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ON THE USE OF PAIRWISE BALANCED DESIGNS AND CLOSURE SPACES IN THE CONSTRUCTION OF STRUCTURES OF DEGREE AT LEAST 3 (*)

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We prove that a set of $v-2$ symmetric idempotent latin squares of order v, such that no two of them agree in an off-diagonal position, exists for all sufficiently large odd v. We describe how the techniques used in the proof relate to techniques used in [17] to construct generalised idempotent ternary quasigroups whose conjugate invariant group contains some specified subgroup. We also show how these techniques fit into the more general context of trying to extend group divisible design methods to combinatorial structures with $t \geq 3$, using closure spaces.

1. Introduction.

Notions used, but not defined, in this introduction will be defined in later sections.

It is well known that PBD-constructions and more general

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group divisible design constructions are among the most important constructive tools for many types of combinatorial structures «with $t = 2$». For analogous structures with $t \geq 3$, there usually are obvious t-wise balanced design constructions, but these are somewhat limited in applicability by the relative scarcity of constructions for t-wise balanced designs with $t \geq 3$.

In [17], we found PBD-constructions for certain structures with $t = 3$. In section 2 of this paper, we give some examples of these constructions. However, we do not go into concrete applications, because such applications are already given in [17].

A naive application of PBD-constructions and group divisible design methods for large sets of disjoint structures with $t = 2$ does not work. However, in section 3 of this paper, we nevertheless use PBD-constructions for large sets of certain structures with $t = 2$. For instance, we describe PBD-constructions for golf designs and related structures. A golf design for v clubs is a collection of $v - 2$ symmetric idempotent latin squares of order v such that no two of them agree in an off-diagonal position. (Golf designs are equivalent to large sets of certain structures with $t = 2$, as we will explain in section 3). Golf designs were introduced in [12] and further studied in [18]. As noticed in [18], any large set of disjoint $S(2,3,v)$ yields a golf design for v clubs. Such large sets exist for all $v \equiv 1$ or 3 (mod 6) with $v \neq 7$ [6, 7, 8]. (However, as reference [8] is unfinished, due to the death of the author, part of the proof for $v \in \{141, 283, 501, 789, 1501, 2365\}$ is missing). No golf design for 5 clubs exists [12]. In [12] a golf design for 7 clubs is given and in [18] a golf design for 17 clubs is constructed. However, the problem remained open for al $v \equiv 5$ (mod 6), $v \notin \{5, 17\}$. In section 3, we prove that a golf design for v clubs exists for all sufficiently large odd v. In section 4, we show that if a golf design for 11 clubs would exist, then they would exist for all $v \equiv 5$ (mod 6), except 5 and possibly 41. The existence of a golf design for 11 clubs remains open, however.

One of the most natural generalizations of PBD-constructions and group divisible design techniques to $t \geq 3$ involves matroids or, more generally, closure spaces. Trying to make this idea yield concrete results for $t \geq 4$ seems very difficult, but may have great potential, because of the fundamental importance of the technique for $t = 2$. In section 5, we explain how the techniques of [17] and sections 2, 3 and 4 of the present paper fit into the general context of trying to extend group divisible design constructions to higher t, using closure spaces.

In this paper, we assume that sets are finite unless they are infinite for obvious reasons.

2. Orthogonal arrays and ordered designs.

A $k-S$-*array*, S a set, k a positive integer, will be subset of S^k. The elements of the array are called *rows*. If we do not want to specify k or S, we use terms such as *k-array, S-array*, or simply *array*. The *conjugate invariant group* $H(q)$ of a $k-S$-*array* q will be the set of all $\sigma \in S_k$ such that $(x_1, \ldots, x_k) \in q$ iff $(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \in q$. We call q *totally symmetric* if $H(q) = S_k$. A *t-subspace* of a $k-S$-*array* q will be a subset D of S such that if $(x_1, \ldots, x_k) \in q$ and $|\{i \in \{1, \ldots, k\}; x_i \in D\}| \geq t$, then $\{x_1, \ldots, x_k\} \subset D$. If q is a $k-S$-*array* and $E \subset S$, we denote by $q|E$ the $k-E$-*array* consisting of all $(x_1, \ldots, x_k) \in q$ with $\{x_1, \ldots, x_k\} \subset E$. If q is a $k-S$-*array* and if q_0 is a $k-D$-*array* with $D \subset S$, then we say that q *contains* (D, q_0) *as a t-subspace* if D is a t-subspace of q and $q|D = q_0$. An *orthogonal array* $OA(t, k, v), t, k, v \in \mathbb{N}, k \geq t+1$, is a $k-S$-*array* q, S a v-set, such that for any t-subset $\{i_1, \ldots, i_t\}$ of $\{1, \ldots, k\}$ and for any t-tuple $x_{i_1}, \ldots, x_{i_t}$ of (not necessarily distinct) elements of S, there is exactly one row $(y_1, \ldots, y_k)$ of q with $y_{i_1} = x_{i_1}, \ldots, y_{i_t} = x_{i_t}$. A *subspace* of an $OA(t, k, v)q$ will be a t-subspace of the array q. If δ is a set of subsets of a set S, then a $\delta - OA(t, k, S)$ will be a $k-S$-*array* q such that, for any $(x_1, \ldots, x_k) \in q$ and for any $C \in \delta$, we have $|\{i \in \{1, \ldots, k\}; x_i \in C\}| < t$ and such that for any t-subset $\{i_1, \ldots, i_t\}$ of $\{1, \ldots, k\}$ and any t-tuple $x_{i_1}, \ldots, x_{i_t}$ of (not necessarily distinct) elements of S, not entirely contained in an element of δ, there is exactly one row $(y_1, \ldots, y_k)$ of q with $y_{i_1} = x_{i_1}, \ldots, y_{i_t} = x_{i_t}$. If q is an $OA(t, k, v)$ on a set S and if δ is a set of subspaces of q, then

$q - \bigcup_{D\in\delta}(q|D)$ is a $\delta - OA(t,k,S)$. However, not every $\delta - OA(t,k,S)$ can be completed into an $OA(t,k,|S|)$ on S. An *ordered design* $OD(t,k,v)$ is defined in the same way as an $OA(t,k,v)$ except that $x_{i_1},\ldots,x_{i_t}$ are assumed to be pairwise distinct and that all rows of the array are assumed to contain k distinct elements of S. If S is a set, we denote by $P(S)$ the set of all subsets of S and by $P_k(S)$ the set of all k-subsets of S. If S is a v-set, $v \geq t-1$, and if $t \geq 2$, then an $OD(t,k,v)$ on S is the same thing as a $P_{t-1}(S) - OA(t,k,S)$. A *subspace* of an $OD(t,k,v)$ is defined to be a t-subspace of the array. An $OA(t,k,v)q$ on a set S is called *idempotent* if $(x,\ldots,x) \in q$ for all $x \in S$, i.e. if $\{x\}$ is a subspace for all $x \in S$. Obviously, if q is an idempotent $OA(2,k,v)$ on a set S, then $q' = q - \{(x,\ldots,x); x \in S\}$ is an $OD(2,k,v)$ with $H(q) = H(q')$. Conversely, if q is an $OD(2,k,v)$ on a set S, then $q' = q \cup \{(x,\ldots,x); x \in S\}$ is an idempotent $OA(2,k,v)$ with $H(q) = H(q')$. Thus, there is a one-to-one correspondence, preserving conjugate invariant groups and subspaces, between idempotent $OA(2,k,v)$ and $OD(2,k,v)$. An $OA(3,4,v)q$ on a set S is called *2-idempotent* if it is idempotent and if (x,x,y,y), (x,y,x,y) and (x,y,y,x) are in q for all $x,y \in S$. In other words, an $OA(3,4,v)q$ on a set S is 2-idempotent iff A is a subspace for all $A \subset S$ with $|A| \leq 2$. Again, there is an obvious one-to-one correspondence, preserving conjugate invariant groups and subspaces, between 2-idempotent $OA(3,4,v)$ and $OD(3,4,v)$.

A $3 - S$-*array* q is called *commutative* if $(12) \in H(q)$. An $OA(2,3,v)$ on a v-set S can also be considered as a latin square with rows and columns indexed by S and with entries in S. The entry in row x and column y is given by z, where z is the unique element of S with $(x,y,z) \in q$. Conversely, every latin square with rows and columns indexed by S and entries in S can be obtained in this way from a unique $OA(2,3,v)$ on S. The $OA(2,3,v)$ is idempotent (commutative, respectively) iff the latin square is idempotent (symmetric, respectively). It is well known that idempotent latin squares of order v exist for all $v \neq 2$ and that symmetric idempotent latin squares of order v, $v \neq 0$, exist iff v is odd. The $OA(2,3,v)$ on a v-set S are also in one-to-one correspondence with the quasigroups on S. Indeed, we can identify the $OA(2,3,v)q$ on S with the quasigroup $(S,\cdot)$ defined by $x \cdot y = z$, where again z is the unique element of S with $(x,y,z) \in q$. The quasigroup is idempotent (commutative) iff the $OA(2,3,v)$ is idempotent (commutative).

A *t-wise balanced design* $S(t,K,v), t,v \in \mathbb{N}, K \subset \mathbb{N}$, is a pair (S,β), where S is a v-set and $\beta \subset P(S)$ such that $|B| \in K$ for all $B \in \beta$ and such that any t-subset of S is contained in exactly one element of β. The elements of S are often called *points* and the elements of β are called *blocks*. If $t=2$, the elements of β are sometimes called *lines*. If $t=2$, and $x,y \in S, x \neq y$, we denote the unique line through x and y by xy. We write $S(t,k,v)$ instead of $S(t,\{k\},v)$. An $S(t,k,v)$ is called a *Steiner system*. There is an obvious one-to-one correspondence between $S(t,t+1,v)$ and totally symmetric $OD(t,t+1,v)$. If (S,β) is an $S(t,K,v)$ and if for every $B \in \beta$ an $OD(t,k,|B|)q_B$ on B is given, then it is easy to check that $\bigcup_{B\in\beta} q_B$ is an $OD(t,k,v)$. Similar constructions can be given for a wide variety of combinatorial structures. We call a construction of this type a t-wise balanced design construction. As 2-wise balanced designs are often called pairwise balanced designs or PBD's, the special case $t=2$ of the construction is called a PBD-construction. The following type of construction is very well known. Again, we state it for $OD(2,k,v)$, but it has analogues for a wide variety of structures «with $t=2$».

Construction 1. Let (S,β) be an $S(2,K,v)$, $|S|>1$. Let X and Y be sets, $X \cap (S \times Y) = \emptyset$, $|X|=w$, $|Y|=u$. If q_X is an $OD(2,k,w)$ on X, if, for each $s \in S, q_s$ is an $OD(2,k,w+u)$ on $X \cup (\{s\} \times Y)$ containing (X,q_X) as a subspace and if for each $B \in \beta$, there is an $OD(2,k,w+u|B|)q_B$ on $X \cup (B \times Y)$ containing $(X \cup (\{s\} \times Y), q_s)$ as a subspace for all $s \in B$, then $q = \bigcup_{B\in\beta} q_B$ is an $OD(2,k,w+uv)$. We have $H(q) \supset \bigcap_{B\in\beta} H(q_B)$. The sets $X, X \cup (\{s\} \times Y)$, $s \in S$, and $X \cup (B \times Y)$, $B \in \beta$, are subspaces of q.

Choosing, in Construction 1, $X=\emptyset$ and $|Y|=1$, yields the PBD-construction described above. Construction 1 has a generalization using ordered designs with holes, which results in $OD(2,k,w+uv)$ in which not all of the sets listed in the last sentence of the construction wind up as subspaces. We will describe this generalization, in the more general context of closure spaces, in section 5. Using closure spaces, Construction 1 can be generalized to arbitrary t, but the resulting construction technique is not exactly easy to use for $t \geq 3$. However, in [17] 2-wise balanced design type constructions and analogues of Construction 1 are used for $OD(3,4,v)$. Potentially, these constructions could also be used to construct $OA(3,4,v)$ with all kinds of prescribed properties. The key ingredient for these constructions is the following proposition and special cases of it.

PROPOSITION1. *[17, Construction 3]: Let* (S,β) *be an* $S(2,K,v)$, $K \cap \{0,1,2\} = \emptyset$, $\beta \neq \emptyset$. *Let* Y *be a* u*-set. Let* $\delta = \{B \times Y; B \in \beta\}$. *Then there is a* $\delta - OA(3,4,S \times Y)q$ *with (13) (24)*$\in H(q)$. *If either* k *is odd for all* $k \in K$ *or* u *is even (or both), then we can choose* q *such that* $H(q) \supset D_4$, *where* $D_4 = \{\sigma \in S_4; \sigma(\{\{1,2\},\{3,4\}\}) = \{\{1,2\},\{3,4\}\}\}$. *(It is easy to see that the group* D_4*, as defined here, is conjugate to the dihedral group on 4 elements. Note that the complement of the graph* $\{\{1,2\},\{3,4\}\}$ *on* $\{1,2,3,4\}$ *is a 4-cycle).*

To make this paper more self-contained, we include a proof of those cases of Proposition 1 that will actually be needed in concrete applications in sections 3 and 4. For each $B \in \beta$, put an idempotent quasigroup (B, o_B) on B. As we are assuming $2 \notin K$, such an idempotent quasigroup exists. If k is odd for all $k \in K$, then we can choose all (B, o_B) to be commutative. Choose $Y = \mathbb{Z}_u$. Put $q = \{((x_1,i_1),(x_2,i_2),(x_3,i_3),(x_4,i_4)) \in (S \times \mathbb{Z}_u)^4;$ $|\{x_1,x_2,x_3,x_4\}| = 4$, $|B \cap \{x_1,x_2,x_3,x_4\}| \leq 2$ for all $B \in \beta$, $i_1+i_2 = i_3+i_4$ and $\{x_1 o_{x_1x_2} x_2\} = \{x_3 o_{x_3x_4} x_4\} = x_1x_2 \cap x_3x_4\}$. It is easy to check taht q is a $\delta - OA(3,4,S \times \mathbb{Z}_u)$ with (13) (24)$\in H(q)$ and that, if all (B, o_B) are commutative, then $D_4 \subset H(q)$. The claim about even u can be proved in a similar way, using commutative quasigroups with holes. As, in this paper, we will not use this claim in concrete applications, we refer to [17] for the details.

Proposition 1 makes the following adaptation of Construction 1 easy to use for $OD(3,4,v)$.

Construction 2. Let (S,β) be an $S(2,K,v)$, $|S|>1$, $K\cap\{0,1\}=\emptyset$. Let X and Y be sets, $X\cap(S\times Y)=\emptyset$, $|X|=w$, $|Y|=u$. Put $\delta=\{B\times Y; B\in\beta\}$. If q_0 is a $\delta-OA(3,4,S\times Y)$, if q_X is an $OD(3,4,w)$ on X, if, for each $s\in S, q_s$ is an $OD(3,4,w+u)$ on $X\cup(\{s\}\times Y)$ containing (X,q_X) as a subspace and if, for each $B\in\beta$, there is an $OD(3,4,w+u|B|)q_B$ on $X\cup(B\times Y)$ containing $(X\cup(\{s\}\times Y),q_s)$ as a subspace for all $s\in B$, then $q=\left(\bigcup_{B\in\beta} q_B\right)\cup q_o$ is an $OD(3,4,w+uv)$. We have $H(q)\supset\left(\bigcap_{B\in\beta} H(q_B)\right)\cap H(q_0)$. The sets $X, X\cup(\{s\}\times Y), s\in S$, and $X\cup(B\times Y), B\in\beta$, are subspaces of q.

Checking the claims in Construction 2 is straightforward. Construction 2 has an obvious generalization to arbitrary k. Also, there are generalizations of Construction 2 in which not all of the listed sets wind up as subspaces. However, we are not trying to maximize generality here, but will defer this to section 5. Applications of Construction 2 are given in [17]. The relevance of Proposition 1 is that if $2\notin K$ and if we are not interested in $H(q)$ or if we only want $H(q)$ to contain (13) (24) or a conjugate permutation, then the q_0 required in Construction 2 always exists. Proposition 1 also makes Construction 2 very powerful as long as one only wants $H(q)$ to contain some group G conjugate to a subgroup of D_4. Even for groups G not conjugate to a subgroup of D_4 and in fact even for $G=S_4$, a q_0 satisfying the properties required in Construction 2 and $H(q_0)\supset G$ can be found for plenty of choices of the $S(2,K,v)$ (S,β) and the cardinality u of Y. Nevertheless, the lack of a very general theorem seems to make Construction 2 less easy to apply for such groups. To get some feeling for the problem of constructing the required q_0 for these groups, it may be useful to look at the special case $G=S_4$ and $|Y|=1$. If (S,β) is an $S(2,K,v)$, $v>1$, then a totally symmetric $\beta-OA(3,4,v)$ on S is easily seen to be equivalent with an $S(3,\{k+1;k\in K\}\cup\{4\},v+1)(S\cup\{\infty\},\beta')$, $\infty\notin S$, such that the elements of β' containing ∞ are exactly the sets $B \cup \{\infty\}$, $B \in \beta$, and such that any block of β' not containing ∞ has exactly 4 elements. It is conjectured that if (S, β) is an $S(2,3,v)$, then such a β' always exists. This conjecture is known to be true for all $v \leq 15$ [3], but finding a general proof seems to be exceedingly difficult. (We refer to [9] for a survey of results on this problem).

Of course, if we want to use Construction 2 to construct totally symmetric $OD(3,4,v)$, or equivalently $S(3,4,v)$, then the special case $K = \{3\}$, $|Y| = 1$ and $|X| = 1$ of the construction becomes trivial.

3. Large sets.

We denote by DS^k the set of all k-tuples of distinct elements of S. A *large set of disjoint* $OA(t,k,v)$, $OD(t,k,v)$ or $S(t,k,v)$, briefly $LA(t,k,v)$, $LD(t,k,v)$ or $LS(t,k,v)$, is a partition of S^k, DS^k or $P_k(S)$ into $OA(t,k,v)$, $OD(t,k,v)$ or $S(t,k,v)$. When we say that a collection $(A_r)_{r \in R}$ is a partition of a set X, we mean that $\{A_r; r \in R\}$ is a partition of X and that $A_{r_1} \neq A_{r_2}$ for $r_1 \neq r_2$. In particular, when we denote a large set by a collection $(q_r)_{r \in R}$, we are assuming that $q_{r_1} \neq q_{r_2}$ for $r_1 \neq r_2$. When denoting the large set by a set $\{q_r; r \in R\}$ we are not making this assumptions.

For a survey of known results about $S(t,k,v)$, we refer to [1]. For a review of results about $OA(t,k,v)$ we refer to [1] for $t = 2$ and to [10] for $t > 2$. An $LA(t,k,v)$ exists iff an $OA(t,k,v)$ exists [15]. For a survey of results about $LS(t,k,v)$, $OD(t,k,v)$ and $LD(t,k,v)$ we refer to [15]. Moreover, in [17] some additional results about $OD(3,4,v)$ are proved.

If δ is a set of subsets of a set S, then a large set of disjoint $\delta - OA(t,k,S)$, or briefly $L\delta - OA(t,k,S)$, is a partition of the set of all $(x_1, \ldots, x_k) \in S^k$ satisfying $|\{i \in \{1, \ldots, k\}; x_i \in D\}| < t$ for all $D \in \delta$, into $\delta - OA(t,k,S)$. If $t \geq 2$ and $v \geq t-1$, an $LD(t,k,v)$ on a set S is the same thing as an $LP_{t-1}(S) - OA(t,k,S)$.

For $LD(2,3,v)$, a naive application of PBD-type constructions or analogues of Construction 1 does not work. However, Proposition 1 will help us out again. We first need some further observations and definitions.

An $LA(t,k,v)$ has v^{k-t} elements. A *subspace* of an $LA(t,k,v)$ $(q_r)_{r\in R}$ on a v-set S is a subset D of S for which there is an $R_0 \subset R$ such that D is a subspace of all q_r with $r \in R_0$ and such that $(q_r|D)_{r\in R_0}$ is an $LA(t,k,|D|)$ on D. We will call a collection $(A_r)_{r\in R}$ of subsets of a set X a *quasipartition* of X if $\bigcup_{r\in R} A_r = X$ and $A_{r_1} \cap A_{r_2} = \emptyset$ whenever $r_1 \neq r_2$. Thus, $(A_r)_{r\in R}$ is a quasipartition of X iff $(A_r)_{r\in R_0}$ is a partition of X, where $R_0 = \{r \in R; A_r \neq \emptyset\}$. If S is a set and $\delta \subset P(S)$, a $\delta - LA(t,k,S)$ will be a collection $(q_r, \delta_r)_{r\in R}$, where, for each $r \in R$, $\delta_r \subset \delta$ and q_r is a $\delta_r - OA(t,k,S)$ such that $(q_r)_{r\in R}$ is a quasipartition of the set $\{(x_1,\dots,x_k) \in S^k$; there is no $D \in \delta$ with $\{x_1,\dots,x_k\} \subset D\}$ and such that, for each $D \in \delta$, there are exactly $|D|^{k-t}$ elements r of R with $D \in \delta_r$. Note that if $(q_r,\delta_r)_{r\in R}$ is a $\delta - LA$ (t,k,S), then we must have $\delta = \bigcup_{r\in R} \delta_r$ or $\delta = \left(\bigcup_{r\in R} \delta_r\right) \cup \{\emptyset\}$ and that a $\delta - LA(t,k,S)$ is exactly the same thing as a $(\delta \cup \{\emptyset\}) - LA(t,k,S)$. (We are assuming $k \geq t+1$). We call a collection $(q_r)_{r\in R}$ of *k-S-arrays* a $(\delta_r)_{r\in R} - LA(t,k,S)$ if $(q_r,\delta_r)_{r\in R}$ is a $\delta - LA(t,k,S)$, where $\delta = \bigcup_{r\in R} \delta_r$. If q is a $\delta - OA(t+1,t+2,S)$, then we can define a $\delta - LA(t,t+1,S)(q_x,\delta_x)_{x\in S}$, where $q_x = \{(x_1,\dots,x_{t+1}) \in S^{t+1}; (x_1,\dots,x_{t+1},x) \in q\}$ and $\delta_x = \{D \in \delta; x \in D\}$. For all $x \in S, H(q_x) \supset (H(q))_{t+2}$, where $(H(q))_{t+2}$ denotes the stabilizer of $t+2$ in $H(q)$. (We consider $(H(q))_{t+2}$ as a permutation group on $\{1,\dots,t+1\}$.) Note that $(12) \in (D_4)_4$. The following proposition is an immediate corollary of Proposition 1 and the preceding observations.

PROPOSITION 2. *Let (S,β) be an $S(2,K,v)$, $K \cap \{0,1,2\} = \emptyset$, $\beta \neq \emptyset$. Let Y be a u-set. For each $(x,y) \in S \times Y$, put $\delta_{(x,y)} = \{B \times Y; x \in B \in \beta\}$. (Of course, $\delta_{(x,y)}$ is idependent of y). Then there is a $(\delta_{(x,y)})_{(x,y)\in S\times Y} - LA(2,3,S\times Y)(q_{(x,y)})_{(x,y)\in S\times Y}$. If either k is odd for all $k \in K$ or u is even (or both), then we can choose all $q_{(x,y)}$ to be commutative.*

Proposition 2 could potentially be used to construct $LA(2,3,v)$ with all kinds of prescribed properties. In the following, we will only emphasize applications to the construction of $LD(2,3,v)$.

The following construction provides a PBD-type construction for $LD(2,3,v)$.

Construction 3. Let (S,β) be an $S(2,K,v), K \cap \{0,1,2\} = \emptyset$, $\beta \neq \emptyset$. Let $\infty_1 \neq \infty_2$, $\{\infty_1,\infty_2\} \cap S = \emptyset$. Assume that for each $B \in \beta$, there is an $LD(2,3,|B|+2)(q_{(x,B)})_{x\in B}$ on $B \cup \{\infty_1,\infty_2\}$ such that, for all

$$x \in B, \{(\infty_1,\infty_2,x),(\infty_1,x,\infty_2),(x,\infty_1,\infty_2),$$

$$(\infty_2,\infty_1,x),(\infty_2,x,\infty_1),(x,\infty_2,\infty_1)\} \subset q_{(x,B)}.$$

Let $(q_{ox})_{x\in S}$ be a $(\beta_x)_{x\in S} - LA(2,3,S)$, where $\beta_x = \{B \in \beta; x \in B\}$. Put $q_x = \left(\bigcup_{B\in\beta_x} q_{(x,B)}\right) \cup q_{0x}$. Then $(q_x)_{x\in S}$ is an $LD(2,3,v+2)$ on $S \cup \{\infty_1,\infty_2\}$ and, for each

$$x \in S, H(q_x) \supset \left(\bigcap_{B\in\beta_x} H(q_{(x,B)})\right) \cap H(q_{ox}).$$

In [5] it is proved that $LD(2,3,v)$ exist for all $v \in \mathbb{N}$, $v \geq 3, v \notin \{6,14,62\}$. No $LD(2,3,6)$ exists. The existence problem for $LD(2,3,14)$ and $LD(2,3,62)$ remains open. Construction 3 provides a construction for $LD(2,3,v)$ not contained in [5], but fails to take care of 14 and 62. Generalizations of Construction 3, of the type we will consider in section 4, do not yield 14 or 62 either.

Robinson [12] asked for which $v \in \mathbb{N}$ there is a collection of $v-2$ symmetric idempotent latin squares of order v such that no two squares in the collection agree in an off-diagonal position. Such a collection is called a *golf design for v clubs*. It is easy to see that a golf design for v clubs is equivalent to a commutative $LD(2,3,v)$. (We call a collection of arrays commutative or totally symmetric if all its elements are commutative or totally symmetric). Totally symmetric $LD(2,3,v)$ are equivalent with $LS(2,3,v)$. A necessary and sufficient condition for the existence of an $S(2,3,v), v \in \mathbb{N}$, is $v \equiv 1$ or $3 \pmod 6$ or $v = 0$ [4]. An $LS(2,3,7)$ does not exist [2]. Lu constructed $LS(2,3,v)$

for all $v \equiv 1$ or $3 \pmod 6$, $v > 7$ [6,7,8]. However, as reference [8] is unfinished, due to the death of the author, part of the proof for the cases $v \in \{141, 283, 501, 789, 1501, 2365\}$ is missing. No commutative $LD(2,3,5)$ exists [12]. In [12], a commutative $LD(2,3,7)$ is given and in [18] a commutative $LD(2,3,17)$ is constructed. However, the existence problem for commutative $LD(2,3,v)$ remained open for all $v \equiv 5 \pmod 6$, $v \notin \{5, 17\}$.

Next, we will give a short proof of the fact that commutative $LD(2,3,v)$ exist for all sufficiently large odd v. In section 4, we will show that if a commutative $LD(2,3,11)$ would exist, then a commutative $LD(2,3,v)$ would exist for all $v \equiv 5 \pmod 6$, except 5 and possibly 41. The existence of a commutative $LD(2,3,11)$ is still in doubt, however.

If $(q_r)_{r\in R}$ is a totally symmetric $LD(2,3,v+2)$ on a set $B \cup \{\infty_1, \infty_2\}$, $\{\infty_1, \infty_2\} \cap B = \emptyset$, $\infty_1 \neq \infty_2$, then, for each $r \in R$, we can change the index r to x, where $(\infty_1, \infty_2, x) \in q_r$. Thus, we obtain a collection $(q_x)_{x\in B}$ of the type required in Construction 3. In particular, a commutative $LD(2,3,2+7)$ and a commutative $LD(2,3,2+11)$ of the type required in Construction 3 both exist. By a famous result of Wilson [19], there is a constant v_0 such that an $S(2,\{7,11\},v)(S,\beta)$ exists for all odd $v \geq v_0$. By Proposition 2, the commutative $(\beta_x)_{x\in S} - LA(2,3,S)(q_{ox})_{x\in S}$ required in Construction 3 exists. Construction 3 then guarantees the existence of a commutative $LD(2,3,v+2)$. Thus, there is an integer v_1 such that commutative $LD(2,3,v)$ exist for all odd $v \geq v_1$.

We close this section by defining and discussing the analogues for $LD(t,k,v)$ of some of the notions defined above for $LA(t,k,v)$. An $LD(t,k,v)$ with $v \geq t$ has $(v-t)(v-t-1)\dots(v-k+1)$ elements. A *subspace* of an $LD(t,k,v)(q_r)_{r\in R}$ on a v-set S is a subset D of S for which there is an $R_0 \subset R$ such that D is a subspace of all q_r with $r \in R_0$ and such that $(q_r|D)_{r\in R_0}$ is an $LD(t,k,|D|)$ on D. Note that the definition of $LD(t,k,v)$ implies that the empty collection is an $LD(t,k,v)$ for all $v \leq k-1$. (Indeed, if $|S| \leq k-1$, then DS^k is empty and the empty collection partitions it). Thus, if $(q_r)_{r\in R}$ is an $LD(t,k,v)$ on a set S, then any subset D of S with $|D| \leq k-1$ is a subspace of

$(q_r)_{r\in R}$. (We only have to choose, in the definition of subspace, $R_0 = \emptyset$). If $t_1, t_2 \in \mathbb{N}$ and S is a set, we put $P_{t_1,t_2}(S) = \{A \subset S; t_1 \leq |A| \leq t_2\}$. A $\delta - OD(t,k,S)$ will be a $(\delta \cup P_{0,t-1}(S)) - OA(t,k,S)q$ such that $q \subset DS^k$. (If $t \geq 2$, every $(\delta \cup P_{0,t-1}(S)) - OA(t,k,S)q$ satisfies $q \subset DS^k$, so that this condition is only relevant for $t = 0$ or $t = 1$). If S is a set and $\delta \subset P(S)$, then a $\delta - LD(t,k,S)$ will be a collection $(q_r, \delta_r)_{r\in R}$ where, for each $r \in R$, $\delta_r \subset \delta$ and q_r is a $\delta_r - OD(t,k,S)$ such that $(q_r)_{r\in R}$ is a quasipartition of the set $\{(x_1,\ldots,x_k) \in DS^k;$ there is no $D \in \delta$ with $\{x_1,\ldots,x_k\} \subset D\}$ and such that, for each $D \in \delta$ with $|D| \geq t$, there are exactly $(|D|-t)(|D|-t-1)\ldots(|D|-k+1)$ elements r of R with $D \in \delta_r$. If $(q_r,\delta_r)_{r\in R}$ is a $\delta - LD(t,k,S)$ then $\bigcup_{r\in R} \delta_r \subset \delta \subset \left(\bigcup_{r\in R}\delta_r\right) \cup P_{0,k-1}(S)$. Moreover, elements of $P_{o,k-1}(S)$ do not play any really essential role in the definition of $\delta - LD(t,k,v)$. We call a collection $(q_r)_{r\in R}$ of $k-S$-*arrays* a $(\delta_r)_{r\in R} - LD(t,k,S)$ if $(q_r,\delta_r)_{r\in R}$ is a $\delta - LD(t,k,S)$ where $\delta = \bigcup_{r\in R}\delta_r$. Obviously, if, for each $r \in R, A_r \subset P_{o,t-1}(S)$, then a $(\delta_r)_{r\in R} - LD(t,k,S)$ is the same thing as a $(\delta_r \cup A_r)_{r\in R} - LD(t,k,S)$.

4. Some further constructions for commutative $LD(2,3,v)$.

A construction of Rosa [13] produces an $LS(2,3,2v+1)$ from an $LS(2,3,v)$ with $v \geq 7$. This construction can be recopied almost literally, replacing unordered triples by ordered triples, for commutative $LD(2,3,v)$. The only change is that, for $v \equiv 5 \pmod 6$, the $S(3,4,v{+}1)(N',B)$ used in [13] has to be replaced by an $OD(3,4,v{+}1)q$ on N' with $(12) \in H(q)$ and the $S(2,3,v)(N'-\{i\},B_i)$ used in [13] have to be replaced by the commutative $OD(2,3,v)q_i = \{(i_1,i_2,i_3); (i_1,i_2,i_3,i) \in q\}$. An $OD(3,4,v)$ with $(12) \in H(q)$ and, in fact, an $OD(3,4,v)$ with $D_4 \subset H(q)$ exists for all even v [17]. Thus, the existence of a commutative $LD(2,3,v), v \geq 7$, implies the existence of a commutative $LD(2,3,2v+1)$.

Next, we construct a totally symmetric $(\delta_{(x,y)})_{(x,y)\in\mathbb{Z}_3^2} - LD(2,3,\mathbb{Z}_3^2 \cup \{\infty_1,\infty_2\})$, where $\infty_1 \neq \infty_2, \{\infty_1,\infty_2\} \cap \mathbb{Z}_3^2 = \emptyset$ and $\delta_{(x,y)} = \{\{\infty_1,\infty_2,$

$(x,0),(x,1),(x,2)\}\}$. For each $(x,y) \in \mathbb{Z}_3^2$, put $A_{(x,y)} = \{\{(0,i),(1,i+y),(2,i+2y+x)\};i \in \mathbb{Z}_3\}$. Put

$$B_{(x,y)} = \{\{(i,j_1),(i,j_2),(x,j_4)\}; i \in \mathbb{Z}_3 - \{x\}, j_1 \neq j_2, \{(i,j_3),(x,j_4)\}$$

is contained in an element of $A_{(x,y)}$, where $\{j_3\} = \mathbb{Z}_3 - \{j_1,j_2\}\}$. Put

$$C_{(x,y)} = \{\{\infty_l,(x+1,j_1),(x+2,j_2+l)\}; l \in \{1,2\}, \{(x+1,j_1),(x+2,j_2)\}$$

is contained in an element of $A_{(x,y)}\}$. Put $D_{(x,y)} = A_{(x,y)} \cup B_{(x,y)} \cup C_{(x,y)}$ and

$$q_{(x,y)} = \{((i_1,j_1),(i_2,j_2),(i_3,j_3)); \{(i_1,j_1),(i_2,j_2),(i_3,j_3)\} \in D_{(x,y)}\}.$$

Then $(q_{(x,y)})_{(x,y)\in\mathbb{Z}_3^2}$ is a totally symmetric $(\delta_{(x,y)})_{(x,y)\in\mathbb{Z}_3^2} - LD(2,3,\mathbb{Z}_3^2 \cup \{\infty_1,\infty_2\})$.

A *group divisible design with block size 3* on a set S will be a triple (S,G,β), where the elements of S are called *points*, the elements of G *groups* and the elements of β *blocks*, such that G is a partition of S, such that $|B| = 3$ for all $B \in \beta$, such that $|B \cap A| \leq 1$ for all $B \in \beta$ and $A \in G$ and such that any two points not contained in a common group are contained in exactly one block.

Construction 4. Let (S,G,β) be a group divisible design with block size 3, S a v-set, and let $\infty_1 \neq \infty_2$, $(S \times \mathbb{Z}_3) \cap \{\infty_1,\infty_2\} = \emptyset$. Assume that, for each $A \in G$ there is an $LD(2,3,3|A|+2)(q_{(x,y)A})_{(x,y)\in A\times\mathbb{Z}_3}$ on $(A \times \mathbb{Z}_3) \cup \{\infty_1,\infty_2\}$. For each $B \in \beta$, let $(q_{(x,y)B})_{(x,y)\in B\times\mathbb{Z}_3}$ be a $(\gamma_{(x,y)})_{(x,y)\in B\times\mathbb{Z}_3} - LD(2,3,(B\times\mathbb{Z}_3)\cup\{\infty_1,\infty_2\})$ where $\gamma_{(x,y)} = (\{x\}\times\mathbb{Z}_3)\cup\{\infty_1,\infty_2\}$. Let $(q_{(x,y)o})_{(x,y)\in S\times\mathbb{Z}_3}$ be a $(\delta_{(x,y)})_{(x,y)\in S\times\mathbb{Z}_3} - LA(2,3,S\times\mathbb{Z}_3)$, where $\delta_{(x,y)} = \{B \times \mathbb{Z}_3; x \in B \in G \cup \beta\}$. For each $(x,y) \in S \times \mathbb{Z}_3$, put

$$q_{(x,y)} = \left(\bigcup_{x\in B\in\beta\cup G} q_{(x,y)B} \right) \cup (q_{(x,y)o}).$$

Then $(q_{(x,y)})_{(x,y)\in S\times\mathbb{Z}_3}$ is an $LD(2,3,3v+2)$ and, for each

$$(x,y) \in S \times \mathbb{Z}_3, H(q_{(x,y)}) \supset \left(\bigcap_{x\in B\in\beta\cup G} H(q_{(x,y)B}) \right) \cap H(q_{(x,y)o}).$$

Construction 4 has the following corollary.

PROPOSITION 3. *If there is a group divisible design* (S, G, β) *with block size 3 such that there is a commutative* $LD(2,3,3|A|+2)$ *for each* $A \in G$*, then there is a commutative* $LD(2,3,3|S|+2)$*. (Note that the existence of a commutative* $LD(2,3,3|A|+2)$ *implies that* $|A|$ *is odd).*

In [11], a $URD(p,k), p, k \in \mathbb{N}$ is defined as an $S(2,\{2,3\},p)(S,\beta)$ provided with a partition $(\beta_r)_{r \in R}$ of β, $|R| = k$, such that $2k-p+1$ of the β_r are partitions of S into 2-subsets and $p-1-k$ of them are partitions of S into 3-subsets. From a $URD(p,k)$ with $p-1-k \geq 1$, we can construct a group divisible design with block size 3 on $2k+1$ points having a group of size $2k-p+1$ and in which all remaining groups have 3 points. Indeed, let R_1 be the set of all $r \in R$ for which β_r consists of 2-subsets. We have $|R_1| = 2k-p+1$ and it is not restrictive to assume $R_1 \cap S = \emptyset$. Let $r_0 \in R - R_1$. Our points will be the elements of $S \cup R_1$, our groups will be R_1 and the elements of β_{r_o} and our blocks will be the elements of $\bigcup_{R-(R_1 \cup \{r_o\})} \beta_r$ as well as all 3-subsets of the type $\{x, y, r\}$, $r \in R_1$, $\{x, y\} \in \beta_r$. In [11], it is proved that if $p \equiv 0 \pmod 6$ and $p/2+1 \leq k \leq p-2$, then there is a $URD(p,k)$ and thus, a group divisible design with block size 3 on $2k+1$ points having a group of size $2k-p+1$ in which all remaining groups have 3 points. Putting $u = 2k-p+1$, $v = p+u$ and retranslating everything in terms of u and v yields the following. For every odd integer $u \geq 3$ and for every $v \geq 2u+3$ with $v-u \equiv 0 \pmod 6$, there is a group divisible design with block size 3 on v points, containing a group of size u in which all remaining groups have 3 points. In particular, a group divisible design on a v-set S with blocks of size 3 in which all groups have size 3 exists for all $v \equiv 3 \pmod 6$. A group divisible design on a v-set S with blocks of size 3 containing a group of size 5 in which all remaining groups have size 3 exists for all $v \equiv 5 \pmod 6$ with $v \geq 17$. A group divisible design on a v-set S with blocks of size 3 containing exactly one group of size 7 in which all remaining groups have size 3 exists for all $v \equiv 1$ (mod

6) with $v \geq 19$. A commutative $LD(2,3,(3 \times 5)+2)$ is constructed in [18]. If a commutative $LD(2,3,11)$ would exist, then a commutative $LD(2,3,(3 \times 7)+2)$ would too, because $(3 \times 7)+2 = 23 = 2(11)+1$. By Proposition 3 this implies that if a commutative $LD(2,3,11)$ exists, then a commutative $LD(2,3,3v+2)$ would exist for all odd $v \geq 15$ as well as for $v = 9$. In other words, a commutative $LD(2,3,v)$ would exist for all $v \equiv 5 \pmod 6$ with $v \geq 3(15)+2 = 47$ as well as for $v = 29$. The cases 5, 17 and 23 have been treated before. A commutative $LD(2,3,35)$ exists, because $35 = (2 \times 17)+1$. Even assuming the existence of a commutative $LD(2,3,11)$, we do not know how to handle the case 41. To summarise, if a commutative $LD(2,3,11)$ exists, then a commutative $LD(2,3,v)$ exists for all $v \equiv 5 \pmod 6$, except 5 and maybe 41. As mentioned before, however, the existence of a commutative $LD(2,3,11)$ is still in doubt.

5. Some general remarks about closure spaces and their constructive use.

A *closure space* is a pair (S,δ), where S is a set and δ is an intersection-stable set of subsets of S. By convention, we put $\bigcap_{C \in \emptyset} C = S$, so that $S \in \delta$. The elements of δ are called *closed sets*. If (S,δ) is a closure space and $A \subset S$, we denote by $\delta(A)$ the intersection of all closed sets containing A. We call $\delta(A)$ the *closure* of A or the closed set *spanned* by A and say that A *spans* $\delta(A)$. If $\delta \neq \{S\}$, we define the *dimension* $\dim(S,\delta)$ of (S,δ) to be the smallest $n \in \mathbb{N}$ such that there is an $(n+1)$-subset A of S with $\delta(A) = S$. We put $\dim(S,\{S\}) = -1$. We call a closure space *simple* if the empty set as well as all singletons are closed. A *matroid* is a closure space (S,δ) such that for any $a, b \in S$ and any subset A of S with $a \in \delta(A \cup \{b\}) - \delta(A)$, we have $b \in \delta(A \cup \{a\})$. The *rank* of a matroid (S,δ) is defined to be $\dim(S,\delta)+1$. A *hyperplane* of a closure space (S,δ) is a maximal element of the poset $(\delta - \{S\}, \subset)$. If (S,δ) is a closure space, we denote the set of all hyperplanes of (S,δ) by δ_h. If (S,δ) is a closure space and $D \subset S$, then (D,δ_D) is a closure space, where $\delta_D = \{C \cap D; C \in \delta\}$. We often identify D with this closure space, especially if $D \in \delta$. For instance, by the dimension $\dim D$ of D, we mean $\dim(D, \delta_D)$, a hyperplane of D is a hyperplane of (D, δ_D) and so on. If $D \in \delta$, then $\delta_D = \{C \in \delta; C \subset D\}$.

In the remainder of this section, we describe in a very general, but also somewhat vague and informal way, the possible uses of closure spaces in the construction of mathematical structures. It is easy to make everything rigorous once one specifies the concrete structures one is interested in. We do not make any claim whatsoever about originality. Most discussed ideas are essentially known, in some form or another.

Suppose we are studying mathematical objects consisting of a structure of a given type, say P, defined on a set S. Quite often, there is a very natural notion of subspace for such structures. The subspaces are subsets of S and the set of all subspaces defines a closure space on S. A structure of type P on a set S will induce a structure of the same type P on any subspace, although not necessarily on any subset of S. Quite often subspaces of structures of type P can even be defined as being those subsets D of S for which the induced structure is itself a structure of type P on D. For instance, if q is an $OA(t,k,v)$ or $OD(t,k,v)$ on a set S, then a subset D of S is a subspace of q iff $q|D$ is an $OA(t,k,|D|)$ or $OD(t,k,|D|)$, respectively.

If δ is a set of subsets of a set S, one can define a structure of type P with holes in δ on S as a structure A that looks exactly like a structure obtained from a structure of type P on S, having the elements of δ as subspaces, by erasing, for each $D \in \delta$, the structure induced on D. However, we do not require that it is actually possible to complete A into a structure of type P on S. This definition is of course imprecise. The formal claim about the discussion below is that it is valid for those types P for which we will state, below, what we mean concretely by a structure of type P with holes. A more general informal claim is that, for very many types P, it becomes valid once structures of type P with holes are defined in the correct way. Very often, it is obvious how this should be done. Sometimes, as for large sets, it may be more tricky. The notions of $\delta - OA(t,k,S)$ and $\delta - OD(t,k,S)$ are the formalizations of the intuitive notions of $OA(t,k,|S|)$ and $OD(t,k,|S|)$ with holes in δ on S. For $LA(t,k,v)$ and $LD(t,k,v)$ one has to be somewhat careful, because some subtleties are involved. For the moment, let us forget about large sets and similar structures. We will come back to them later in this section.

If $S \in \delta$, then there is exactly one structure of type P with holes in δ on S, which we will call the *empty structure of type P on S*. (Depending on the formalism, this may be the empty set or the pair $(S, \emptyset)$ or something similar).

If $\delta \subset P(S)$, then $c(\delta)$ will denote the set of all intersections of families of elements of δ. Obviously, $(S, c(\delta))$ is a closure space. If $\delta \subset P(S)$, $S \notin \delta$, then there is no difference between a structure of type P with holes in δ and a structure of type P with holes in $c(\delta)_h$ or with holes in $c(\delta) - \{S\}$. Note that, if $S \notin \delta$, then $c(\delta)_h$ consists of all elements of δ that are maximal for inclusion. Thus, when talking about structures of type P with holes in δ on a set S, it is not restrictive to assume that $\delta = \gamma_h$ or that $\delta = \gamma - \{S\}$, where (S,γ) is a closure space.

The *degree* of a given type P will be the smallest integer n such that, for any closure space (S,δ) with $\dim(S,\delta) \geq n$, there is exactly one structure of type P with holes in δ_h, namely the empty structure of type P on S. If no such n exists, then we say that the type P has infinite degree. For instance, if P consists of the class of all $OA(t_0, k_0, v)$, where t_0 and k_0 are fixed, but v is not, then P has degree t_0. The same holds for the class of all $OD(t_0, k_0, v)$.

Assume that we are given a closure space (S,δ) and want to construct a structure of type P on S such that all elements of δ are subspaces of this structure. This problem splits completely into many subproblems that can be solved independently. These problems are to find, for each $C \in \delta$, a structure of type P having the hyperplanes of C as holes. If P has finite degree t, then this problem is trivial for those $C \in \delta$ with $\dim C \geq t$, so that we only have to solve it for the $C \in \delta$ with $\dim C < t$. We will refer to this use of closure spaces, where every closed set of the closure space we started from becomes a subspace of the constructed structure, as a *strict use of closure spaces*.

One can use closure spaces in a more general way, which we will refer to in the sequel as an *opportunistic use*. Let (S, δ) be a closure space. Let $(\delta_1, \ldots, \delta_n)$ be an n-tuple of pairwise disjoint subsets of δ, such that $\delta_n = \{S\}$, such that any two elements of δ_1 are disjoint and such that, for any $i \in \{2, \ldots, n-1\}$ and for any two distinct elements C_1 and C_2 of δ_i, the set $C_1 \cap C_2$ is contained in an element of $\delta_1 \cup \ldots \cup \delta_{i-1}$. If we can put a structure of type P on each of the elements of δ_1 and put on each $C \in \delta_i$, $i \in \{2, \ldots, n\}$, a structure of type P with holes in $\{C \cap A; A \in \delta_1 \cup \ldots \cup \delta_{i-1}\}$, then putting all these structures together will produce a structure of type P on S. The elements of δ_1 will be subspaces, but this will not necessarily be the case for all elements of δ. If P has finite degree t and $\dim(S, \delta) \geq t$, then, in many applications, we will have $\dim(S, c(\delta_1 \cup \ldots \cup \delta_{n-1})) \geq t$, in which case one of the problems we have to solve, namely finding a stucture of type P with holes in $\delta_1 \cup \ldots \cup \delta_{n-1}$ on S, becomes trivial.

One of the advantages of using closure spaces is that one can construct a limited class of structures of type P with holes and then construct closure spaces for which all the structures needed for a strict or opportunistic use belong to that class.

The reason why we did not try to maximize generality in Constructions 1, 2, 3 and 4 is that any such a type of construction can be completely described by specifying the structures we want to construct, the kind of closure spaces we want to use and whether we want to use these closure spaces in a strict or in an opportunistic way.

Let us call a closure space *1-equicardinal* if all sets $\delta(\{x\})$, $x \in S - \delta(\emptyset)$, have the same cardinality. Obviously, every simple closure space is 1-equicardinal. For $OD(2, k, v)$ a strict use of 1-equicardinal matroids of rank 3 corresponds to applying Construction 1. An opportunistic use corresponds to the technical generalization of Construction 1 referred to in Section 2. Rank 3 matroids which are not 1-equicardinal have also been used for several combinatorial structures, both in a strict and an opportunistic way. If (S, β) is an $S(t,K,v)$, $|\beta| > 1$, $k \geq t$ for all $k \in K$, then the closure space $(S,\delta_\beta) = \{S\} \cup \{A \subset S; |A| \leq t-1\} \cup \beta$ is a matroid of dimension t or, equivalently, rank $t+1$. We call (S,δ_β) the *matroid associated with* (S,β). Every simple rank 3 matroid is associated with an $S(2,K,v)$. The t-wise balanced design construction described in section 2 corresponds to a strict use of the matroid associated with the $S(t,K,v)$. Construction 2 corresponds to a strict use of 1-equicardinal rank 3 matroids for $OD(3,4,v)$. An opportunistic use yields the technical generalization of Construction 2 referred to in section 2.

For large sets and other collections of individual structures, it would seem that we have to adapt the above discussion, replacing single closure spaces by collections of closure spaces. However, the notions and constructions discussed in this section seem to be of such a general validity that everything should work in a completely literal way for structures of any reasonable type. The reason why they do not work out literally for large sets and other collections of structures on a set S is that, for collections of structures, the essence of the notion of subspace can quite often not be completely captured by considering subspaces to be subsets of S. Our feeling is that, for the kind of purposes described in this section, collections of structures on a set S should be considered to be defined on $S \cup R$, where R is the indexing set.

A $\delta - LA(t,k,S,R)$, $R \cap S = \emptyset$, $\delta \subset P(S \cup R)$, will be a collection q of $(k+1)$-tuples $(x_1,\ldots,x_k,r)$, $x_1,\ldots,x_k \in S$, $r \in R$, such that:

(i) for any t-subset $\{i_1,\ldots,i_t\}$ of $\{1,\ldots,k\}$ and for any t-tuple $x_{i_1},\ldots,x_{i_t}$ of (not necessarily distinct) elements of S and any $r \in R$, the number of elements $(y_1,\ldots,y_k,r)$ of q with $y_{i_1} = x_{i_1},\ldots,y_{i_t} = x_{i_t}$ equals 0 or 1 depending on whether $\{x_{i_1},\ldots,x_{i_t},r\}$ is contained in an element of δ or not.

(ii) for any k-tuple $x_1,\ldots,x_k$ of (not necessarily distinct) elements of S, there are either 0 or 1 elements r of R with $(x_1,\ldots,x_k,r) \in q$ depending on whether $\{x_1,\ldots,x_k\}$ is contained in an element of δ or not.

Then, one can define $LA(t,k,S,R)$ as $\emptyset - LA(t,k,S,R)$. If S and

R are sets, put

$$\delta(t,S,R) = \{A \subset S \cup R; |A \cap S| \leq t-1, |A \cap R| = 1\}.$$

A $\delta - LD(t,k,S,R)$ will be a

$$(\delta \cup P_{0,k-1}(S) \cup \delta(t,S,R)) - LA(t,k,S,R).$$

An $LD(t,k,S,R)$ will be a $\emptyset - LD(t,k,S,R)$. A *subspace* of an LA (or LD) $(t,k,S,R)q$ is defined to be a subset D of $S \cup R$ such that $q|D$ is an LA (or LD)$(t,k,S \cap D, R \cap D)$.

Obviously, an $LA(t,k,S,R)$ or $LD(t,k,S,R)$ is completely equivalent to an $LA(t,k,|S|)$ or $LD(t,k,|S|)$ on S with index set R. If we consider $\delta - LA(D)(t,k,S,R)$ as $LA(D)(t,k,S,R)$ with holes in δ, then our entire discussion about structures of type P with holes applies literally. Thus, it also applies literally to $LA(D)(t,k,v)$, as long as we are willing to retranslate everything in terms of $LA(D)(t,k,S,R)$. For fixed t_o and k_0, $LA(t_o,k_o,S,R)$ and $LD(t_o,k_o,S,R)$ are structures of degree k_o. (Remember that we are assuming $k \geq t+1$). In the above, the set $S \cup R$ has a preassumed structure, namely the ordered partition (S,R), which induces a preassumed structure on all subsets X of S, namely the quasipartition $(X \cap S, X \cap R)$. This represents no problem, because our discussion about closure spaces and holes did by no means preclude the existence of such a preassumed structure.

How do the notions of $\delta - LA(t,k,S,R)$ and $\delta - LD(t,k,S,R)$ relate to the notions of $\delta - LA(t,k,S)$ and $\delta - LD(t,k,S)$ defined earlier? Let us call a $\delta - LA(t,k,S,R)$ *strict* if $|D \cap R| = |D \cap S|^{k-t}$ for all $D \in \delta$. Similarly, we call a $\delta - LD(t,k,S,R)$ *strict* if $|D \cap R| = (|D \cap S| - t)(|D \cap S| - t - 1)\ldots(|D \cap S| - k + 1)$ for all $D \in \delta$ with $|D \cap S| \geq t$. Strict uses of closure spaces in the construction of $LA(t,k,v)$ or $LD(t,k,v)$ obviously require strict $\delta - LA(t,k,S,R)$ or strict $\delta - LD(t,k,S,R)$, but this is not necessarily the case for opportunistic uses. A strict $\delta - LA(D)(t,k,S,R)$ is easily seen to be equivalent with a $(\delta_r)_{r \in R} - LA(D)(t,k,S)$, where $\delta_r = \{D \cap S; D \in \delta, r \in D\}$.

Let $\delta \subset P(S)$ and put $\delta' = \{D \times \{0,1\}; D \in \delta\}$. Then a $\delta' - LA(t,t+1,S \times \{0\}, S \times \{1\})$ is equivalent to a $(\delta_x)_{x \in S} - LA(t,t+1,S)$

where $\delta_x = \{D \in \delta; x \in D\}$. Obviously, it is also equivalent to a $\delta - OA(t+1, t+2, S)$. We already used these equivalences in section 3 to obtain Proposition 2 from Proposition 1. Let T be a t-subset disjoint from S and from $S \times \{0,1\}$, and put $\delta_T = \{D \cup T; D \in \delta\}$ and $\delta'_T = \{(D \times \{0,1\}) \cup T; D \in \delta\}$. Then a $\delta'_T - LD(t, t+1, (S \times \{0\}) \cup T, S \times \{1\})$ is equivalent to a $(\delta_x)_{x \in S} - LD(t, t+1, S \cup T)$, where $\delta_x = \{D \in \delta_T; x \in D\}$. If $\dim(S, c(\delta)) \geq t$, then a $\delta'_T - LD(t, t+1, (S \times \{0\}) \cup T, S \times \{1\})$ is also equivalent to a $\delta' - LA(t, t+1, S \times \{0\}, S \times \{1\})$ and thus, to a $\delta - OA(t+1, t+2, S)$ and a $(\delta_x)_{x \in S} - LA(t, t+1, S)$. This is why, in Constructions 3 and 4, we were able to use $(\delta_x)_{x \in S} - LA(2, 3, S)$ in the construction of $LD(2, 3, v)$. For $LD(2, 3, v)$, an opportunistic use of the closure spaces $\delta'_{\{\infty_1, \infty_2\}}$, where (S, δ) is a 1-equicardinal rank 3 matroid with $\delta(\emptyset) = \emptyset$, $\infty_1 \neq \infty_2$ and $S \cap \{\infty_1, \infty_2\} = \emptyset$, yields a natural generalization of Construction 4.

The notions of $\delta - LS(t, k, S, R)$, $LS(t, k, S, R)$ and subspace of an $LS(t, k, S, R)$ can be defined in the obvious way. Most notions used in this paper have obvious generalizations to «higher λ». It may be worth mentioning that $LD(t, t+1, S, R)$ and $LS(t, t+1, S, R)$ are special cases of more general structures used in [14,16] to construct non-trivial t-designs without repeated blocks for all t. Part of the difficulty in using $LS(\lambda; t, k, v)$ with $k > t+1$ in the same way as $LS(\lambda; t, t+1, v)$ were used in the inductive construction of designs with larger t is that in the definition of $LS(t, k, S, R)$ two classes of subsets have to be covered by $(k+1)$-subsets, members of one class having size $t+1$ and members of the other having size k. For $k = t+1$, these two sizes coincide, but not for $k > t+1$.

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