On the Use of High-Order Moment Matching to Approximate the Generalized-*K* Distribution by a Gamma Distribution

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***Abstrac*[1]—Composite fading takes place in several communication channels due to the random variations of the local average power of the received multipath-faded signal. The generalized-*K* (Gamma–Gamma) probability density function (PDF) has been proposed recently to model composite fading in wireless channels. However, further derivations using the generalized-*K* PDF have shown to be quite involved due to the computational and analytical difficulties associated with the arising special functions. In this paper, the approximation of the generalized-*K* PDF by a Gamma PDF using the moment matching method is explored. As expected, matching positive and negative moments leads to a better approximation in the upper and lower tail regions, respectively. However, due to arising limitations for small values of the multipath fading and shadowing parameters, and the higher level of accuracy sought, the use of an adjustable form for the expressions of the approximating Gamma PDF parameters, obtained by matching the first two positive moments, is devised. The optimal values of the adjustment factor for different integer and non-integer values of the fading and shadowing parameters are given. The introduced approximation may simplify performance analysis in distributed antenna systems (DASs), network MIMO, multihop relay networks, radar, and sonar systems.**

Keywords: composite fading channels, generalized-*K* distribution, Gamma distribution, moment matching method, negative and positive moments, amount of fading, sum of generalized-*K* random variables.

I. INTRODUCTION

Statistical modeling of communication channels plays an essential role in the design and analysis of different communication schemes over such channels. In wireless channels, modeling the composite fading phenomenon resulting from the simultaneous occurrence of multipath fading and shadowing is relevant to many communication problems such as performance analysis of distributed antenna systems, relay networks and radar systems. The Gamma-Gamma (generalized-*K*) model was first introduced in the literature on the scattering in radar and sonar systems [1] and is recently proposed to model composite fading in wireless channels as a substitute to the analytically intractable lognormal-based models [2-4]. The theoretical and experimental validations for the use of this model are presented in [1]. However, further analysis using the generalized-*K* model has resulted in analytical and numerical difficulties due to the special functions encountered [5, 6].

In [7], a region-wise approximation of the generalized-*K* PDF by the familiar Gamma probability density function (PDF) was introduced using an adjustable form of the well-known first two moment matching method. Numerical results for different values of the composite fading parameters have shown that such an approximation can be made accurate enough in both the lower and the upper tail regions.

In this paper, the approximation of the generalized-*K* PDF by a Gamma PDF is further explored by matching both the positive and negative moments. The obtained results have shown that matching higher order moments leads to a good approximation, up to a certain level of accuracy, in both the upper and lower tail regions, and may lead to lower and upper bounds on the approximated cumulative distribution function (CDF). However, matching higher order moments has two main limitations; (i) it results in involved expressions that are difficult to solve and complicated to draw insights from and (ii) negative moments may not exist for small values of the multipath fading and shadowing parameters. To bypass these limitations, the use of the adjustable form for the expressions obtained by matching the first two positive moments, as introduced in [7], is devised and the optimal adjustment factors are computed for relevant values of the composite fading channel parameters.

II. THE GENERALIZED-*K* COMPOSITE FADING MODEL

The PDF of the instantaneous received power γ in a composite fading channel can be expressed, using the generalized-*K* model, as [1, 2]

$$p_\gamma(x) = \frac{2}{\Gamma(m_s)\Gamma(m_m)}\left(\frac{b}{2}\right)^{m_s+m_m} x^{\left(\frac{m_s+m_m}{2}\right)-1} K_{m_s-m_m}\left(b\sqrt{x}\right), \quad (1)$$

$$, x \geq 0, m_m \geq 0.5, m_s > 0$$

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where $\Gamma(\cdot)$ is the Gamma function, m_m and m_s are the Nakagami multipath fading and shadowing parameters, respectively. $K_{m_s-m_m}(\cdot)$ is the modified Bessel function of the second kind and order (m_s-m_m), $b=2\sqrt{\frac{m_m m_s}{\Omega_0}}$, and Ω_0 is the mean of the local average power.

The CDF of the generalized-*K* random variable (RV) and the moment generating function as obtained in [3] contain special functions, namely the hyper-geometric and the Whittaker functions. These functions have numerical instabilities and are analytically cumbersome to handle [5].

III. THE APPROXIMATION USING HIGHER ORDER MOMENTS

The moment matching method is widely used to fit distributions to empirical data or to approximate an analytically complicated distribution by a more tractable one. In this paper, moment matching is used to approximate the generalized-*K* distribution by the more tractable Gamma distribution.

The *n*th moment of the generalized-*K* distribution can be derived as [3]

$$E\left[X^n\right]=\frac{\Gamma(m_m+n)\Gamma(m_s+n)}{\Gamma(m_m)\Gamma(m_s)}\left(\frac{\Omega_0}{m_m m_s}\right)^n . \quad (2)$$

The *n*th moment of the Gamma distribution whose scale and shape parameters are θ and k, respectively is [8].

$$E\left[X^n\right]=\frac{\Gamma(k+n)\theta^n}{\Gamma(k)} . \quad (3)$$

Now, using the expressions in (2) and (3), the first and second positive moments of the generalized-*K* distribution and the approximating Gamma distribution can be matched as

$$k\theta=\Omega_0 , \quad (4)$$

$$\theta^2 k(k+1)=\frac{(m_m+1)(m_s+1)}{m_m m_s}\Omega_0^{\,2} . \quad (5)$$

Solving (4) and (5) for θ and k results in

$$\theta_{1,2}=(K_1-1)\Omega_0 , \quad \theta_{1,2}>0 , \quad (6\text{-a})$$

$$k_{1,2}=\frac{1}{K_1-1}, \quad k_{1,2}>0 , \quad (6\text{-b})$$

where $\theta_{i,j}$ and $k_{i,j}$ denote the scale and shape parameters of the Gamma PDF obtained by matching the *i*th and the *j*th moments and $K_1=\frac{(m_m+1)(m_s+1)}{m_m m_s}=1+\frac{1}{m_m}+\frac{1}{m_s}+\frac{1}{m_m m_s}$. Note that $k_{1,2}$ converges to m_m as m_s increases and vice versa.

Similarly, matching the first positive moment, as given in (4), and third positive moment as

$$\theta^3(k+2)(k+1)(k)=K_1K_2\Omega_0^{\,3} , \quad (7)$$

results in

$$\theta_{1,3}=\frac{\left(-3+\sqrt{9+8(K_1K_2-1)}\right)\Omega_0}{4}, \theta_{1,3}>0 , \quad (8\text{-a})$$

$$k_{1,3}=\frac{4}{-3+\sqrt{9+8(K_1K_2-1)}}, k_{1,3}>0 , \quad (8\text{-b})$$

where $K_2=\frac{(m_m+2)(m_s+2)}{m_m m_s}=1+\frac{2}{m_m}+\frac{2}{m_s}+\frac{4}{m_m m_s}$.

Note again that if m_m and m_s are large (i.e., $K_1\to 1$ and $K_2\to 1$), then $\theta_{1,3}\to 0$ as expected.

Matching the second positive moment, as in (5), and the third positive moment, as in (7), results in

$$k_{2,3}=\frac{\left(-4+\frac{K_2^{\,2}}{K_1}\right)-\sqrt{\left(\frac{K_2^{\,2}}{K_1}\right)^2+8\frac{K_2^{\,2}}{K_1}}}{2\left(1-\frac{K_2^{\,2}}{K_1}\right)}, k_{2,3}>0 , \quad (9\text{-a})$$

$$\theta_{2,3}=\sqrt{\frac{K_1}{\left(k_{2,3}^{\,2}+k_{2,3}\right)}}\Omega_0, \theta_{2,3}>0 . \quad (9\text{-b})$$

On the other hand, negative moments, as defined in [9], of the generalized-*K* and the Gamma PDFs can be expressed using (2) and (3). Matching again the first positive moment, as in (4), and the first negative moment as

$$\frac{1}{\theta(k-1)}=\frac{m_m m_s}{(m_m-1)(m_s-1)\Omega_0}, m_m>1,\ m_s>1, k>1 , \quad (10)$$

results in

$$\theta_{1,-1}=(1-K_{-1})\Omega_0, \theta_{1,-1}>0 , \quad (11\text{-a})$$

$$k_{1,-1}=\frac{1}{1-K_{-1}}, \ k_{1,-1}>1 , \quad (11\text{-b})$$

where $K_{-1}=1-\frac{1}{m_m}-\frac{1}{m_s}+\frac{1}{m_m m_s}$.

Similarly, matching the first positive moments, as in (4), and the second negative moments, $E\left[X^{-2}\right]$, as

$$\theta^2(k-2)(k-1)=K_{-1}K_{-2}\Omega_0^{\,2}, k>2, m_m>2, m_s>2 , \quad (12)$$

results in

$$\theta_{1,-2}=\frac{\left(3-\sqrt{9+8(K_{-1}K_{-2}-1)}\right)\Omega_0}{4}, \ \theta_{1,-2}>0 , \quad (13\text{-a})$$

$$k_{1,-2}=\frac{4}{3-\sqrt{9+8(K_{-1}K_{-2}-1)}}, \ k_{1,-2}>2 , \quad (13\text{-b})$$

where $K_{-2} = 1 - \frac{2}{m_m} - \frac{2}{m_s} + \frac{4}{m_m m_s}$.

Finally, matching the first and the second negative moments results in

$$\theta_{-1,-2} = \left(\frac{1}{m_m} + \frac{1}{m_s} - \frac{3}{m_m m_s}\right)\Omega_0, \; m_m > 2, m_s > 2, \theta_{-1,-2} > 0, \tag{14-a}$$

$$k_{-1,-2} = \frac{K_{-1}\Omega_0}{\theta_{-1,-2}} + 1, \; k_{-1,-2} > 2. \tag{14-b}$$

Now, from the expressions given in (6) to (14), the following may be stated:

- The scale parameter of the approximating Gamma PDF obtained by matching the positive moments is larger than the one obtained by matching negative moments. Since the negative moments characterize a distribution at the origin [9] (the lower tail for a positive RV) and the positive moments characterize a distribution at the upper tail, *we may conclude that the generalized-K PDF can be approximated by a Gamma distribution whose scale and shape parameters depend on the region of the PDF (CDF) of interest*. A similar region-wise (piece-wise) approximation was used in [10] to well-approximate the sum of lognormal RVs by a single lognormal RV.
- Matching moments for $n \geq 2$ will lead to involved expressions as seen in (9-a) and (9-b). Moreover, not including the first positive moment in matching any two moments results in an approximating Gamma PDF that does not have the same mean as the approximated generalized-*K* PDF (i.e., the two PDFs may have different means of the local power).
- Matching the negative moments is not possible for small values of m_m and m_s since these moments may not exist as indicated in (11-a) to (14-b).
- The scale and shape parameters of the approximating Gamma distribution are dependent on the fading parameters in the sense that as m_m and m_s increase, the difference between the predicted scale and shape parameters decreases and hence the difference between the approximating PDFs (CDFs) becomes small. So, for small values of m_m and/or m_s (while m_m and m_s>2), the difference between the two approximating Gamma CDFs might be large enough to bound the approximated CDF as seen in Fig. 1. On the other hand, matching the same moments for large values of m_m and m_s does not result in a good approximation if higher accuracy is sought as seen in Fig.s 2-3 since the approximating CDFs are too close to each other.

Note: in Fig.s 1-3, the complementary cumulative distribution function (CCDF) and particularly the region corresponding to $P(X \geq x) \leq 0.1$ is shown to obtain more illustrative results for the upper tail region. The notation $\mu_{i,j}$ in these figures denotes that the *i*th and *j*th moments are matched.

IV. THE MOMENT MATCHING METHOD WITH ADJUSTMENT

In order to bypass the limitations explained before on the use of the moment matching for higher order moments, we may consider an adjustable form for the parameters of the approximating Gamma PDF obtained by matching the first two positive moments since (i) the expressions in (6-a) and (6-b) are simple and valid for any value of m_m and m_s and (ii) the first positive moment is included in the matching.

The expressions in (6-a) and (6-b) may be re-written as

$$\theta_{1,2} = \left(\frac{1}{m_m} + \frac{1}{m_s} + \frac{1}{m_m m_s}\right)\Omega_0 = [AF]\Omega_0, \; 0 \leq AF \leq AF_{\max}, \tag{15-a}$$

$$k_{1,2} = \frac{1}{AF}, \qquad 0 \leq AF \leq AF_{\max}. \tag{15-b}$$

In the above, $\frac{1}{m_m} + \frac{1}{m_s} + \frac{1}{m_m m_s}$ is the amount of fading (*AF*) in the composite fading channel as derived in [2]. The value of $AF_{\max}$ is finite since the smallest values of m_m and m_s in real propagation channels are non-zero. However, the approximating PDF as given by (15-a) and (15-b) results in a poor fit in the lower and upper tail regions since matching only the first and second moments yields a good fit only around the mean. To overcome this limitation, we may consider the following adjustable form of the expressions in (15-a) and (15-b)

$$\theta'_{1,2} = [AF - \varepsilon]\Omega_0, \; 0 \leq AF \leq AF_{\max}, -\varepsilon_0 \leq \varepsilon \leq AF, \theta'_{1,2} > 0, \tag{16-a}$$

$$k'_{1,2} = \frac{1}{AF - \varepsilon}, \; 0 \leq AF \leq AF_{\max}, -\varepsilon_0 \leq \varepsilon \leq AF, k'_{1,2} > 0. \tag{16-b}$$

Since the amount of fading "added" to the scale parameter of the approximating Gamma PDF can not exceed the original amount of fading of the approximated PDF (i.e., $\varepsilon_0 \leq AF$), the value of ε should be bounded as $-AF \leq \varepsilon \leq AF$. Due to the fact that the relevant practical range of *AF* is from zero (for non-fading channels) to around 8 (for severe multipath fading and shadowing conditions where m_m=0.5 and m_s=0.5)[2], the relevant range of the adjustment factor ε becomes $-8 \leq \varepsilon \leq 8$.

The optimal values of the adjustment factor can be computed through minimizing a numerical measure of the

[2] Smaller values of m_m and m_s may take place in land mobile satellite channels [11].

difference between the approximated and the approximating PDFs (CDFs). A common measure is the absolute value of the difference between the approximated and the approximating PDFs (CDFs) that is similar to the well-known Kolmogorov-Smirnov test on the difference between the CDFs of two continuous distributions [8]. Also, in this paper, the CDFs rather than the PDFs are considered since the Gamma PDF goes to infinity near the origin for k<1 [8] which causes numerical instabilities in the lower tail region. The plots of the optimal adjustment factor versus the multipath fading and shadowing parameters are shown in Fig.s 4 and 5 for values of both m_m and m_s ranging from 0.5 to 10. It is observed from the plots that the adjustment factor decreases as either or both m_m and m_s increase. The decrease of the adjustment factor as both m_m and m_s increase is interesting since it indicates that the product of two Gamma PDFs can be approximated by a Gamma PDF using the method of matching the first two moments. The amount of fading for $m_m = m_s = m$ can be expressed as $AF = (2m+1)/m^2$ which is approximately $2/m$ for moderate values of m, and it converges to zero for very large values of m. However, if a high level of accuracy is sought, then the magnitude of the adjustment factor increases. Similar plots can be obtained for other regions of interest. The optimal values of the adjustment factor can be tabulated as in Table I to be available for use.

V. On Approximating the PDF of the Sum of Generalized-K RVs

The distribution of the sum of N independent generalized-K RVs is needed to analyze the performance of maximal ratio combining (MRC), the outage capacity and the ergodic capacity in composite fading channels. The adjustable form of the first two moments matching can be used to approximate the PDF of the sum of N independent generalized-K RVs and the 3-D plots of ε versus m_m and m_s for each value of N>1 can be obtained. Moreover, it is found [12] that the PDF of the sum of N independent and identically distributed (i.i.d.) generalized-K RVs is closely approximated by the PDF of another generalized-K RV, ζ, whose amount of fading is $AF_\zeta = \frac{1}{Nm_m} + \frac{1}{Nm_s} + \frac{1}{Nm_m m_s}$; the corresponding $m_{m,\zeta}$ and $m_{s,\zeta}$ can be computed by matching the AF and setting $\frac{m_{m,\varsigma}}{m_{s,\varsigma}} = \frac{m_m}{m_s}$, and $\Omega_{0,\zeta} = N\Omega_0$. It can also be shown that the PDF of the sum of N generalized-K RVs whose shadowing components are identically distributed and fully correlated and multipath components are i.i.d. is identical to the PDF of another generalized-K RV whose parameters are $m_{m,sum} = Nm_m$ and $m_{s,sum} = m_s$, and vice versa. Furthermore, the approximation in Section IV can be used to approximate the PDF of the sum of N generalized-K RVs by a single Gamma RV.

VI. Conclusions

In this paper, the approximation of the generalized-K distribution by the more tractable Gamma distribution using the moment matching method is investigated. The obtained results have shown that matching the positive and negative moments can yield a good approximation in both the lower and upper tail regions. However, the associated limitations have led to the proposal of an adjusted form of the expressions of the parameters of the approximating Gamma PDF obtained by matching the first two positive moments. This simple and sufficiently accurate approximation can be further used to approximate the PDF of the sum of both independent and correlated generalized-K RVs. Extensions of this work may include considering matching non-integer moments and approximating the PDF of the sum of non-i.i.d. generalized-K RVs using the proposed method.

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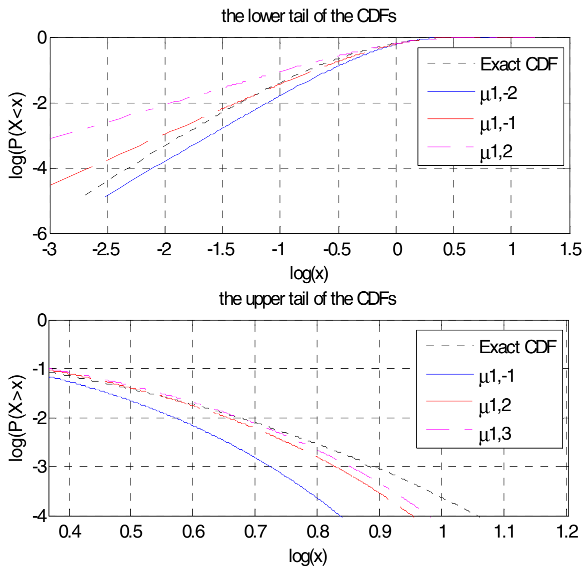

Figure 1. The CDFs corresponding to the generalized-*K* and the approximating Gamma PDFs for m_m=2.5 and m_s=2.5 using the moment matching method where $\mu_{i,j}$ denotes that the *i*th and *j*th moments are matched.

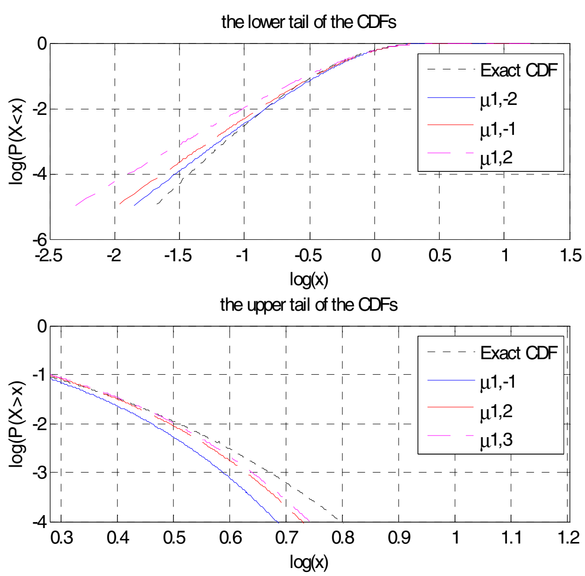

Figure 2. The CDFs corresponding to the generalized-*K* and the approximating Gamma PDFs for m_m=7 and m_s=4 using the moment matching method.

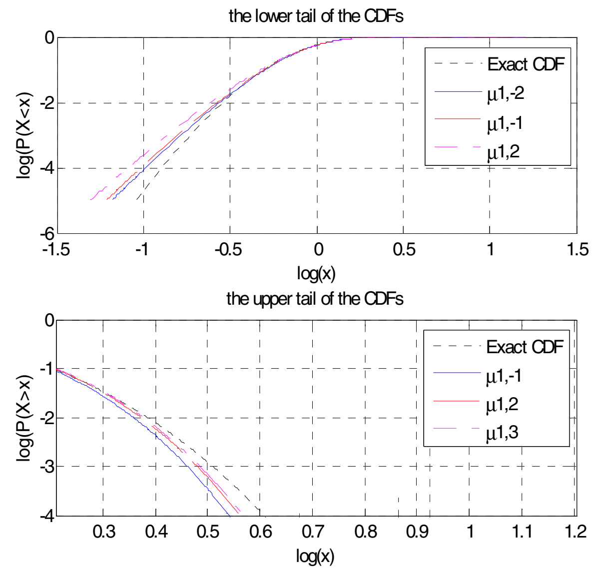

Figure 3. The CDFs corresponding to the generalized-*K* and the approximating Gamma PDFs for m_m=10 and m_s=10 using the moment matching method.

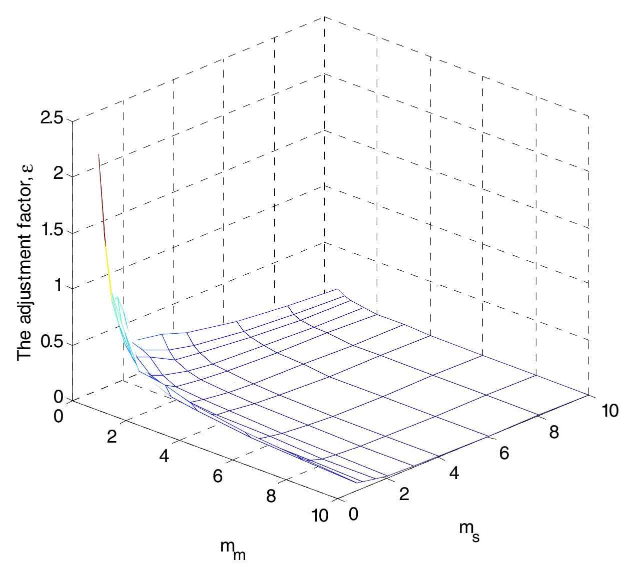

Figure 4. The plot of the adjustment factor that minimizes the absolute value of the difference between the approximated generalized-*K* and the approximating Gamma distributions over the entire CDF.

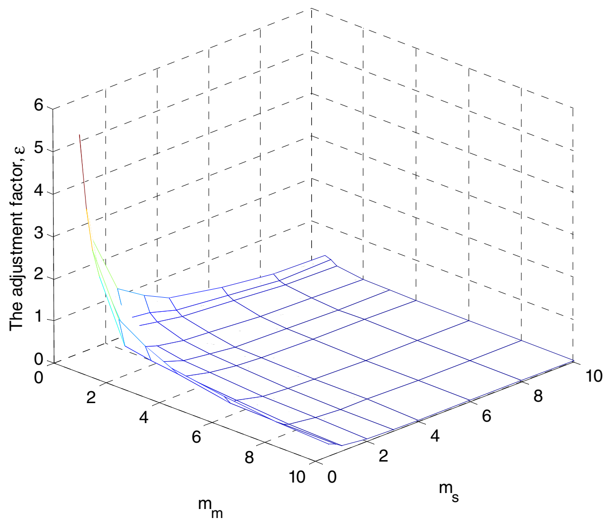

Figure 5. The plot of the adjustment factor that minimizes the absolute value of the difference between the approximated generalized-*K* and the approximating Gamma distributions for the lower tail (CDF<0.1).

Table I

The optimal values of the adjustment factor that minimize the absolute value of the difference between the generalized-*K* and the Gamma distributions over the entire CDF for different values of m_m and m_s (refer to Fig. 4).

m_m / m_s	**0.5**	**0.75**	**1**	**1.5**	**2**	**3**	**4**	**6**	**8**	**10**
0.5	2.20	1.4	1.0	0.70	0.40	0.36	0.28	0.20	0.16	0.12
0.75	1.4	0.9	0.70	0.46	0.36	0.24	0.19	0.129	0.10	0.08
1	1.0	0.7	0.50	0.34	0.30	0.18	0.15	0.10	0.075	0.06
1.5	0.70	0.46	0.34	0.22	0.18	0.12	0.09	0.07	0.05	0.04
2	0.40	0.34	0.30	0.18	0.12	0.09	0.08	0.05	0.04	0.03
3	0.36	0.24	0.18	0.12	0.09	0.06	0.05	0.035	0.025	0.02
4	0.28	0.19	0.15	0.09	0.08	0.05	0.03	0.02	0.018	0.015
6	0.20	0.129	0.10	0.07	0.05	0.035	0.02	0.018	0.013	0.012
8	0.16	0.10	0.075	0.05	0.04	0.025	0.02	0.013	0.009	0.0075
10	0.12	0.08	0.06	0.04	0.03	0.02	0.015	0.012	0.0075	0.006