ON THE USE OF THE INTERPOLATION POLYNOMIAL FOR SOLUTIONS OF SINGULAR INTEGRAL EQUATIONS*

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Abstract. On the basis of integration of singular integral equations by means of Gaussian quadrature, it is demonstrated how to obtain the corresponding approximate polynomial solution. For some special cases compact formulas are given for the strength of the singularities at the endpoints of the integration interval.

1. Introduction. For singular integral equations of the form

$$af(x) + \frac{b}{\pi} \int_{-1}^{1} f(t) \frac{dt}{t - x} + \int_{-1}^{1} k(t, x)f(t) \, dt = g(x), \qquad -1 < x < 1, \qquad (1.1)$$

where $g(x)$ and $k(t, x)$ are known Hölder-continuous functions and a and b are real constants, two effective ways of numerical solution have been devised [1–4]. They both result in an approximate solution of the form

$$f(x) = w(x)F(x) \simeq w(x) \sum_{j=0}^{n-1} c_i p_i(x), \qquad (1.2)$$

where $w(x)$ is the fundamental function of (1.1), and p_i, $j = 0, \cdots, n - 1$, are the orthogonal polynomials associated with $w(x)$. It is easily shown [5] that $w(x)$ is of the form

$$w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}, \qquad -1 < \alpha, \beta < 1, \qquad (1.3)$$

and accordingly $p_i(x)$ are the Jacobi polynomials denoted by $P_i^{(\alpha,\beta)}(x)$ [6].

Although the two methods yield results of the same form, the underlying principles are different. The first method is based upon the development of $F(x)$ as an infinite series

$$F(x) = \sum_{j=0}^{\infty} C_j P_j^{(\alpha,\beta)}(x). \qquad (1.4)$$

By use of the orthogonality relation [6]

$$\int_{-1}^{1} P_i^{(\alpha,\beta)}(t)P_k^{(\alpha,\beta)}(t)(1 - t)^{\alpha}(1 + t)^{\beta} \, dt = h_i \, \delta_{ik}, \qquad (1.5)$$

$$h_i = \frac{2^{\alpha+\beta+1}}{2j + \alpha + \beta + 1} \frac{\Gamma(j + \alpha + 1)\Gamma(j + \beta + 1)}{j! \, \Gamma(j + \alpha + \beta + 1)}, \qquad (1.6)$$

an infinite system of linear equations is obtained for the determination of the coefficients C_j [1, 4]. As a rule numerical evaluation of integrals must be used extensively. By trunca-

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tion an approximate system of equations is obtained for the determination of the n coefficients C_j, $j = 0, 1, \cdots, n - 1$.

The second method starts with an approximating polynomial of finite degree,

$$F(x) \simeq \sum_{j=0}^{n-1} c_j P_j^{(\alpha,\beta)}(x). \tag{1.7}$$

By applying the quadrature formulas given in [2] and [3], approximate values of $F(x)$ are obtained in the points x_i determined by $P_n^{(\alpha,\beta)}(x_i) = 0$, $i = 1, 2, \cdots, n^*$. It is important to note that this is accomplished without performing numerical evaluation of integrals.

The present paper gives simple summation formulas for the determination of the coefficients c_j in (1.7). For the important cases $(\alpha, \beta) = (\pm 1/2, \pm 1/2)$, which correspond to $a = 0$, special formulas are derived in terms of Chebyshev polynomials. These include formulas for direct determination of $F(1)$ and $F(-1)$.

2. General formulas. Polynomials orthogonal over a finite interval satisfy a recurrence relation which may be put in the form [6, 8]

$$a_{j-1}p_{j-1}(x) + b_j p_j(x) + a_j p_{j+1}(x) = x p_j(x). \tag{2.1}$$

In the following, $p_j(x)$, $j = 0, 1, \cdots, n - 1$, will be considered as normalized. By choosing x as

$$p_n(x_i) = 0, \quad i = 1, 2, \cdots, n, \tag{2.2}$$

and defining $p_{-1}(x) \equiv 0$, (2.1) may be written using a symmetrical tridiagonal matrix

$$\begin{bmatrix} b_0 & a_0 & & & & \\ a_0 & b_1 & a_1 & & & \\ & a_1 & b_2 & a_2 & & \\ & & & a_{n-3} & b_{n-2} & a_{n-2} \\ & & & & a_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} p_0(x_i) \\ p_1(x_i) \\ p_2(x_i) \\ \\ p_{n-1}(x_i) \end{bmatrix} = x_i \begin{bmatrix} p_0(x_i) \\ p_1(x_i) \\ p_2(x_i) \\ \\ p_{n-1}(x_i) \end{bmatrix}. \tag{2.3}$$

The vectors $[p_0(x_i), p_1(x_i), \cdots, p_{n-1}(x_i)]$, $i = 1, 2, \cdots, n$, are eigenvectors corresponding to the eigenvalues x_i. These vectors are normalized by calculating

$$\sum_{j=0}^{n-1} p_j^2(x_i) = \frac{1}{\lambda_{in}}. \tag{2.4}$$

By using the Christoffel–Darboux formula [6], the λ_{in}, $i = 1, 2, \cdots, n$, are seen to be the weights from the corresponding Gaussian quadrature formula. As the matrix $\{\sqrt{\lambda_{in}}\, p_j(x_i)\}$ is orthogonal with respect to both rows and columns, we obtain the formula

$$\sum_{i=1}^{n} \lambda_{in} p_j(x_i) p_k(x_i) = \delta_{jk}. \tag{2.5}$$

Just as C_j, $j = 0, 1, \cdots$ are determined by the continuous orthogonality relation, c_j, $j = 0, 1, \cdots, n - 1$, are determined by use of the discrete orthogonality relation (2.5). The result is

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* For $\alpha = \beta = \pm 1/2$ the connection between C_j and c_j is discussed in [7].

$$F(x) \simeq \sum_{j=0}^{n-1} c_i p_i(x), \qquad c_i = \sum_{i=1}^{n} \lambda_{in} p_i(x_i) F(x_i). \tag{2.6}$$

For the Jacobi polynomials (2.6) is slightly modified:

$$F(x) \simeq \sum_{j=0}^{n-1} c_i P_i^{(\alpha,\beta)}(x), \qquad c_i = \frac{1}{h_j} \sum_{i=0}^{n} \lambda_{in} P_i^{(\alpha,\beta)}(x_i) F(x_i), \tag{2.7}$$

where h_j is given by (1.6) and λ_{in} by [3, 4]

$$\lambda_{in} = -2^{\alpha+\beta} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(n+1)! \, \Gamma(n+\alpha+\beta+2)} \frac{2n+\alpha+\beta+2}{P_n^{(\alpha,\beta)\prime}(x_i)P_{n+1}^{(\alpha,\beta)}(x_i)}. \tag{2.8}$$

3. Special cases. The cases $(\alpha, \beta) = (\pm 1/2, \pm 1/2)$ are special because the corresponding Jacobi polynomials can be expressed by Chebyshev polynomials. This gives the possibility of direct determination of x_i from (2.2) by means of trigonometric functions [4]. Furthermore, the index j may be summed analytically in the calculation of $F(1)$ and $F(-1)$. These values are of special physical importance, for example, in evaluation of stress intensity factors [4] and [9–11].

In the following the summation sign $\sum'$ means that the first term must be multiplied by 1/2. Use of $\sum''$ implies that both first and last terms must be multiplied by 1/2. The derivation of the formulas is based on the trigonometric definition of the Chebyshev polynomials and the two relations

$$\sum_{j=0}^{n} {}'' \cos(j\theta) = \tfrac{1}{2} \sin(n\theta) \cot(\tfrac{1}{2}\theta), \tag{3.1}$$

$$\sum_{j=0}^{n-1} \cos[(2j+1)\theta] = \frac{1}{2} \frac{\sin(2n\theta)}{\sin\theta}. \tag{3.2}$$

$\alpha = \beta = -1/2$:

$$F(x) \simeq \sum_{j=0}^{n-1} {}' c_i T_i(x),$$

$$c_i = \frac{2}{n} \sum_{i=1}^{n} T_i(x_i)F(x_i) = \frac{2}{n} \sum_{i=1}^{n} \cos\left[\frac{2i-1}{2n} j\pi\right] F(x_i),$$

$$x_i = \cos\left[\frac{2i-1}{2n}\pi\right]. \tag{3.3}$$

$$F(1) \simeq \frac{1}{n} \sum_{i=1}^{n} U_{2n-2}[(1+x_i)/2]^{1/2}F(x_i)$$

$$\frac{1}{n} \sum_{i=1}^{n} \frac{\sin\left[\dfrac{2n-1}{4n}(2i-1)\pi\right]}{\sin\left[\dfrac{2i-1}{4n}\pi\right]} F(x_i). \tag{3.4}$$

$$F(-1) \simeq \frac{1}{n} \sum_{i=1}^{n} U_{2n-2}[(1-x_i)/2]^{1/2}F(x_i)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{\sin\left[\dfrac{2n-1}{4n}(2i-1)\pi\right]}{\sin\left[\dfrac{2i-1}{4n}\pi\right]} F(x_{n+1-i}). \tag{3.5}$$

$\alpha = \beta = 1/2$:

$$F(x) \simeq \sum_{i=0}^{n-1} c_i U_i(x),$$

$$c_i = \frac{2}{n+1} \sum_{i=1}^{n} (1 - x_i^2) U_i(x_i) F(x_i)$$

$$= \frac{2}{n+1} \sum_{i=1}^{n} \sin\left[\frac{i}{n+1}\,\pi\right] \sin\left[\frac{i}{n+1}(j+1)\pi\right] F(x_i),$$

$$x_i = \cos\left[\frac{i}{n+1}\,\pi\right]. \tag{3.6}$$

$$F(1) \simeq \sum_{i=1}^{n} (1 + x_i) U_{n-1}(x_i) F(x_i)$$

$$= \sum_{i=1}^{n} \cot\left[\frac{i}{n+1}\frac{\pi}{2}\right] \sin\left[\frac{n}{n+1}\,i\pi\right] F(x_i). \tag{3.7}$$

$$F(-1) \simeq \sum_{i=1}^{n} (1 - x_i) U_{n-1}(-x_i) F(x_i)$$

$$= \sum_{i=1}^{n} \cot\left[\frac{i}{n+1}\frac{\pi}{2}\right] \sin\left[\frac{n}{n+1}\,i\pi\right] F(x_{n+1-i}). \tag{3.8}$$

$\alpha = -\beta = 1/2$:

$$F(x) \simeq \sum_{i=0}^{n-1} c_i U_{2i}[(1+x)/2]^{1/2},$$

$$c_i = \frac{2}{2n+1} \sum_{i=1}^{n} (1 - x_i) U_{2i}[\sqrt{(1+x_i)/2}] F(x_i)$$

$$= \frac{4}{2n+1} \sum_{i=1}^{n} \sin\left[\frac{i}{2n+1}\,\pi\right] \sin\left[\frac{i}{2n+1}(2j+1)\pi\right] F(x_i).$$

$$x_i = \cos\left[\frac{2i}{2n+1}\,\pi\right]. \tag{3.9}$$

$$F(1) \simeq \sum_{i=1}^{n} U_{2n-2}[\sqrt{(1+x_i)/2}] F(x_i)$$

$$= \sum_{i=1}^{n} \frac{\sin\left[\dfrac{2n-1}{2n+1}\,i\pi\right]}{\sin\left[\dfrac{i}{2n+1}\,\pi\right]} F(x_i) \tag{3.10}$$

$$F(-1) \simeq \frac{2}{2n+1} \sum_{i=1}^{n} \sqrt{(1-x_i)/2}\; U_{2n-1}[\sqrt{(1-x_i)/2}] F(x_i)$$

$$= \frac{2}{2n+1} \sum_{i=1}^{n} \cot\left[\frac{2i-1}{2n+1}\frac{\pi}{2}\right] \sin\left[\frac{n}{2n+1}(2i-1)\pi\right] F(x_{n+1-i}). \tag{3.11}$$

$\alpha = -\beta = -1/2$:

$$F(x) \simeq \sum_{j=0}^{n-1} c_j \sqrt{2/(1+x)}\, T_{2j+1}[\sqrt{(1+x)/2}],$$

$$c_i = \frac{4}{2n+1} \sum_{i=1}^{n} \sqrt{(1+x_i)/2}\, T_{2j+1}[\sqrt{(1+x_i)/2}]F(x_i)$$

$$= \frac{4}{2n+1} \sum_{i=1}^{n} \cos\left[\frac{2i-1}{2n+1}\frac{\pi}{2}\right]\cos\left[\frac{2i-1}{2n+1}(2j+1)\frac{\pi}{2}\right]F(x_i),$$

$$x_i = \cos\left[\frac{2i-1}{2n+1}\pi\right]. \tag{3.12}$$

$$F(1) \simeq \frac{2}{2n+1} \sum_{i=1}^{n} \sqrt{(1+x_i)/2}\, U_{2n-1}[\sqrt{(1+x_i)/2}]F(x_i)$$

$$= \frac{2}{2n+1} \sum_{i=1}^{n} \cot\left[\frac{2i-1}{2n+1}\frac{\pi}{2}\right]\sin\left[\frac{n}{2n+1}(2i-1)\pi\right]F(x_i). \tag{3.13}$$

$$F(-1) \simeq \sum_{i=1}^{n} U_{2n-2}[\sqrt{(1-x_i)/2}]F(x_i)$$

$$= \sum_{i=1}^{n} \frac{\sin\left[\dfrac{2n-1}{2n+1}i\pi\right]}{\sin\left[\dfrac{i}{2n+1}\pi\right]} F(x_{n+1-i}) \tag{3.14}$$

4. Example. The equation for this example is taken from [2]:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt - \lambda\int_{-1}^{x}f(t)\,dt \doteq 1, \qquad -1 < x < 1. \tag{4.1}$$

$f(x)$ is the shear stress acting on the interface between an elastic coverplate and an elastic half-space, when the half-space is loaded parallel to the plate at infinity. Of special interest is the stress intensity factor at $x = 1$ represented by $F(1)$. In [2] values of $F(1)$ were obtained by three-point extrapolation for $\lambda = 10/3$, $1/3$, and $n = 20$, 40, 60. In Table 1 these values are given together with the similar values obtained by use of (3.4). The difference is not great, a fact that follows from the smoothness of $F(x)$. That greater differences must be expected for less smooth functions is clearly seen from the weights in (3.4), which are given in Table 2 for $n = 20$. In particular, when inferences are made from the values of $F(1)$ concerning the convergence of the method of quadrature, the formulas presented here must be used.

TABLE 1

The strength of the stress singularity, $F(1)$.

$F(1)$	$\lambda = 10/3$		$\lambda = 1/3$	
n	[1]	(3.4)	[1]	(3.4)
20	0.4061	0.4076	0.8323	0.8325
40	0.4104	0.4108	0.8331	0.8332
60	0.4115	0.4117	0.8340	0.8341

TABLE 2

Weights for the calculation of $F(1)$ for $\alpha = \beta = -1/2$ and $n = 20$ (3.4).

i	$U_{2n-2}((1 + x_i)/2)^{1/2}$	i	$U_{2n-2}((1 + x_i)/2)^{1/2}$
1	25.4517	11	0.9244
2	−8.4490	12	−0.7883
3	5.0273	13	0.6682
4	−3.5457	14	−0.5600
5	2.7106	15	0.4610
6	−2.1692	16	−0.3689
7	1.7856	17	0.2820
8	−1.4966	18	−0.1989
9	1.2685	19	0.1184
10	−1.0818	20	−0.0393

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