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ON THE USE OF SOME PROPERTIES OF LEONTIEFF'S MATRICES IN THEIR INVERSION

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1. INTRODUCTION

The complicated relations in the economic system were the reason for forming models which try to describe the reality both as accurately and as simply as possible.

Let us assume that the economic system we intend to deal with is composed of n branches.

We require that the quantity of the i-th product produced by the i-th branch be sufficient to cover, on the one hand, the demand of this product in all branches (including the i-th branch) and, on the other hand, the final demand. We introduce the following notation:

w_i the whole production of the i-th product,
f_i the final demand for the i-th product,
w_{ij} the quantity of the i-th product necessary for the production of w_j,

all taken for a chosen unit of time (e.g. one year).
w_i, w_{ij} and I_i are expressed in the same units (for example monetary units).

The above mentioned quantities are related as follows:

$$
\begin{aligned}
w_1 &= w_{11} + w_{12} + \ldots + w_{1n} + f_1 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
w_i &= w_{i1} + w_{i2} + \ldots + w_{in} + f_i \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
w_n &= w_{n1} + w_{n2} + \ldots + w_{nn} + f_n
\end{aligned} \tag{1}
$$

and it is clear from their introduction that all the quantities are non-negative.

Leontieff considered the linear relation between w_i and w_{ij}

$$w_{ij} = a_{ij} w_j \tag{2}$$

where every a_{ij} is non-negative and less than 1.

The relations (1) and (2) together form Leontieff's static economic model. If we further assume that

(3) $$\sum_{i=1}^{n} a_{ij} < 1$$

holds for every j, we deal with the open model. In what follows we shall assume that (3) holds.

Substituting (2) into (1) we get the system of linear equations for the unknowns w_i:

(4)
$$\begin{aligned} w_1 &= a_{11}w_1 + \ldots + a_{1n}w_n + f_1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w_i &= a_{i1}w_1 + \ldots + a_{in}w_n + f_i \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w_n &= a_{n1}w_1 + \ldots + a_{nn}w_n + f_n \end{aligned}$$

where a_{ij} are given constants whose values are determined by means of econometrics, and also f_i are given.

It is possible to express this system in the matrix form. Let $\boldsymbol{A}$ be the matrix with elements a_{ij}, $\boldsymbol{W}$ the column vector with elements w_i, $\boldsymbol{F}$ the column vector with elements f_i, and $\boldsymbol{E}$ the unit matrix. Instead of (4), we can use then this matrix equation:

(5) $$(\boldsymbol{E} - \boldsymbol{A})\,\boldsymbol{W} = \boldsymbol{F}\,.$$

From the knowledge of $\boldsymbol{A}$ and $\boldsymbol{F}$ we are, by means of (5), to determine $\boldsymbol{W}$ so that $\boldsymbol{W} \geqq \boldsymbol{O}$, $\boldsymbol{W} \neq \boldsymbol{O}$. The condition (3) guarantees the existence of $\boldsymbol{W}$ with these required properties.

The matrix $\mathrm{M} = \boldsymbol{E} - \boldsymbol{A}$ is called Leontieff's matrix. It is evident that it has the following properties:

a) The diagonal elements of $\boldsymbol{M}$ are positive.
b) The off-diagonal elements are non-positive.
c) A certain norm of $\boldsymbol{M}$ is less than 1.
d) The inverse matrix $\boldsymbol{M}^{-1}$ exists and $\boldsymbol{M}^{-1} \geqq \boldsymbol{O}$.

The property b) implies that matrix $\boldsymbol{M}$ belongs to the class of $\boldsymbol{Z}$-matrices ($m_{ij} \leqq 0$ for $i \neq j$), and property d) together with the property b) implies that it belongs to the class of $\boldsymbol{K}$-matrices [1].

I shall mention here, without proof, several theorems which hold for $\boldsymbol{K}$-matrices. The proofs are given in [1].

Let be $\boldsymbol{P} \in \boldsymbol{K}$. Then

1) All real eigenvalues of $\boldsymbol{P}$ are positive.
2) All principal minors of $\boldsymbol{P}$ are positive.
3) There exists a vector $\boldsymbol{x} = \boldsymbol{O}$ so that $\boldsymbol{Px} > \boldsymbol{O}$.
4) If $\boldsymbol{B} \in \boldsymbol{Z}$, $\boldsymbol{B} \geqq \boldsymbol{P}$ (i.e. $b_{ij} \geqq p_{ij}$ for every i, j), then $\boldsymbol{B}^{-1}$ exists and $\boldsymbol{O} \leqq \boldsymbol{B}^{-1} \leqq \leqq \boldsymbol{P}^{-1}$.

5) There exists a diagonal matrix $\boldsymbol{D}$ with $d_i > 0$ such that the matrix $\boldsymbol{W} = \boldsymbol{PD}$ has a dominant positive principal diagonal (i.e. $w_{ii} > \sum_{i \neq j} |w_{ij}|$).

Note that matrices $\boldsymbol{A}$ originating from econometrics are mostly of high orders (several hundreds and more) and most of their elements are zero.

2. ECONOMY STORING OF MATRICES WITH MANY NULL-ELEMENTS

The accuracy and speed of computation in a computer is usually not essentially affected by operations where one (or both) operands are zero. This is because normally the wiring in the computer skips these operations. A different matter is the occupation of storage. It seems obvious to store only the non-null elements. This leads to the rather interesting problem how to store in the best way and most economically their position in the matrix. It appears that a system convenient for binary computers is less convenient for decimal computers and that the necessary subroutines are rather complicated (viz [5], [7], [13], [15]). Decisive for the choice of the system of Economy Storing will be the volume of storage needed for the subroutines as well as for the matrix elements and their position, along with the computing time necessary for handling the elements in the system.

3. TRANSFORMATION OF A MATRIX INTO THE QUASITRIANGULAR FORM

From the way the problem was put it is evident that the numeration of the branches does not change the problem. For the matrix $\boldsymbol{A}$ (and therefore for the matrix $\boldsymbol{M}$ as well) a numeration means a simultaneous permutation of rows and columns. On the other hand it is well known that inverting a quasitriangular matrix is essentially easier than inverting a full matrix [2]. We shall concern ourselves with the problem how to transform a matrix into the quasitriangular form by a simultaneous permutation of rows and columns.

The transformation into this form is, by itself, meaningful for the economists. From it, they can see directly certain relations between particular branches.

One of the possibilities how to solve this problem is given by the graph theory.

Let us assign one node of a certain oriented graph to every row of the matrix. In this graph $\boldsymbol{G}$, the edge from the i-th node to the k-th one exists iff the element with the row index i and column index k is non-null. Now we shall assign to the graph $\boldsymbol{G}$ another graph $\boldsymbol{G}'$, the reduced graph corresponding to $\boldsymbol{G}$, whose nodes correspond to the quasicomponents of $\boldsymbol{G}$. By a "quasicomponent" we mean a subgraph the transitive closure of which is complete [4]. We number the nodes in $\boldsymbol{G}'$ so that its edges point always from a node with a lower number to a node with a higher number. This is always possible because the reduced graph is acyclic [4].

After that we shall carry out a simultaneous permutation of rows and columns of the matrix so that the rows which belong to one quasicomponent adjoin each other (the ordering of rows within each quasicomponent is irrelevant), while the quasi-components are ordered in an increasing order. As a result we get an upper (right) quasitriangular matrix with irreducible diagonal blocks.

In [4] this algorithm is described in considerable detail and is given in a form applicable to computers.

4. INVERSION OF A QUASITRIANGULAR MATRIX

Let us assume that the matrix $\mathbf{M}$ is quasitriangular and has the following form:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{N}_1 & & & \\ & \mathbf{M}_2 & \mathbf{N}_2 & & \\ & & \cdots\cdots\cdots & & \\ & & & \mathbf{M}_{k-1} & \mathbf{N}_{k-1} \\ & & & & \mathbf{M}_k \end{bmatrix}. \tag{6}$$

The matrix $\mathbf{M}$ is of the type (n, n), the matrix $\mathbf{M}_i$ is of the type (n_i, n_i), the matrix $\mathbf{N}_i$ is of the type $(\sum_{s=1}^{i} n_s, n_{i+1})$ where $\sum_{s=1}^{k} n_s = n$.

We shall describe here a method for the inversion of the matrix $\mathbf{M}$; it is essentially a generalization of a method for the inversion of a matrix divided into blocks using the quasitriangular form of $\mathbf{M}$.

If we invert a quasitriangular matrix $\mathbf{U}$ divided into 4 blocks as follows:

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{O} & \mathbf{U}_4 \end{bmatrix} \tag{7}$$

we obtain $\mathbf{Z}$, the inverse to $\mathbf{U}$, in the form

$$\mathbf{U}^{-1} = \mathbf{Z} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{O} & \mathbf{Z}_4 \end{bmatrix} \tag{8}$$

where $\mathbf{Z}_i$ is a matrix of the same type as $\mathbf{U}_i$, and

$$\left\{ \begin{aligned} \mathbf{Z}_1 &= \mathbf{U}_1^{-1} \\ \mathbf{Z}_2 &= -\mathbf{U}_1^{-1}\mathbf{U}_2\mathbf{U}_4^{-1} \\ \mathbf{Z}_4 &= \mathbf{U}_4^{-1} \end{aligned} \right. \tag{9}$$

holds.

Now we shall use relations (9) for constructing finite iterative formulae for the inverse of the matrix $\mathbf{M}$.

If we introduce the matrices $\mathbf{T}_i$ by

(10) $$\mathbf{T}_1 = \mathbf{M}_1^{-1}$$

and

(11) $$\mathbf{T}_i = \begin{bmatrix} \mathbf{M}_1 & \mathbf{N}_1 & & & \\ & \mathbf{M}_2 & \mathbf{N}_2 & & \\ & & \cdots\cdots & & \\ & & & \mathbf{M}_{i-1} & \mathbf{N}_{i-1} \\ & & & & \mathbf{M}_i \end{bmatrix}^{-1}$$

and therefore

(12) $$\mathbf{T}_k = \mathbf{M}^{-1},$$

then, using (9), we obtain

(13) $$\mathbf{T}_{i+1} = \begin{bmatrix} \mathbf{T}_i & -\mathbf{T}_i \mathbf{N}_i \mathbf{M}_{i+1}^{-1} \\ & \mathbf{M}_{i+1}^{-1} \end{bmatrix}.$$

Then there holds that

(14) $$\mathbf{T}_i = \begin{bmatrix} \mathbf{M}_1^{-1} & \mathbf{C}_1 & & & \\ & \mathbf{M}_2^{-1} & \mathbf{C}_2 & & \\ & & \cdots\cdots & & \\ & & & \mathbf{M}_{i-1}^{-1} & \mathbf{C}_{i-1} \\ & & & & \mathbf{M}_i^{-1} \end{bmatrix}$$

where the matrices $\mathbf{C}_j$ are of the same type as the corresponding $\mathbf{N}_j$. We may write

(15) $$\mathbf{C}_j = -\begin{bmatrix} \mathbf{M}_1^{-1} & \mathbf{C}_1 & & & \\ & \mathbf{M}_2^{-1} & \mathbf{C}_2 & & \\ & & \cdots\cdots & & \\ & & & \mathbf{M}_{j-1}^{-1} & \mathbf{C}_{j-1} \\ & & & & \mathbf{M}_j^{-1} \end{bmatrix} \mathbf{N}_j \mathbf{M}_{j+1}^{-1},$$

and therefore the inverse matrix $\mathbf{M}^{-1} = \mathbf{T}_k$ has the following form:

(16) $$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{M}_1^{-1} & \mathbf{C}_1 & & & \\ & \mathbf{M}_2^{-1} & \mathbf{C}_2 & & \\ & & \cdots\cdots & & \\ & & & \mathbf{M}_{k-1}^{-1} & \mathbf{C}_{k-1} \\ & & & & \mathbf{M}_k^{-1} \end{bmatrix}.$$

5. INVERSION OF LEONTIEFF'S MATRICES

It was shown in the preceding paragraphs how to reduce the problem of inverting a matrix of order n to the problem of inverting several matrices the sum of whose orders equals n. It is evident that these matrices are again of Leontieff's type .

For the inversion of the $\boldsymbol{M}_i$ defined in (16) it is possible to use several different methods. Here the Hotelling method will be described.

The Hotelling method is an iterative method given by the formulae

$$\boldsymbol{X}^{(i+1)} = \boldsymbol{X}^{(i)} + \boldsymbol{X}^{(i)}\boldsymbol{R}^{(i)}, \tag{17}$$

$$\boldsymbol{R}^{(i)} = \boldsymbol{E} - \boldsymbol{M}\boldsymbol{X}^{(i)}. \tag{18}$$

This method is convergent iff the initial approximation $\boldsymbol{X}^{(0)}$ is so close to the solution that a certain norm of the initial residual matrix $\boldsymbol{R}^{(0)}$ is less than one:

$$\|\boldsymbol{R}^{(0)}\| < 1. \tag{19}$$

We choose for the initial approximation the unit matrix:

$$\boldsymbol{X}^{(0)} = \boldsymbol{E}. \tag{20}$$

For $\boldsymbol{R}^{(0)}$, we obtain the relation

$$\boldsymbol{R}^{(0)} = \boldsymbol{A}. \tag{21}$$

From the property c) of paragraph 1 there follows that the condition (19) is satisfied.

If we use formulae (17) and (18) we can watch in each step of the iteration, without any further computation, the residual matrix $\boldsymbol{R}^{(i)}$ (which determines how the product $\boldsymbol{M}\boldsymbol{X}^{(i)}$ differs from the unit matrix) and also the product $\boldsymbol{X}^{(i)}\boldsymbol{R}^{(i)}$ (which determines how two consecutive iterations differ from each other).

The process can be stopped when the obtained accuracy is either equal to the required one, or is the best obtainable (that means when the matrix $\boldsymbol{X}^{(i)}$ does not change when adding the matrix $\boldsymbol{X}^{(i)}\boldsymbol{R}^{(i)}$).

It may be shown that for the i-th approximation $\boldsymbol{X}^{(i)}$ of the matrix $\boldsymbol{M}^{-1}$

$$\boldsymbol{X}^{(i)} = \sum_{k=0}^{2^i-1} \boldsymbol{A}^k \tag{22}$$

holds and therefore

$$\boldsymbol{M}^{-1} - \boldsymbol{X}^{(i)} = \boldsymbol{A}^{2^i}\sum_{k=0}^{\infty} \boldsymbol{A}^k = \boldsymbol{A}^{2^i}\boldsymbol{M}^{-1}. \tag{23}$$

By denoting $\|\boldsymbol{A}\| = \mu$, $\mu < 1$ we obtain for the norm $\|\boldsymbol{M}^{-1}\|$ the relation

$$\|\boldsymbol{M}^{-1}\| \leqq \frac{1}{1-\mu}. \tag{24}$$

Using (23) we get

$$\|\boldsymbol{X}^{(i)} - \boldsymbol{M}^{-1}\| \leqq \frac{\mu^{2^i}}{1-\mu} \tag{25}$$

where

$$\|\mathbf{R}^{(i)}\| \leqq \mu^{2^i}. \tag{26}$$

It is, of course, possible to obtain the inverse matrix by other methods as well. First of all the Gaussian elimination without pivoting must be considered. The diagonal elements of $\mathbf{M}^{-1}$ are the quotients of principal minors of $\mathbf{M}$ which are positive in view of property 2) of $\mathbf{K}$-matrices (cf. paragraph 1).

Further, there is also the Gauss-Seidel iteration.

It might be well worth trying to construct some new methods using the remarkable properties of Leontieff's matrices, if only for comparison with Gaussian elimination, the Hotelling method and other known methods. This comparison would concern accuracy, speed and storage requirements. For higher accuracy, the use of double-arithmetics in certain parts would be suitable.

6. CONCLUSION

From this paper there follows that for the inversion of Leontieff's matrices the following process can be recommended:

1. The elements of the matrix $\mathbf{A}$ are stored in a special form.
2. The matrix is transformed into the quasitriangular form using the method of [4].
3. The algorithm of paragraph 4 is used.
4. The matrices $\mathbf{M}_i^{-1}$ of the relation (16) are obtained by the Hotelling method. The unit matrix is used as the initial approximation. This algorithm is explained in paragraph 5.

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Souhrn

O VYUŽITÍ NĚKTERÝCH VLASTNOSTÍ LEONTJEVOVSKÝCH MATIC PŘI JEJICH INVERTOVÁNÍ

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V článku je popsán postup vhodný pro invertování leotjevovských matic vysokých řádů na samočinných počítačích. V § 1 je zadání problému, § 2 se nabývá úsporným ukládáním prvků matice do paměti počítače, § 3 stručně popisuje převedení matice do kvazitrojúhelníkového tvaru. Jádro práce je v § 4, kde je odvozena nová metoda pro inverz i kvazitrojúhelníkové matice. V § 5 je popsána Hotellingova metoda, které se použije při aplikaci metody popsané v § 4. § 6 je závěrečný.

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