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ON THE USE OF THE ENERGY-MOMENTUM PRINCIPLE IN GENERAL RELATIVITY

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Abstract

The primary purpose of this article is to obtain from the general relativity form of the energy-momentum principle certain new consequences which are needed for later work that the author has in mind. In addition, it is the intention to give at the same time a somewhat comprehensive and coherent treatment of the principle and its consequences, which it is hoped will increase the confidence and facility of physicists in the use of this important part of the general theory of relativity. In carrying out the investigation, it has seemed desirable for English readers, to take Eddington's "Mathematical Theory of Relativity" as a starting point, and this has incidentally led to a new form of deduction for certain consequences of the energy-momentum principle that were already known.

After presenting the energy-momentum principle in the form discovered by Einstein and showing its application to the case of the conservation of energy in an isolated system, an important expression is derived which gives the total densities of energy and momentum in the form of a divergence. This expression is equivalent to one previously obtained by Einstein but on account of the starting point adopted is derived and expressed in terms of the quantities $\mathfrak{g}^{\mu\nu}$ and $\mathfrak{g}_{\alpha}^{\mu\nu}$ instead of the $g^{\mu\nu}$ and $g_{\alpha}^{\mu\nu}$. Following this, the limiting values at large distances from an isolated material system are obtained for the quantities $\mathfrak{g}^{\alpha\beta}\partial\mathfrak{L}/\partial\mathfrak{g}_{\gamma}^{\alpha\beta}$ and $\mathfrak{g}^{\alpha 4}\partial\mathfrak{L}/\partial\mathfrak{g}_{\gamma}^{\alpha 4}$. These values, which have considerable use, have not previously received explicit expression. This is followed by a deduction from our present starting point of Einstein's famous relation $U=m$ between the energy and gravitational producing mass of an isolated system. An important expression is then obtained which gives the energy of a quasi-static isolated system in the form of an integral which has to be extended only over the portion of space actually occupied by matter or radiation. This expression has not previously received a satisfactory derivation. The result is used to obtain an expression for the energy of a spherical distribution of a perfect fluid, and it is then shown that this expression, in the case of a sphere of ordinary material, approaches in a sufficiently weak field to the classical expression for energy including the potential gravitational energy. This result is not only intrinsically useful, but also shows for a particular case that a higher order of approximation to the general relativity value for total energy is obtained by including the classical gravitational energy than by going at once to flat space-time as is often done. Finally, a general consideration is given to the problem of determining the conditions imposed on those changes from one static state to another which could occur in a non-isolated system forming part of a larger static system, without changing the distribution of matter and radiation outside the boundary and without contravening the energy-momentum principle as applied to the system as a whole.

§1. Introduction

IN ACCORDANCE with Einstein's development of the general theory of relativity, the relativity analogues of the classical principles of the conservation of energy and momentum are to be obtained with the help of integration from the well-known differential equation[1]

$$\frac{\partial}{\partial x_\nu}(\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu) = 0 \tag{1}$$

where $\mathfrak{T}_\mu^\nu$ is the tensor density of material energy and momentum and $\mathfrak{t}_\mu^\nu$ is the psuedo-tensor density of potential gravitational energy and momentum. Considering x_4 to be the time-like coordinate, equation (1) leads to the three equations for the conservation of momentum with $\mu = 1, 2, 3$, and leads to the equation for the conservation of energy with $\mu = 4$.

Taking for illustration the latter case, multiplying the equation by $dx_1\, dx_2\, dx_3$, and integrating over the system of interest, we obtain with some rearrangement of terms

$$\frac{\partial}{\partial x_4}\int\int\int(\mathfrak{T}_4^4+\mathfrak{t}_4^4)dx_1dx_2dx_3$$
$$= -\int\int\int\left[\frac{\partial}{\partial x_1}(\mathfrak{T}_4^1+\mathfrak{t}_4^1)+\frac{\partial}{\partial x_2}(\mathfrak{T}_4^2+\mathfrak{t}_4^2)+\frac{\partial}{\partial x_3}(\mathfrak{T}_4^3+\mathfrak{t}_4^3)\right]dx_1dx_2dx_3$$

and by performing the indicated integrations on the right hand side this can be rewritten in the form

$$\frac{\partial}{\partial x_4}\int\int\int(\mathfrak{T}_4^4+\mathfrak{t}_4^4)dx_1dx_2dx_3$$
$$= -\int\int\Big|\mathfrak{T}_4^1+\mathfrak{t}_4^1\Big|_{x_1}^{x_1'}dx_2dx_3 - \int\int\Big|\mathfrak{T}_4^2+\mathfrak{t}_4^2\Big|_{x_2}^{x_2'}dx_1dx_3 - \int\int\Big|\mathfrak{T}_4^3+\mathfrak{t}_4^3\Big|_{x_3}^{x_3'}dx_1dx_2 \tag{2}$$

where the limits of integration are denoted by x_1, x'_1 etc. The result states that the rate of change with the time (x_4) in the value of the integral on the left hand side of the equation can be calculated from the conditions prevailing at the boundary of the system of interest as given by the right hand side of the equation. The equation can thus be regarded as a statement of the energy principle provided we define the energy of the system by the expression

$$U = \int\int\int(\mathfrak{T}_4^4+\mathfrak{t}_4^4)dx_1dx_2dx_3. \tag{3}$$

And a similar treatment for the components of momentum can be obtained by taking $\mu = 1, 2, 3$.

[1] See for example Eddington, "The Mathematical Theory of Relativity." Cambridge 1923, equations (59.2) and (59.3).

Owing to the fact that equations (1) and (2) are not tensor equations and that the quantity $\mathfrak{t}_\mu^\nu$ is not a true tensor density, considerable doubt as to the validity of the above formulation of the energy principle was at one time expressed[2] and perhaps to some extent still exists.[3] It can be shown, nevertheless, that the equations have the necessary fundamental property of being true in all sets of coordinates, and a completely satisfactory justification of the formulation was finally given by Einstein in 1918.[4]

Since that time, however, the interest of mathematical physicists has been largely turned to other matters, and the methods and results of applying the energy-momentum principle have not been particularly investigated. It is the purpose of the present article to consider some of these methods and results not only because of their intrinsic importance, but also because of certain further applications which the writer has in mind.

In carrying out the investigation we shall use, as far as may be, the notation adopted by Eddington in his "Mathematical Theory of Relativity", and shall base our deductions on equations given by him. This choice of starting point necessitates some duplication of results which have previously been obtained by other methods, but seems desirable owing to the familiarity of English readers with Eddington's treatise and owing to the excellence of the detailed and coherent treatment which he has given.

In the immediately following section, §2, we shall first give Einstein's method of applying the above energy principle to an isolated material system. In §3 we shall then obtain a very useful formula which expresses the total density of energy and momentum as an ordinary divergence; the formula is equivalent in import to one already obtained by Einstein but because of our choice of starting point differs in method of derivation and form of expression. Following this, in §4, we shall deduce the limiting values at large distances from such an isolated system of certain functions of the gravitational field; these values are necessary for our further work and have not previously received explicit expression. In §5 we shall then be able to give a deduction from our present basis of Einstein's relation between the total energy and gravitational mass of an isolated material system. In §6 we shall deduce an extremely important equation expressing the total energy of an isolated system by an integral which has to be extended only over that portion of space which is actually occupied by matter or radiation; this equation

[2] Schroedinger, Phys. Zeits. **19,** 4 (1918); Bauer, ibid. **19,** 163 (1918).

[3] Thus, for example, Eddington (reference 1, pp. 135–136) objects to the fundamental significance of the equation $\partial/\partial x_\nu(\mathfrak{T}_\mu^\nu+\mathfrak{t}_\mu^\nu)=0$ on the ground that $\mathfrak{t}_\mu^\nu$ is not a true tensor density, and appears to regard the introduction of the equation as an unfortunate pandering to an immoral desire to obtain by hook or crook some kind of conservation laws in the mechanics of general relativity. It should be remarked, however, that the appropriate criterion for the fundamental significance of equations should not be that they are written in tensorial form but that they are written in covariant form so as to be true in all sets of coordinates. All tensor equations are indeed covariant equations, but this does not exclude the possibility of covariant equations, such as the one above, which are not tensorial. To assume the contrary would be the fallacy of the Dormouse in Alice in Wonderland, who said:—"I breathe when I sleep" is the same thing as "I sleep when I breathe."

[4] Einstein, Berl. Ber. 1918, p. 448.

has not previously received a satisfactory derivation. In the following section, §7, we shall use this equation to obtain an expression for the total energy of a distribution of perfect fluid having spherical symmetry. And in §8 we shall show that, on making the gravitational field weaker, the above expression approaches the classical expression for the energy of such a sphere including the classical value for its potential gravitational energy; the result is an intrinsically useful one and also shows for a particular case that a higher order of approximation to the general relativity value for total energy is obtained by including the classical gravitational energy than by neglecting the gravitational energy entirely as has hitherto often been done. Finally in §§9, 10 we shall consider the application of the general energy-momentum principle to changes occurring within a region which forms part of a larger system, and in §11 shall make some concluding remarks.

§2. The Conservation of Energy in an Isolated System

Let us now first consider the application of the energy principle to those changes which can take place within a limited system without producing any changes in the gravitational field outside of a sufficiently distant boundary located in the free space surrounding the system. Such a system will be called an isolated one. In this case, we can easily show that the general energy principle, as given by equation (2), can be interpreted as leading to the conservation of energy within the boundary taken.

For the tensor density of matter and energy $\mathfrak{T}_\mu^\nu$, occurring in equation (2), we can write from the equation of definition[5]

$$\mathfrak{T}_\mu^\nu = \sqrt{-g}\, g_{\alpha\mu} T^{\alpha\nu} = \sqrt{-g}\, g_{\alpha\mu}\rho_0 \frac{dx_\alpha}{ds}\,\frac{dx_\nu}{ds} \tag{4}$$

where ρ_0 is the proper density of matter. And since we take the boundary which encloses the system as located in free space the density ρ_0 will there be zero, so that it is evident that the quantities $\mathfrak{T}_4^1$, $\mathfrak{T}_4^2$ and $\mathfrak{T}_4^3$, occurring on the right hand side of equation (2), will themselves have the value zero at the boundary.

Furthermore, the pseudo-tensor density of potential energy is defined in terms of the Lagrangian function $\mathfrak{L}$ and the cosmological constant Λ by the equation[6]

$$\mathfrak{t}_\mu^\nu = \frac{1}{16\pi}\left\{ g_\mu^\nu \mathfrak{L} - \mathfrak{g}_\mu^{\alpha\beta} \frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\nu^{\alpha\beta}} \right\} + \frac{\Lambda}{8\pi} g_\mu^\nu \sqrt{-g}. \tag{5}$$

The quantity g_μ^ν, however, is equal to zero with $\mu \neq \nu$, and the symbol $\mathfrak{g}_4^{\alpha\beta}$ is a short hand for $\partial(g^{\alpha\beta}\sqrt{-g})/\partial x_4$ and hence is equal to zero at the boundary enclosing our system, if the gravitational field at that boundary is not changing with the time as postulated. Hence the quantities $\mathfrak{t}_4^1$, $\mathfrak{t}_4^2$, and $\mathfrak{t}_4^3$,

[5] See Eddington, reference 1, equation (53.1).

[6] See Eddington, reference 1, equation (59.4). The additional term in Λ, when the cosmological term is not neglected, can easily be shown necessary. Compare Einstein, reference 4, equation (18).

occurring on the right hand side of equation (2), will also be zero at the boundary with which we have enclosed the system.

Hence using a system of coordinates such that the limits of integration coincide with the boundary enclosing the system,[7] it is evident that the terms on the right hand side of equation (2) will all of them become zero and the energy principle for this particular case will reduce to the conservation of energy

$$\frac{\partial}{\partial x_4}\iiint(\mathfrak{T}_4^4+\mathfrak{t}_4^4)dx_1dx_2dx_3=0$$

or

$$U=\iiint(\mathfrak{T}_4^4+\mathfrak{t}_4^4)dx_1dx_2dx_3=\text{const.} \tag{6}$$

§3. The Densities of Momentum and Energy Expressed as Divergences

In order to make use of this interesting result, it will now be necessary to make a rather lengthy digression by deducing certain useful lemmas in this and in the following section. In the present section we shall show the possibility of expressing the total density of momentum or energy as an ordinary divergence in accordance with the equation[8]

$$8\pi(\mathfrak{T}_\mu^\nu+\mathfrak{t}_\mu^\nu)=\frac{\partial}{\partial x_\gamma}\left(-\mathfrak{g}^{\alpha\nu}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\alpha}_{\gamma}}+\frac{1}{2}g_\mu^\nu\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{\gamma}}\right). \tag{7}$$

To prove this equation will be a very tiresome business, justified only by the importance of the result. To carry out the demonstration we shall have to make use of a large number of well established results which we shall now give.

For the density of material energy and momentum we shall use the fundamental equation connecting it with the metrical properties of the field[9]

$$-8\pi\mathfrak{T}_\mu^\nu=\mathfrak{G}_\mu^\nu-\tfrac{1}{2}g_\mu^\nu\mathfrak{G}+\Lambda\mathfrak{g}_\mu^\nu \tag{8}$$

where $\mathfrak{G}_\mu^\nu$ is the tensor density corresponding to the contracted Riemann-Christoffel tensor and Λ the cosomological constant. For the density of potential energy and momentum we shall use the equation of definition already given above

$$16\pi\mathfrak{g}_\mu^\nu=g_\mu^\nu\mathfrak{L}-\mathfrak{g}_\mu^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{\nu}}+2\Lambda\mathfrak{g}_\mu^\nu \tag{9}$$

[7] With a system of coordinates chosen so that some of the limits of integration do not fall on the boundary, the quantities $\left|\mathfrak{T}_4^1+\mathfrak{t}_4^1\right|x_1'$ etc. will not necessarily be zero. Compare §9.

[8] This equation is equivalent in import to equation (18) given by Einstein, Berl. Ber. 1916, p. 1115. It differs in form since we are regarding $\mathfrak{L}$ as a function of the $\mathfrak{g}^{\mu\nu}$ and $\mathfrak{g}_\alpha^{\mu\nu}$ instead of as a function of the $g^{\mu\nu}$ and $g_\alpha^{\mu\nu}$.

[9] See Eddington, reference 1, equation (54.71).

where the Lagrangian function $\mathfrak{L}$ is defined in terms of the Christoffel symbols by the equation[10]

$$\mathfrak{L}=\sqrt{-g}\, g^{\alpha\beta}\Big(\{\alpha\gamma,\epsilon\}\{\beta\epsilon,\gamma\}-\{\alpha\beta,\gamma\}\{\gamma\epsilon,\epsilon\}\Big). \tag{10}$$

In accordance with this definition it is possible to show that $\mathfrak{L}$ can also be expressed as a function of the quantities

$$\mathfrak{g}^{\alpha\beta}=g^{\alpha\beta}\sqrt{-g} \quad\text{and}\quad \mathfrak{g}^{\alpha\beta}_{\gamma}=\frac{\partial}{\partial x_\gamma}(g^{\alpha\beta}\sqrt{-g}) \tag{11}$$

and will then have the differential coefficients[11]

$$\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}}=-\Big(\{\alpha\gamma,\epsilon\}\{\beta\epsilon,\gamma\}-\{\alpha\beta,\gamma\}\{\gamma\epsilon,\epsilon\}\Big) \tag{12}$$

and[12]

$$\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{\gamma}}=-\{\alpha\beta,\gamma\}+\frac{1}{2}g^{\gamma}_{\alpha}\{\beta\epsilon,\epsilon\}+\frac{1}{2}g^{\gamma}_{\beta}\{\alpha\epsilon,\epsilon\}. \tag{13}$$

Further important properties of the Lagrangian function, relating it to the metrical properties of the field, are given by the equation[13]

$$G_{\mu\nu}=\frac{\partial}{\partial x_\gamma}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\nu}_{\gamma}}-\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\nu}} \tag{14}$$

and[14]

$$\mathfrak{G}=\frac{\partial}{\partial x_\gamma}\left(\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{\gamma}}\right)-\mathfrak{L}. \tag{15}$$

Finally, the quantity $\mathfrak{g}^{\alpha\beta}_{\gamma}$ may be expressed in terms of the Christoffel symbols by the equation[15]

$$\mathfrak{g}^{\alpha\beta}_{\gamma}=\sqrt{-g}\Big(-\{\delta\gamma,\alpha\}g^{\delta\beta}-\{\delta\gamma,\beta\}g^{\delta\alpha}+\{\gamma\delta,\delta\}g^{\alpha\beta}\Big). \tag{16}$$

We are now ready to proceed to the derivation of equation (7). Combining equations (8) and (9) we have

$$8\pi(\mathfrak{T}^{\nu}_{\mu}+\mathfrak{t}^{\nu}_{\mu})=-\,\mathfrak{G}^{\nu}_{\mu}+\frac{1}{2}g^{\nu}_{\mu}\mathfrak{G}+\frac{1}{2}g^{\nu}_{\mu}\mathfrak{L}-\frac{1}{2}\mathfrak{g}^{\alpha\beta}_{\mu}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{\nu}}$$

[10] See Eddington, reference 1, equation (58.1).

[11] See Eddington, reference 1, equation (58.51).

[12] See Eddington, reference 1, equation (58.52). We have written the expression in a symmetrical form which is equivalent to Eddington's unsymmetrical form.

[13] See Eddington, reference 1, equation (58.6).

[14] See Eddington, reference 1, equation (58.8).

[15] See Eddington, reference 1, the equation immediately following (58.72).

Substituting (14) and (15) we obtain

$$8\pi(\mathfrak{T}_\mu^\nu+\mathfrak{t}_\mu^\nu)=-\mathfrak{g}^{\alpha\nu}\frac{\partial}{\partial x_\gamma}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\gamma^{\mu\alpha}}+\mathfrak{g}^{\alpha\nu}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\alpha}}+\frac{1}{2}g_\mu^\nu\frac{\partial}{\partial x_\gamma}\left(\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\gamma^{\alpha\beta}}\right)-\frac{1}{2}\mathfrak{g}_\mu^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\nu^{\alpha\beta}}$$

and this can evidently be rewritten in the form

$$8\pi(\mathfrak{T}_\mu^\nu+\mathfrak{t}_\mu^\nu)=\frac{\partial}{\partial x_\gamma}\left(-\mathfrak{g}^{\alpha\nu}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\gamma^{\mu\alpha}}+\frac{1}{2}g_\mu{}^\nu\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\gamma^{\alpha\beta}}\right)$$
$$+\mathfrak{g}_\gamma^{\alpha\nu}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\gamma^{\mu\alpha}}+\mathfrak{g}^{\alpha\nu}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\alpha}}-\frac{1}{2}\mathfrak{g}_\mu^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\nu^{\alpha\beta}}\cdot\qquad(17)$$

Comparing this result with equation (7), we see that our deduction will be completed if we can now show that the sum of the last three terms is equal to zero. To accomplish this we must substitute for the quantities occurring in these three terms their explicit values in terms of the Christoffel symbols as given by equations (12), (13) and (16). Doing so we obtain

$$\mathfrak{g}_\gamma^{\alpha\nu}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\gamma^{\mu\alpha}}+\mathfrak{g}^{\alpha\nu}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\alpha}}-\frac{1}{2}\mathfrak{g}_\mu^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\nu^{\alpha\beta}}$$
$$=\sqrt{-g}[-\{\delta\gamma,\alpha\}g^{\delta\nu}-\{\delta\gamma,\nu\}g^{\delta\alpha}+\{\gamma\delta,\delta\}g^{\alpha\nu}][-\{\mu\alpha,\gamma\}+\tfrac{1}{2}g_\mu^\gamma\{\alpha\epsilon,\epsilon\}+\tfrac{1}{2}g_\alpha^\gamma\{\mu\epsilon,\epsilon\}]$$
$$+\sqrt{-g}\,g^{\alpha\nu}[-\{\alpha\gamma,\epsilon\}\{\mu\epsilon,\gamma\}+\{\alpha\mu,\gamma\}\{\gamma\epsilon,\epsilon\}]$$
$$-\tfrac{1}{2}\sqrt{-g}[-\{\delta\mu,\alpha\}g^{\delta\beta}-\{\delta\mu,\beta\}g^{\delta\alpha}+\{\mu\delta,\delta\}g^{\alpha\beta}][-\{\alpha\beta,\nu\}+\tfrac{1}{2}g_\alpha^\nu\{\beta\epsilon,\epsilon\}$$
$$+\tfrac{1}{2}g_\beta^\nu\{\alpha\epsilon,\epsilon\}]$$
$$=\sqrt{-g}[\overset{(1)}{\{\delta\gamma,\alpha\}}\{\mu\alpha,\gamma\}g^{\delta\nu}-\tfrac{1}{2}\overset{(2)}{\{\delta\mu,\alpha\}}\{\alpha\epsilon,\epsilon\}g^{\delta\nu}-\tfrac{1}{2}\overset{(3)}{\{\delta\alpha,\alpha\}}\{\mu\epsilon,\epsilon\}g^{\delta\nu}$$
$$+\overset{(4)}{\{\delta\gamma,\nu\}}\{\mu\alpha,\gamma\}g^{\delta\alpha}-\tfrac{1}{2}\overset{(5)}{\{\delta\mu,\nu\}}\{\alpha\epsilon,\epsilon\}g^{\delta\alpha}-\tfrac{1}{2}\overset{(6)}{\{\delta\alpha,\nu\}}\{\mu\epsilon,\epsilon\}g^{\delta\alpha}$$
$$-\overset{(7)}{\{\gamma\delta,\delta\}}\{\mu\alpha,\gamma\}g^{\alpha\nu}+\tfrac{1}{2}\overset{(8)}{\{\mu\delta,\delta\}}\{\alpha\epsilon,\epsilon\}g^{\alpha\nu}+\tfrac{1}{2}\overset{(3)}{\{\alpha\delta,\delta\}}\{\mu\epsilon,\epsilon\}g^{\alpha\nu}$$
$$-\overset{(1)}{\{\alpha\gamma,\epsilon\}}\{\mu\epsilon,\gamma\}g^{\alpha\nu}+\overset{(7)}{\{\alpha\mu,\gamma\}}\{\gamma\epsilon,\epsilon\}g^{\alpha\nu}$$
$$-\tfrac{1}{2}\overset{(4)}{\{\delta\mu,\alpha\}}\{\alpha\beta,\nu\}g^{\delta\beta}+\tfrac{1}{4}\overset{(5)}{\{\delta\mu,\nu\}}\{\beta\epsilon,\epsilon\}g^{\delta\beta}+\tfrac{1}{4}\overset{(2)}{\{\delta\mu,\alpha\}}\{\alpha\epsilon,\epsilon\}g^{\delta\nu}$$
$$-\tfrac{1}{2}\overset{(4)}{\{\delta\mu,\beta\}}\{\alpha\beta,\nu\}g^{\delta\alpha}+\tfrac{1}{4}\overset{(2)}{\{\delta\mu,\beta\}}\{\beta\epsilon,\epsilon\}g^{\delta\nu}+\tfrac{1}{4}\overset{(5)}{\{\delta\mu,\nu\}}\{\alpha\epsilon,\epsilon\}g^{\delta\alpha}$$
$$+\tfrac{1}{2}\overset{(6)}{\{\mu\delta,\delta\}}\{\alpha\beta,\nu\}g^{\alpha\beta}-\tfrac{1}{4}\overset{(8)}{\{\mu\delta,\delta\}}\{\beta\epsilon,\epsilon\}g^{\beta\nu}-\tfrac{1}{4}\overset{(8)}{\{\mu\delta,\delta\}}\{\alpha\epsilon,\epsilon\}g^{\alpha\nu}]$$
$$=0$$

where the value zero arises, after some changes in dummy suffixes, from the mutual cancellation of all terms, as can easily be verified by noting that the terms have been labelled in such a way that those which destroy each other are given the same number.

Combining this result with equation (17), we now complete the derivation of the original equation (7), which it was the purpose of this section to prove.

§4. The Limiting Values of Certain Quantities at Large Distances from an Isolated Material System

As a further preparation for our later considerations we shall now obtain the limiting values at large distances from an isolated material system for the quantities $\mathfrak{g}^{\alpha\beta}(\partial\mathfrak{L}/\partial\mathfrak{g}^{\alpha\beta}_{\gamma})$ and $\mathfrak{g}^{\alpha 4}(\partial\mathfrak{L}/\partial\mathfrak{g}^{\alpha 4}_{\gamma})$ which occur on the right hand side of equation (7). To do this we recall that we have defined an isolated system in §2, in such a way that the changes taking place within the system do not produce changes in the gravitational field outside of a sufficiently distant boundary. Therefore, at a sufficient distance from an isolated system the gravitational field will be static and spherically symmetrical, and we can use for it the well known Schwarzschild solution. Hence placing the system in the neighborhood of the origin of a set of coordinates x, y, z, t, which approach Galilean coordinates at large distances we can write for the line element the approximate Schwarzschild expression[16]

$$ds^2 = -(1+2m/r)(dx^2+dy^2+dz^2)+(1-2m/r)dt^2$$

where the constant m is the mass of the system and r is an abbreciation for $(x^2+y^2+z^2)^{1/2}$.

Since the above expression gives the form taken by the Schwarzschild solution in a particular kind of quasi-Galilean coordinates, it is evident that later results, which are dependent on the present section, will also be originally derived in the form which they assume in these particular coordinates, and their translation into the language of other systems of coordinates must be undertaken with due cognizance of their method of derivation. Furthermore, the above expression is an approximate one, valid at distances large enough so that terms of the order $(m/r)^2$ can be neglected, and yet at the same time small enough so that the curvature of the universe as a whole, as given by the cosmological term, can be neglected. Hence those later considerations which are dependent on this section will primarily apply only to systems which are small compared to the total dimensions of the universe, even though of course still very large compared with ordinary terrestrial dimensions.

Returning to our approximate expression for the Schwarzschild line element we may now write for the components of the fundamental metrical tensor, the values

$$\begin{aligned} &g_{11}=g_{22}=g_{33}=-[1+2m/r] && g_{44}=[1-2m/r] \\ &g^{11}=g^{22}=g^{33}=-\frac{1}{[1+2m/r]} && g^{44}=\frac{1}{[1-2m/r]} \\ &g_{\mu\nu}=g^{\mu\nu}=0 \qquad (\mu\neq\nu) && \sqrt{-g}=[1+2m/r]^{3/2}[1-2m/r]^{1/2} \end{aligned} \tag{18}$$

To calculate the desired quantities from these expressions for the metrical tensor, we shall first need the values of the Christoffel symbols. These are defined by the equation [17]

[16] See Eddington, reference 1, equation (46.15).

[17] See Eddington, reference 1, equation (27.2).

$$\{\mu\nu,\sigma\} = \frac{1}{2} g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda}\right)$$

which evidently reduces under our circimstances to

$$\{\mu\nu,\sigma\} = \frac{1}{2} g^{\sigma\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right) \qquad \text{(not summed)}$$

and taking μ, ν, and σ as *different* indices we obtain the four cases

$$\begin{aligned} \{\mu\mu,\mu\} &= \frac{1}{2} g^{\mu\mu} \frac{\partial g_{\mu\mu}}{\partial x_\mu} \\ \{\mu\mu,\nu\} &= -\frac{1}{2} g^{\nu\nu} \frac{\partial g_{\mu\mu}}{\partial x_\nu} \\ \{\nu\mu,\mu\} &= \{\mu\nu,\mu\} = \frac{1}{2} g^{\mu\mu} \frac{\partial g_{\mu\mu}}{\partial x_\nu} \\ \{\mu\nu,\sigma\} &= 0. \end{aligned} \qquad (19)$$

We are now ready to proceed to our calculations. Under our circumstances we may evidently write

$$\mathfrak{g}^{\alpha\beta}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{\alpha\beta}_1} = \sqrt{-g}\left(g^{11}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{11}_1} + g^{22}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{22}_1} + g^{33}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{33}_1} + g^{44}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{44}_1}\right).$$

Substituting the values given by equation (13), this becomes

$$\mathfrak{g}^{\alpha\beta}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{\alpha\beta}_1} = \sqrt{-g}\Big(-g^{11}\{11,1\} + \frac{1}{2}g^{11}\{1\epsilon,\epsilon\} + \frac{1}{2}g^{11}\{1\epsilon,\epsilon\}$$
$$-g^{22}\{22,1\} - g^{33}\{33,1\} - g^{44}\{44,1\}\Big)$$

and introducing the values of the Christoffel symbols given by equations(19) it reduces to

$$\mathfrak{g}^{\alpha\beta}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{\alpha\beta}_1} = \frac{1}{2}\sqrt{-g}\Big(g^{11}g^{22}\frac{\partial g_{22}}{\partial x} + g^{11}g^{33}\frac{\partial g_{33}}{\partial x} + g^{11}g^{44}\frac{\partial g_{44}}{\partial x}$$
$$+ g^{22}g^{11}\frac{\partial g_{22}}{\partial x} + g^{33}g^{11}\frac{\partial g_{33}}{\partial x} + g^{44}g^{11}\frac{\partial g_{44}}{\partial x}\Big).$$

Finally, substituting the values for the components of the metrical tensor given by equations (18), and, because of the large distances under consideration, neglecting quantities of the order m/r in comparison with unity, we easily obtain

$$\mathfrak{g}^{\alpha\beta}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_1{}^{\alpha\beta}} = -2\frac{\partial}{\partial x}\left(\frac{2m}{r}\right) + \frac{\partial}{\partial x}\left(\frac{2m}{r}\right) = \frac{2m}{r^2}\frac{\partial r}{\partial x}\,.$$

From considerations of symmetry it is evident that we shall also have similar expressions in y and z, and introducing the direction cosines for the radius vector we may evidently write

$$\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{1}}=\frac{2m}{r^2}\cos(nx)\,,\quad \mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{2}}=\frac{2m}{r^2}\cos(ny)\,,\quad \mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{3}}=\frac{2m}{r^2}\cos(nz)\quad(20)$$

while the value of the fourth quantity $\mathfrak{g}^{\alpha\beta}(\partial\mathfrak{L}/\partial\mathfrak{g}^{\alpha\beta}_4)$ will not be needed for our later work.

Turning now to the second of the quantities of interest mentioned at the beginning of this section, we may evidently write with the help of equations (18), (13) and (19)

$$\mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha 4}_{1}}=\sqrt{-g}\,g^{44}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{44}_{1}}=-\sqrt{-g}\,g^{44}\{44,1\}=\frac{1}{2}\sqrt{-g}\,g^{44}g^{11}\frac{\partial g_{44}}{\partial x}$$

or, to the same order of approximation as before, we obtain

$$\mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha 4}_{1}}=-\frac{m}{r^2}\cos(nx)\,,\quad \mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha 4}_{2}}=-\frac{m}{r^2}\cos(ny)\,,\quad \mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha 4}_{3}}=-\frac{m}{r^2}\cos(nz)\,.\quad(21)$$

§5. Relation Between the Energy and Mass of an Isolated System

We are now ready to return to our discussion of the energy principle by giving a deduction from our present basis of Einstein's relation between the energy and mass of an isolated material system. Using the quasi-Galilean coordinates described in the last section, we may write for the energy of the system, in accordance with equation (6),

$$U=\iiint(\mathfrak{T}^4_4+\mathfrak{t}^4_4)\,dx\,dy\,dz$$

and this quantity will be constant independent of the time t as shown in §2, provided the boundary of the region of integration is taken sufficiently distant from the system. Substituting the value for total energy density given by equation (7), we can now rewrite our expression for the energy in the form

$$U=\frac{1}{8\pi}\iiint\frac{\partial}{\partial x_\gamma}\left(-\mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha 4}_{\gamma}}+\frac{1}{2}\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{\gamma}}\right)dx\,dy\,dz\,.$$

Taking the boundary of the region of integration as a sphere with its center at the origin of coordinates, using Gauss's theorem to transform to a surface integral, and substituting at the distant boundary the three values for the term in parenthesis found in the previous section as given by equations (20) and (21), this can evidently be rewritten in the form

$$U=\frac{1}{8\pi}\iint\frac{2m}{r^2}[\cos^2(nx)+\cos^2(ny)+\cos^2(nz)]\,dS$$
$$+\frac{1}{8\pi}\frac{\partial}{\partial t}\iiint\left(-\mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha 4}_{4}}+\frac{1}{2}\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{4}}\right)dx\,dy\,dz\,.$$

Since, however, both U and the surface integral on the right hand side of the equation are constants independent of the time,[18] it is evident that the second term on the right hand side of the equation must be zero, as the volume integral in the second term could not continue to change permanently at a constant rate with the time. Hence evaluating the surface integral, we easily obtain for an isolated system the simple relation

$$U=m \tag{22}$$

where U is Einstein's expression for the energy of the system, and m is the mass which must be substituted into the Schwarzschild solution to give the gravitational field at large distances. This appropriate result is itself no mean justification for Einstein's formulation of the energy principle.

§6. The Energy of a Quasi-static Isolated System Expressed by an Integral Extending Only Over the Occupied Space

For certain purposes both of the expressions for the energy of an isolated system, $\iiint(\mathfrak{T}_4^4+\mathfrak{t}_4^4)dxdydz$ and m, are somtimes unsatisfactory, the first because the integration has to be extended over a region large compared with the actual system, owing to the fact that $\mathfrak{t}_4^4$ is in general not zero in free space, and the second because it gives no method of computing the energy from the actual distribution of matter or radiation within the system. For a particular class of systems, which we shall call quasistatic, a more usable expression can be obtained.

Starting once more with our fundamental equation (6) for the energy of an isolated system, we write

$$U=\iiint(\mathfrak{T}_4^4+\mathfrak{t}_4^4)dxdydz$$

where we again use the quasi-Galilean coordinates defined in §4. Substituting the expression for the density of potential energy $\mathfrak{t}_4^4$ given by equation (9) this can be rewritten in the form[19]

$$U=\iiint\left(\mathfrak{T}_4^4+\frac{\mathfrak{L}}{16\pi}-\frac{1}{16\pi}\mathfrak{g}_4^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_4^{\alpha\beta}}\right)dxdydz$$

and introducing the expression for $\mathfrak{L}$ given by equation (15) this becomes

$$U=\iiint\left[\mathfrak{T}_4^4-\frac{\mathfrak{G}}{16\pi}+\frac{1}{16\pi}\frac{\partial}{\partial x_\gamma}\left(\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_\gamma^{\alpha\beta}}\right)-\frac{1}{16\pi}\mathfrak{g}_4^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_4^{\alpha\beta}}\right]dxdydz.$$

[18] The quantity m must be a constant, since it determines the gravitational field at the distante boundary, and by our definition an isolated system produces a constant gravitational field at the distant boundary.

[19] We omit the cosmological term in the expressions for $\mathfrak{t}_4^4$ and $\mathfrak{T}$ since our present considerations already contain the assumption that the system is small compared with the total dimensions of the universe.

Substituting now for $\mathfrak{G}$ the well-known expression[19]

$$\mathfrak{G}=8\pi\mathfrak{T}=8\pi(\mathfrak{T}_1^1+\mathfrak{T}_2^2+\mathfrak{T}_3^3+\mathfrak{T}_4^4)$$

writing the third term of the integrand out in full, and combining with the last term, we then easily obtain

$$\begin{aligned} U=&\frac{1}{2}\iiint(\mathfrak{T}_4^4-\mathfrak{T}_1^1-\mathfrak{T}_2^2-\mathfrak{T}_3^3)dxdydz \\ &+\frac{1}{16\pi}\iiint\left[\frac{\partial}{\partial x}\left(\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_1^{\alpha\beta}}\right)+\frac{\partial}{\partial y}\left(\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_2^{\alpha\beta}}\right)+\frac{\partial}{\partial z}\left(\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_3^{\alpha\beta}}\right)\right]dxdydz \qquad (23) \\ &+\frac{1}{16\pi}\iiint\mathfrak{g}^{\alpha\beta}\frac{\partial}{\partial t}\left(\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_4^{\alpha\beta}}\right)dxdydz. \end{aligned}$$

The second integral on the right hand side of this equation can be evaluated, however, by taking the boundary of the region of integration as a sphere, transforming to a surface integral and substituting at the distant boundary the values given by equations (20). We thus easily obtain $m/2$ as the value of the second integral and since for an isolated system this is itself equal by equation (22) to $U/2$ as shown in the preceding section, we can rewrite equation (23) in the form

$$U=\iiint(\mathfrak{T}_4^4-\mathfrak{T}_1^1-\mathfrak{T}_2^2-\mathfrak{T}_3^3)dxdydz+\frac{1}{8\pi}\iiint\mathfrak{g}^{\alpha\beta}\frac{\partial}{\partial t}\left(\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_4^{\alpha\beta}}\right)dxdydz \qquad (24)$$

Finally, let us now define a quasi-static system as one in which changes with the time are taking place sufficiently slowly so that the last term in this equation can be neglected in comparison with the others, as will of course be exactly true in case we are only interested in the energy of the system at times when it is in a quiescent state of temporary or permanent equilibrium. We can then write as the desired expression[20] for the energy of an isolated quasi-static system

$$U=\iiint(\mathfrak{T}_4^4-\mathfrak{T}_1^1-\mathfrak{T}_2^2-\mathfrak{T}_3^3)dxdydz \qquad (25)$$

[20] This expression for energy was first given by Nordström, Proc. Amster. Acad. **202,** 1080 (1918). The derivation given for the expression, however, was unsatisfactory since it was made to depend on an equation which the author ascribed to von Laue without any citation of the place of publication nor statement as to its range of validity. I have myself not been able to find the source of this expression and Professor von Laue has informed me by letter that he has been unable to find such an expression in the second volume of his book on relativity, does not know how he could have arrived at such an equation, now where else he might have set it forth, and is inclined to believe that Nordström must have been in error concerning the reference. Attention should also be called in this connection to the limited range as to the nature of the system and the choice of coordinates which will make equation (25) valid.

where it is evident from the equation of definition

$$T^{\mu\nu}=\rho_0\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}$$

that the integration has to be extended only over the region actually occupied by matter or radiation.

§7. Energy of a Sphere of Perfect Fluid.

We may now use equation (25) to calculate the energy of a spherical distribution of perfect fluid in a static state of stable or metastable equilibrium.

Continuing to use our quasi-Galilean coordinates x, y, z, t, we may evidently write the line element for our spherically symmetrical and static system in the form

$$ds^2=-e^\mu\,(dx^2+dy^2+dz^2)+e^\nu dt^2 \tag{26}$$

where μ and ν are functions of $r=(x^2+y^2+z^2)^{1/2}$ and independent of t. In accordance with this line element we have for the components of the fundamental metrical tensor

$$\begin{aligned} &g_{11}=g_{22}=g_{33}=-e^\mu && g_{44}=e^\nu \\ &g^{11}=g^{22}=g^{33}=-e^{-\mu} && g^{44}=e^{-\nu} \\ &g_{\rho\sigma}=g^{\rho\sigma}=0 \quad (\rho\neq\sigma) && \sqrt{-g}=e^{\frac{3\mu+\nu}{2}} \end{aligned} \tag{27}$$

For the tensor of energy and momentum for a perfect fluid we have the well known equation[21]

$$T^{\rho\sigma}=(\rho_{00}+p_0)\frac{dx_\rho}{ds}\frac{dx_\sigma}{ds}-g^{\rho\sigma}p_0 \tag{28}$$

where ρ_{00} and p_0 are the proper *macroscopic* density and proper pressure of the fluid as measured by local observers, and the quantities (dx_ρ/ds) are *macroscopic* velocities. For our case the macroscopic velocities will all be zero except for the case $\rho=\sigma=4$ and we shall then have

$$\left(\frac{dt}{ds}\right)^2=g^{44}=e^{-\nu}$$

so that the surviving components of the energy tensor will be

$$T^{11}=T^{22}=T^{33}=e^{-\mu}p_0 \qquad T^{44}=e^{-\nu}\rho_{00}$$

or lowering suffixes

$$T_1^1=T_2^2=T_3^3=-p_0 \qquad T_4^4=\rho_{00} \tag{29}$$

Multiplying these results by $\sqrt{-g}$ to change to tensor densities and substituting in equation (25), we now obtain for the energy of a steady spherical distribution of perfect fluid

[21] See Eddington, reference 1, equations (54.81) and (54.82).

$$U=\int\int\int(\rho_{00}+3p_0)e^{\frac{3\mu+\nu}{2}}dxdydz. \tag{30}$$

We also have, moreover, the general relation

$$dV_0ds = \sqrt{-g}dx_1dx_2dx_3dx_4$$

where dV_0 is the element of proper three dimensional volume, and in our case this reduces to

$$dV_0 = e^{\frac{3\mu+\nu}{2}}dxdydz\frac{dt}{ds} = e^{3\mu/2}\,dxdydz$$

so that equation (30) can be rewritten in the extremely simple form

$$U=\int(\rho_{00}+3p_0)\,e^{\nu/2}\,dV_0=\int(\rho_{00}+3p_0)\sqrt{g_{44}}dV_0. \tag{31}$$

§8. Approximate Expression for the Energy of a Sphere of Fluid in a Weak Gravitational Field

We shall now investigate the value of this expression, for the energy of a sphere of perfect fluid, under circumstances where the gravitational field is weak enough so that the Newtonian theory of gravitation is approximately valid. Such a condition can of course be achieved by taking the quantity of matter in the sphere sufficiently small.

Under these circumstances the line element will approach that for flat space-time and we may write the components of the metrical tensor in the form

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \tag{32}$$

where the quantities $\delta_{\mu\nu}$ are the Galilean values for the components of the metrical tensor, ± 1 or 0, and the quantities $h_{\mu\nu}$ are *small* deviations therefrom. With these values for the components of the metrical tensor, however, we may easily obtain a well-known relation between h_{44} and the Newtonian gravitational potential Ψ.

To do this let us consider the behavior of a test particle placed at the point of interest and then allowed to move freely under the action of the gravitational field. In accordance with the theory of relativity the motion of this free particle must correspond to a geodesic in space-time and hence be governed by the equation[23]

$$\frac{d^2x_\alpha}{ds^2} + \{\mu\nu,\alpha\}\frac{dx\mu}{ds}\frac{dx_\nu}{ds} = 0. \tag{33}$$

Since the particle starts from rest, however, the initial "velocities" dx_μ/ds and dx_ν/ds will all be zero except with μ or $\nu=4$, and taking first the case of $\alpha=1$, the above equation will reduce at the beginning of the motion to

[22] See Eddington, reference 1, equation (49.42).

[23] See Eddington, reference 1, equation (28.5).

$$\frac{d^2x}{ds^2} + \{44,1\}\left(\frac{dt}{ds}\right)^2 = 0. \tag{34}$$

Furthermore, we have for a particle at rest

$$\begin{aligned} ds^2 &= g_{44}dt^2 = (1 + h_{44})dt^2 \\ &\approx dt^2 \end{aligned} \tag{35}$$

and in general

$$\{44,1\} = \frac{1}{2}g^{\lambda 1}\left(\frac{\partial g_{4\lambda}}{\partial x_4} + \frac{\partial g_{4\lambda}}{\partial x_4} - \frac{\partial g_{44}}{\partial x_\lambda}\right)$$

which for our case reduces with sufficient approximation to

$$\{44,1\} = \frac{1}{2}\frac{\partial g_{44}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{h_{44}}{2}\right). \tag{36}$$

So that substituting (35) and (36) in (34) and writing the analogous equations with $\alpha = 2$ and 3, we easily obtain for the initial acceleration of the test particle the equations

$$\frac{d^2x}{dt^2} = -\frac{\partial}{\partial x}\left(\frac{h_{44}}{2}\right), \qquad \frac{d^2y}{dt^2} = -\frac{\partial}{\partial y}\left(\frac{h_{44}}{2}\right), \qquad \frac{d^2z}{dt^2} = -\frac{\partial}{\partial z}\left(\frac{h_{44}}{2}\right). \tag{37}$$

We note at once that $(h_{44}/2)$ satisfies the same differential equations of motion as the gravitational potential Ψ in the ordinary Newtonian theory, and since h_{44} tends to zero at large distances may now write

$$\frac{h_{44}}{2} = \Psi \tag{38}$$

provided we make the usual convention that the gravitational potential shall be zero at infinity. This equation connects the relativity and Newtonian methods of treatment, providing the gravitational field is weak enough to permit the use of the older method.

In addition to this equation, we shall also need to use a value for a certain integral which was known in the Newtonian theory but is sufficiently unfamiliar so that we shall now derive it. The integral in question is $3\int p dV$, where p is the pressure within the fluid and the integration is to be taken over the volume V of the whole sphere. We write

$$3\int p dV = 3\int_0^R 4\pi r^2 p\, dr$$

where R is the radius of the sphere, and by partial integration obtain

$$\begin{aligned} 3\int p dV &= 4\pi r^3 p\Big|_0^R - \int_0^R 4\pi r^3 dp \\ &= -\int_0^R 4\pi r^3\, dp \end{aligned}$$

where the first term has been dropped since r is zero at one limit and p at the other. It is evident, however, that the quantity $-4\pi r^2 dp$ is the total radial force acting outwards on the spherical shell of material dM_r lying between the radii r and $r+dr$ and hence can be equated to the gravitational attraction acting on this shell which gives us

$$3\int p dV = \int_0^R \frac{M_r}{r} dM_r.$$

Moreover, the quantity on the right hand side of this equation is obviously the work that would be necessary to remove the material of the sphere to infinity and hence the negative of its potential energy. So that we can finally write

$$3\int p dV = -\int \frac{\rho\Psi}{2} dV \tag{39}$$

where the integral on the right hand side of the equation is a well-known expression for potential energy, ρ being the density of material and Ψ the gravitational potential.

We are now ready to return to equation (31) which gave as an exact expression for the total energy of the sphere

$$U = \int (\rho_{00}+3p_0)\sqrt{g_{44}}\, dV_0.$$

In accordance with equations (32) and (38) we can write as approximately valid in the case of a weak field

$$\sqrt{g_{44}} = (1+h_{44})^{1/2} \approx 1 + \frac{h_{44}}{2} = 1+\Psi.$$

And this can be substituted above to give

$$U = \int \rho_{00} dV_0 + \int \rho_{00}\Psi dV_0 + 3\int p_0 dV_0 + 3\int p_0 \Psi dV_0.$$

In the case of a weak gravitational field, however, the potential Ψ is everywhere small compared with unity and for the case of ordinary matter in a weak gravitational field[24] p_0 is small compared with ρ_{00}. Hence the last term in the above expression may be dropped entirely and the next two preceding terms simplified by omitting the subscripts ($_0$) which specify a proper system of coordinates for the measuring of quantities. The result then becomes

$$U = \int \rho_{00} dV_0 + \int \rho\Psi dV + 3\int p dV$$

[24] This is not true for radiation where $p_0 = \rho_{00}/3$, so that our present considerations apply to a sphere in which the density of ordinary matter is large compared with that of radiation, which is of course the case for which the Newtonian potential energy was known.

and introducing the value of $3\int p dV$ given by equation (39), we finally obtain as an approximate expression for the energy of a sphere of perfect fluid in a weak gravitational field

$$U = \int \rho_{00} dV_0 + \int \frac{\rho \Psi}{2} dV. \tag{40}$$

In satisfactory agreement with older theory, the energy thus consists of two parts,—the first being the total proper energy of the material out of which the sphere is composed, and the second the well-known Newtonian expression for its potential gravitational energy.

As far as the writer is aware, this is the first case in which it has been shown that Einstein's exact relativity expression for the energy of a system is more closely approximated by including the Newtonian potential energy than by going at once to flat space-time. It is hoped that the reasonableness of this result will lead to increased confidence in the use of the energy momentum principle in general relativity.

§9. The Interpretation and Use of the Energy-Momentum Principle in the Case of Non-Isolated Systems

So far, the applications of the energy-momentum principle, which we have considered in the foregoing, have dealt with the conservation of energy for an isolated system enclosed within a distant boundary located in the free space surrounding the system, and these applications have been made largely in order to illustrate the reasonableness of Einstein's formulation of the principle. In applying the theory of relativity to the phenomena of nature, however we may often be interested in using the energy-momentum principle to give us information as to the changes which could take place within a limited region which forms part of a larger system. Hence in the present section we shall consider the interpretation and use of the energy-momentum principle when applied to such non-isolated systems, and in the following section we shall discuss specifically the changes which the energy-momentum principle would allow in the distribution of matter within a certain kind of non-isolated system.

Let us return to the fundamental differential equation, true in all sets of coordinates, which was given in §1

$$\frac{\partial}{\partial x_\nu} (\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu) = 0 \tag{41}$$

where $\mathfrak{T}_\mu^\nu$ is the tensor density of material energy and momentum and $\mathfrak{t}_\mu^\nu$ the pseudo-tensor density of potential gravitational energy and momentum. Taking x_4 to be the time like coordinate, this equation will lead to the momentum principles with $\mu = 1, 2, 3$ and to the energy principle with $\mu = 4$. For our present purposes, however, we shall leave μ unspecified since the following considerations are general enough to apply to the energy-momentum principle as a whole.

Multiplying the above equation by $dx_1dx_2dx_3$, and integrating over the system of interest we obtain with some rearrangement of terms

$$\frac{\partial}{\partial x_4}\iiint(\mathfrak{T}_\mu^4+\mathfrak{t}_\mu^4)dx_1dx_2dx_3$$
$$=-\iiint\left[\frac{\partial}{\partial x_1}(\mathfrak{T}_\mu^1+\mathfrak{t}_\mu^1)+\frac{\partial}{\partial x_2}(\mathfrak{T}_\mu^2+\mathfrak{t}_\mu^2)+\frac{\partial}{\partial x_3}(\mathfrak{T}_\mu^3+\mathfrak{t}_\mu^2)\right]dx_1dx_2dx_3$$

and, by performing the indicated integrations on the right hand side, this can be rewritten in the form

$$\frac{\partial}{\partial x_4}\iiint(\mathfrak{T}_\mu^4+\mathfrak{t}_\mu^4)dx_1dx_2dx_3$$
$$=-\iint\left|\mathfrak{T}_\mu^1+\mathfrak{t}_\mu^1\right|_{x_1}^{x_1'}dx_2dx_3-\iint\left|\mathfrak{T}_\mu^2+\mathfrak{t}_\mu^2\right|_{x_2}^{x_2'}dx_1dx_3-\iint\left|\mathfrak{T}_\mu^3+\mathfrak{t}_\mu^3\right|_{x_3}^{x_3'}dx_1dx_2 \tag{42}$$

where the limits of integration for the spatial variables x_1, x_2, x_3 are to be chosen so as to include the system of interest.

Equation (42) as written is true in all sets of coordinates, owing to its immediate dependence on the covariant equation (41). The interpretation and use of the equation, however, are often simplified if we choose coordinates in such a way that the limits of integration which must be taken in order to include the system of interest actually lie on the boundary surface which separates the region in question from its surroundings. Thus for example quasi-Galilean coordinates x, y, z with the limits of integration x to x', y to y', and z to z' lying on the boundary of the system, are usually preferable for our present purposes to polar coordinates r, θ, ϕ, with the origin inside the system and the limits of integration 0 to r, 0 to π and 0 to 2π, in which case r is the only limit actually lying on the boundary. The increased simplicity of the properly chosen coordintes arises from the fact that the right hand side of equation (42) is then completely determined solely by the values assumed at the boundary by the quantities $\mathfrak{T}_\mu^1$, $\mathfrak{t}_\mu^1$ etc. and is not dependent on their values within the system.

Having chosen coordinates in the way suggested, the interpretation of equation (42) becomes very simple. The equation now states that the rate of change with time x_4 of the volume integral on the left hand side of the equation is equal to the quantity on the right hand side, whose value is entirely determined by the conditions prevailing at the boundary of the system. The left hand side of the equation can then be interpreted as the rate of change with the time of a component of the total momentum of the system, with $\mu=1$, 2, 3, or the rate of change of the energy of the system, with $\mu=4$; and the right hand side of the equation can be interpreted as the flux of momentum or energy through the boundary.

The use of equation (42) will also be facilitated by the suggested choice of coordinates, when we are interested in some process which takes place within our system under circumstances such that we have definite information as to the values of the quantities $\mathfrak{T}_\mu^1$, $\mathfrak{t}_\mu^1$ etc. at the boundary of the system, but do not have definite information as to the values they may assume in the interior of the system during the course of the process which interests us. In the following section we shall use coordinates of the kind suggested without further remark.

§10. Application of the Energy-Momentum Principle to the Static States of a System

Having thus obtained an indication as to the interpretation and use of the energy-momentum principle in the case of non-isolated systems, we shall now apply the principle to determine what restriction it would impose on the form of line element within a system which could exist in different static states. To solve this problem we apply the energy-momentum principle to the following process.

We start with a non-isolated system which together with its surroundings is originally in some given static state such that none of the components of the metrical tensor are changing with the time x_4. Without altering the metric or the distribution of matter and radiation outside of the system, we then assume a change to take place in the distribution of matter and energy inside the system in such a way that the system ultimately arrives in some new possible static state. In the absence of the detailed knowledge, which would permit us to describe the exact mechanism of the internal process that takes place, we now inquire into the restrictions which the energy-momentum principle, as applied to the system as a whole, would impose on the changes in the form of line element inside the system which could accompany such a process.

The first condition on the possible changes in the line element is imposed by the hypothesis that the metric and the distribution of matter and radiation outside the system are not to be changed by the process, this hypothesis being introduced since our interest will lie in those changes which could take place solely within the system without affecting anything in the outside surroundings. As an immediate result of the assumption that the metric is not to be changed outside of the system, it is at once evident that the process must produce no change in the values at the boundary of the components $g_{\mu\nu}$ of the metrical tensor and their first differential coefficients $\partial g_{\mu\nu}/\partial x_\alpha$. And as a result of the assumption that the distribution of matter and radiation is not to be changed outside of the system, it is evident that the energy-momentum tensor $T^{\mu\nu}$ will remain unchanged at the boundary since it is completely determined by the distribution of matter and radiation in accordance with the equation of definition

$$T^{\mu\nu} = \rho_0 \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} \cdot \qquad (43)$$

The second condition on the possible changes in the line element is imposed by the requirement that the process taking place within the system shall agree with the energy-momentum principle as applied to the system as a whole. This condition is given by our previous equation

$$\frac{\partial}{\partial x_4}\iiint(\mathfrak{T}_\mu^4+\mathfrak{t}_\mu^4)dx_1dx_2dx_3$$
$$=-\iint\left|\mathfrak{T}_\mu^1+\mathfrak{t}_\mu^1\right|_{x_1}^{x_1'}dx_2dx_3-\iint\left|\mathfrak{T}_\mu^2+\mathfrak{t}_\mu^2\right|_{x_2}^{x_2'}dx_1dx_3-\iint\left|\mathfrak{T}_\mu^2+\mathfrak{t}_\mu^3\right|_{x_3}^{x'_3}dx_1dx_2. \tag{44}$$

To apply this equation, we note that at the start of the process when the system is in its original static state, the left hand side of the equation is obviously equal to zero since none of the quantities involved can be changing with the time. Thus at the start of the process the values at the boundary of the quantities $\mathfrak{T}_\mu^1 \cdots \mathfrak{t}_\mu^3$ must be such as to make the right hand side also equal to zero. The quantities $\mathfrak{T}_\mu^1$, $\mathfrak{T}_\mu^2$ and $\mathfrak{T}_\mu^3$, however, are determined by the $g_{\mu\nu}$ and $T^{\mu\nu}$ and the quantities $\mathfrak{t}_\mu^1$, $\mathfrak{t}_\mu^2$ and $\mathfrak{t}_\mu^3$ by the $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial x_\alpha$, and hence in accordance with the last paragraph the values which they assume at the boundary will not change during the process. The result is that both sides of equation (44) remain equal to zero throughout the process, and the condition imposed by the energy-momentum principle reduces to the requirement of constant energy and components of momentum for the system as given by the equation

$$\iiint(\mathfrak{T}_\mu^4+\mathfrak{t}_\mu^4)dx_1dx_2dx_3=\text{const.} \tag{45}$$

To investigate the effect of this condition on the possible changes in line element which could occur, we shall now reexpress the integrand by substituting for it the expression given by equation (7) in §3. We thus obtain

$$\iiint\frac{\partial}{\partial x_\gamma}\left(-\mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\alpha}_{\gamma}}+\frac{1}{2}g_\mu^4\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{\gamma}}\right)dx_1dx_2dx_3=\text{const.} \tag{46}$$

In this equation, however, the suffix γ is a dummy and the term with $\gamma=4$ will be zero at the beginning and end of our process, since by hypothesis the system is at these times a static one and the rate of change of all quantities with respect to the time x_4 is zero. Hence, writing the summation out in full and performing the possible integrations with respect to x_1, x_2 and x_3 which are indicated, we can now express the condition imposed on the form of the line element by stating that for the static states at the beginning and end of the process we must have the relation

$$\begin{aligned}&\iint\left|-\mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\alpha}_{1}}+\frac{1}{2}g_\mu^4\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{1}}\right|_{x_1}^{x_1'}dx_2dx_3\\&+\iint\left|-\mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\alpha}_{2}}+\frac{1}{2}g_\mu^4\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{2}}\right|_{x_2}^{x_2'}dx_1dx_3\\&+\iint\left|-\mathfrak{g}^{\alpha 4}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\mu\alpha}_{3}}+\frac{1}{2}g_\mu^4\mathfrak{g}^{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}_{3}}\right|_{x_3}^{x_3'}dx_1dx_2=\text{const.}\end{aligned} \tag{47}$$

where the limits of integration x_1, x_1' etc. lie on the boundary of the system. The quantities occurring in the integrands of this equation, however, are completely determined by the values of the components $g_{\mu\nu}$ of the metrical tensor and their first differential coefficients, as can be verified from equations already given in this article. Hence the condition imposed by the energy-momentum principle has already been met by our previous requirement necessary to preserve the metric outside that the values at the boundary for the $g_{\mu\nu}$ and their first differential coefficients should not be changed by the process.

Hence, noting the restriction given in §9 as to the kind of coordinate systems that we employ, we can now state the following otherwise general conclusion. Let us start with a non-isolated system, which together with its surroundings is originally in some given static state, and consider that some process then takes place which leaves the metric and the distribution of matter and radiation outside the system unaltered but changes the matter and radiation inside the system in such a way that we finally arrive in some new static state. Assuming no detailed knowledge as to the nature of the internal process that has occurred, but applying the energy-momentum principle to the system as a whole, we then find that the requirements imposed by the energy-momentum principle on the possible changes in line element within the system are to be met by the condition that the components $g_{\mu\nu}$ of the metrical tensor and their first differential coefficients $\partial g_{\mu\nu}/\partial x_\alpha$ are to retain their values unaltered at the boundary.[25]

§11. Conclusion

This concludes the material which it was desired to present in the present article. It is hoped that the general coherence of the treatment and the specific satisfactory result, as to the inclusion of the ordinary Newtonian expression for potential gravitational energy in the case of a fluid sphere in a weak gravitational field, will increase the confidence with which the energy principle is used in general relativity.

Some apology should perhaps be offered for the great length of the two preceding sections on the application of the energy principle to the changes that may take place within a limited region which forms part of a larger system. There are, however, a number of points connected with the development which have seemed puzzling enough to warrant a detailed exposition and the final conclusion is one of considerable usefulness.

It should perhaps also be remarked that we have not treated in the foregoing any questions which involve the energy of the closed universe as a whole since these deserve separate consideration.[26]

[25] In connection with the formulation of this section, I have been greatly helped by discussions with my colleague Dr. J. Robert Oppenheimer, for which I wish to thank him, also in this place.

[26] See Einstein, Berl. Ber. 1918, p. 452, and Tolman, Proc. Nat. Acad. **14**, 348 (1928).