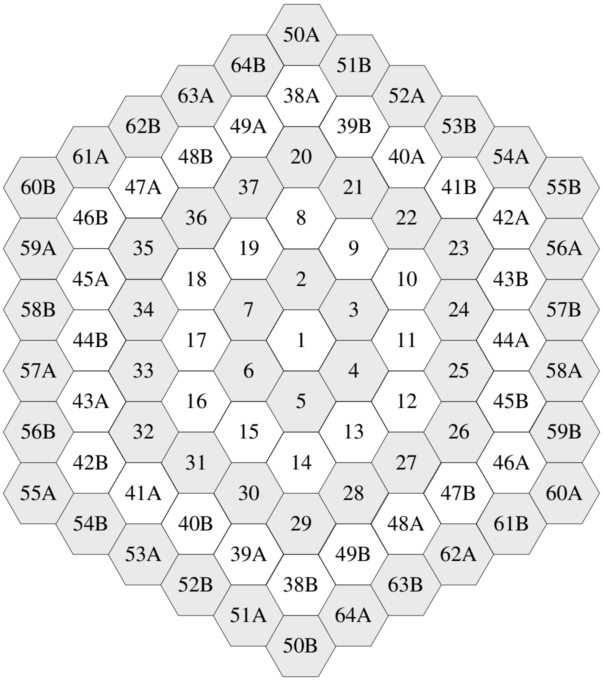

Fig. 2. Hexagonal constellation with 91 signal points.

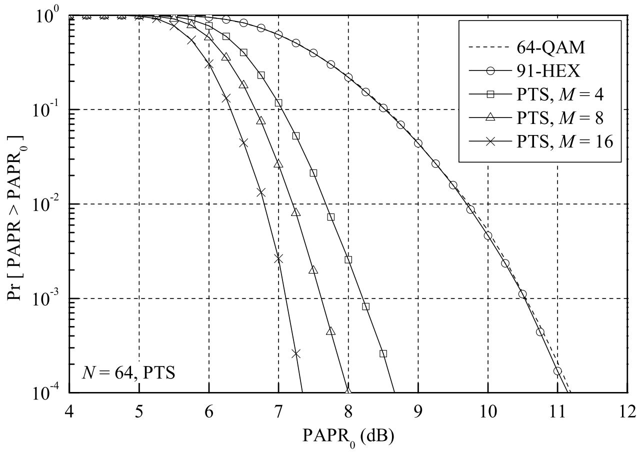

Fig. 3. CCDF of the PTS technique with hexagonal constellation with 91 signal points for an OFDM system with 64 subcarriers.

In designing hexagonal constellation, the average power is also important. We can design the hexagonal constellation such that the average power of hexagonal constellation is not larger than that of QAM constellation under consideration. For example, the average power of the signal points of the 91-HEX is $10.36d^2$ while that of the 64-QAM is $10.50d^2$. So there is no average power increase from using 91-HEX instead of 64-QAM. The average power can be reduced even further if we reduce the number of signal points with more than 1 representation.

D. Computational Complexity

Here, we consider the computational complexity of the PTS and SLM techniques with hexagonal constellation. We also compare the computational complexity of the proposed techniques with that of the similar techniques in [8] where PTS and SLM techniques without side information were proposed. Although details are different, the concept of eliminating side information is the same for the proposed techniques and the techniques in [8]. So, it is meaningful to compare the proposed techniques with those in [8] in terms of PAPR reduction capability, error performance, and computational complexity. Here, we concentrate only on the computational complexity. PAPR reduction capability and error performance will be discussed in Section V. Table 1 summarizes the computationl requirements for the proposed techniques and the techniques in [8] both at the transmitter and the receiver.

It is shown that the computational complexity is almost the same at the transmitter side. At the receiver side, the proposed techniques are much simpler than the techniques in [8] provided that we can demodulate the hexagonal constellation with comparable complexity to QAM. Fortunately, we do have a simple hexagonal constellation demodulation technique whose complexity is not much larger than that for the conventional QAM [9]. See Appendix for details of efficient demodulation technique for hexagonal constellation. So, the proposed technique has much smaller computational complexity at the receiver side.

V. Numerical Results and Discussions

We assume an OFDM system with 64 subcarriers ($N = 64$) with 64-QAM or hexagonal constellation with 91 signal points (91-HEX) as shown in Fig. 2. Symbol '1', '2', $\cdots$, '37' in 64-QAM have 1 representation in 91-HEX and symbol '38', '39', $\cdots$, '64' in 64-QAM have 2 representations in 91-HEX. For the PTS technique, the 64 subcarriers are divided into M subblocks with N/M contiguous subcarriers in each subblock. For the SLM technique, we randomly generate U phase sequences. The transmitted signal is oversampled by a factor of 4 ($L = 4$) and 100,000 random OFDM blocks were generated to obtain the complementary cumulative density functions (CCDFs) of PAPR.

Figure 3 shows the CCDFs of PAPR of the PTS technique with hexagonal constellation. The CCDF of PAPR of the unmodified OFDM signal is also shown for comparison. It is shown that the unmodified OFDM signal has a PAPR which exceeds 10.6 dB for less than 0.1 percent of the blocks. We can lower this 0.1 percent PAPR by 2.4 dB, 3.0 dB, and 3.5 dB with the proposed scheme for $M = 4$, 8, and 16, respectively. Note that there is no necessity to transmit side information between the transmitter to the receiver, which may cause data rate loss. Figure 4 shows the CCDFs of PAPR of the SLM technique with hexagonal constellation. The CCDF of PAPR of the unmodified OFDM signal is also shown for comparison. We can lower the 0.1 percent PAPR by 2.4 dB, 3.2 dB, and 3.6 dB with the proposed scheme for $U = 4$, 8, and 16, respectively. We also found that the PAPR reduction capability of the techniques in [8] is almost identical to that of the PTS and SLM techniques with hexagonal constellation under same simulation settings.

It is also of interest to see how the proposed techniques will perform under amplifier nonlinearity at the transmitter side. To approximate the effect of nonlinear power amplifier, we adopt Rapp's model for amplitude conversion [10]. The

TABLE I

COMPUTATIONAL OF COMPUTATIONAL REQUIREMENTS AT THE TRANSMITTER (TX) AND RECEIVER (RX).

PTS	Proposed	Tx	· $(M+1)$ IDFTs · Search for M phase factors
		Rx	· Demodulation of hexagonal constellation
	[8]	Tx	· M IDFTs · Search for $(M-1)$ phase factors
		Rx	· Decision metric D^m_{PTS} [8] calculation for each phase factor · Demodulation of QAM constellation
SLM	Proposed	Tx	· U IDFTs and find the minimum
		Rx	· Demodulation of hexagonal constellation
	[8]	Tx	· U IDFTs and find the minimum
		Rx	· Decision metric D_{SLM} [8] calculation · Demodulation of QAM constellation

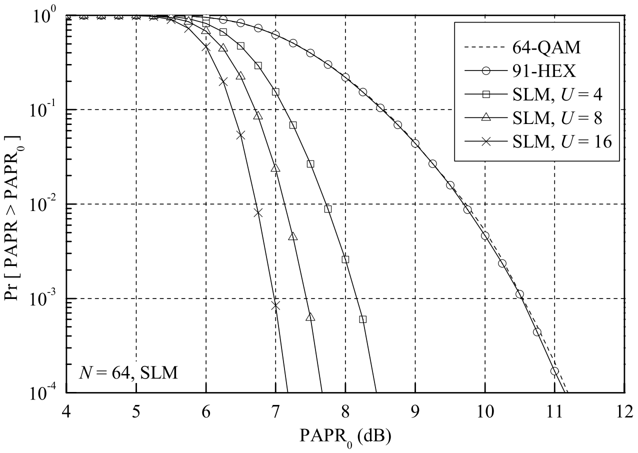

Fig. 4. CCDF of the PTS technique with hexagonal constellation with 91 signal points for an OFDM system with 64 subcarriers.

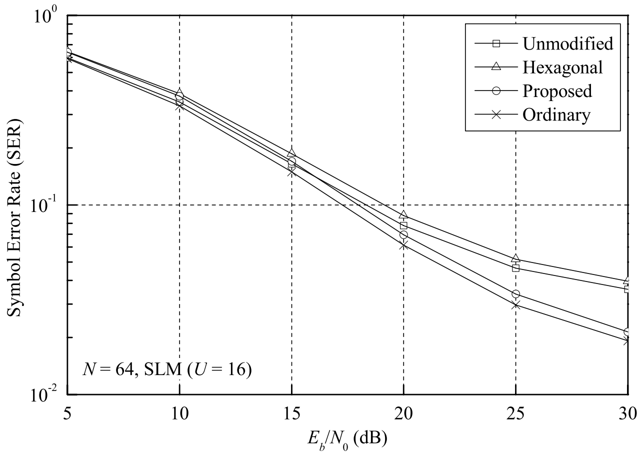

Fig. 5. SER of the SLM technique with hexagonal constellation with 91 signal points for an OFDM system with 64 subcarriers in a Rayleigh fading channel.

relation between amplitude of the normalized input signal A and amplitude of the normalized output signal $g(A)$ of the nonlinear power amplifier is given by

$$g(A) = \frac{A}{(1 + A^{2p})^{1/(2p)}} \tag{8}$$

where p is a parameter that represent the nonlinear characteristic of the power amplifier. The power amplifier approaches linear amplifier as p gets larger. We chose $p = 3$ which is a good approximation of a general power amplifier [10]. The phase conversion of the power amplifier is neglected in this paper. The input signal is normalized by a normalization factor to appropriately fit the input signal into the desired range in the input-output relation curve [11]. The normalized output signal is processed back into original scale before normalization. The amount of nonlinear distortion depends on the output back-off (OBO) which is defined as $\mathrm{OBO} = P_{\mathrm{o,max}}/P_{\mathrm{o,avg}}$ where $P_{\mathrm{o,max}}$ is the output power at the saturation point and $P_{\mathrm{o,avg}}$ is the average power of the output signal.

Figure 5 shows the symbol error rate (SER) of the SLM techniques with hexagonal constellation and that of the SLM technique in [8] both with 16 phase sequences ($U = 16$) and OBO = 7 dB in a Rayleigh fading channel. It is shown that the proposed scheme achieves lower SER than unmodified OFDM signal under transmitter nonlinearity due to its reduced dynamic range. It is also shown that the SER of the proposed technique is within 0.5 dB of the ordinary SLM technique when the phase sequence estimation is perfect. It is shown in [8] that the performance gap of the SLM technique without side information and that of the perfect phase sequence estimation case is similar to that of the SLM technique with hexagonal constellation. So it can be concluded that the SER performance of both techniques is not much different.

VI. CONCLUSIONS

In this paper, we proposed the use of hexagonal constellation for PAPR reduction and applied the proposed technique to eliminate the side information in the PTS and SLM techniques. It is shown that we can achieve significant reduction in PAPR with no data rate loss from side information and achieve lower SER than unmodified OFDM signal due to reduced dynamic range. It is also shown that the proposed techniques achieve almost identical PAPR reduction and SER performance to

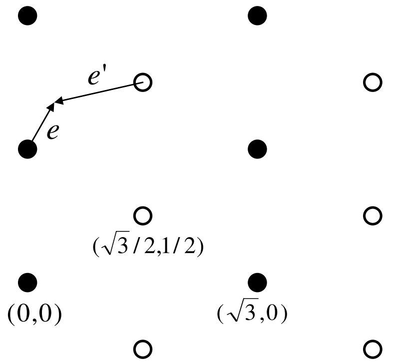

Fig. A1. Hexagonal lattice as a disjoint union of a rectangular lattice and its coset [9].

the techniques in [8] with much reduced complexity at the receiver.

Appendix
Efficient Demodulation of Hexagonal Constellation

Here we will briefly summarize the algorithm for efficient demodulation of hexagonal constellation, which was proposed in [9].

One main difficulty of hexagonal constellation is its complicate data decision operations. From Fig. 1 and Fig. 2, it is obvious that the nearest neighbor region of every lattice point is simply a regular hexagon centering at that point. The trivial and straightforward method is to compute the distances of all lattice points from the query points.

The new demodulation algorithm is stemmed from the idea that a lattice may be represented as a disjoint union of a sublattice and its cosets. Thus fast decoding of the constituent sublattice leads to fast decoding of the original lattice. In addition, we need the important observation that a hexagonal lattice can be expressed as disjoint union of a rectangular lattice and its coset.

We define the hexagonal lattice A_2 as the set of all vectors $u_1(0,1)+u_2(\sqrt{3}/2,1/2)$ where u_1 and u_2 are integers. Define $B = \sqrt{3}\mathbf{Z}\bigoplus\mathbf{Z}$ where $\mathbf{Z}$ is the set of integers. Then we have $A_2 = B\bigcup((\sqrt{3}/2,1/2)+B)$. Figure A1 illustrates graphically how A_2 is partitioned into B and its translated version. Since B is a rectangular lattice, its decoding algorithm involves only scaling and rounding operations. Let $\Phi(x)$ be the closest point of x in B. The new decoding algorithm of is now obvious.

Algorithm 1 – Given input vector $x \in \mathbf{R}^2$, compute $\Phi(x)$ and $\Phi(x-(\sqrt{3}/2,1/2))+(\sqrt{3}/2,1/2)$. Compare the two resultant vectors with x and choose the closest.

Further simplification to Algorithm 1 is possible. Let $e = x-\Phi(x)$ and $e' = x-\Phi(x-(\sqrt{3}/2,1/2))-(\sqrt{3}/2,1/2)$. In other words, e and e' are the error vectors resulting from quantizing x by lattice B and $(\sqrt{3}/2,1/2)+B$, respectively. From Fig. A1, it can be seen that

$$e - e' = \left(\frac{\sqrt{3}}{2}\mathrm{sgn}(e_1), \frac{1}{2}\mathrm{sgn}(e_2)\right) \qquad \text{(A1)}$$

where $(e_1,e_2) = x-\Phi(x)$ and the sign function $\mathrm{sgn}(\cdot)$ is defined as

$$\mathrm{sgn}(y) \equiv \begin{cases} 1, & y \geq 0, \\ -1, & y < 0. \end{cases}$$

Algorithm 1 suggest to output $x-e'$ if $||e||^2 > ||e'||^2$ and output $x-e$ otherwise. By (A1), $||e||^2 > ||e'||^2 \iff \sqrt{3}|e_1|+|e_2| > 1$. This suggest another version of the demodulation algorithm.

Algorithm 2 – Given input vector $x \in \mathbf{R}^2$, compute $(e_1,e_2) = x-\Phi(x)$. Output $\Phi(x)+(\sqrt{3}/2\cdot\mathrm{sgn}(e_1), 1/2\cdot\mathrm{sgn}(e_2))$ as the closest vector if $\sqrt{3}|e_1|+|e_2| > 1$, and output $\Phi(x)$ otherwise.

By using Algorithm 2, demodulation of hexagonal constellation is greatly simplified because it consists of normal QAM demodulation and a comparator.

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