On the value distribution of composite meromorphic functions

Walter Bergweiler and Chung-Chun Yang

Dedicated to Professor Klaus Habetha
on the occasion of his 60th birthday

1 Introduction and main result

Let f and g be transcendental entire functions and let p be a nonconstant polynomial. A recent result of Bergweiler [2] says that the function $f(g(z)) - p(z)$ has infinitely many zeros, confirming a conjecture of Gross [11] dealing with the special case $p(z) = z$. The case that $f(g)$ is of finite order follows from either of the earlier results of Gol'dberg and Prokopovich [8], Goldstein [9], Gross and Yang [15], and Mues [19]. In fact, various generalizations are obtained in these papers. In particular, it follows from each of these papers that if f and g are entire, p is a nonconstant polynomial, and $f(g(z)) - p(z)$ has only finitely many zeros, then either f is linear or there exists a polynomial q such that $p = q(g)$, provided $f(g)$ is transcendental and of finite order (the restriction on the order not being essential, as shown in [4]). This latter result does not hold for meromorphic f, even if $f(g)$ has finite order, as shown by the example $f(z) = i\sqrt{z}\tan\sqrt{z}$, $g(z) = z^2$, and $p(z) = z$.

It is natural to conjecture, however, that the function $f(g(z)) - R(z)$ has infinitely many zeros, if f is meromorphic and transcendental, g is entire and transcendental, and R is rational and nonconstant. As proved in [3], this is in fact the case if $R(z) = z$ and hence for any Möbius transformation R. The method used in [3], however, does not seem to be suitable to handle the case that the degree of R is greater than one.

In this paper, we give an affirmative answer to the above question in the case that $f(g)$ has finite order.

Theorem *Let f be meromorphic and transcendental, let g be entire and transcendental, and let R be rational and nonconstant. If $f(g)$ has finite order,* *then the function $f(g(z)) - R(z)$ has infinitely many zeros.*

Our proof is based on a theorem of Steinmetz [21]. This result has proved to be a powerful tool in the study of questions of the above type (cf. [1, 12, 18, 23, 24]), if the functions under consideration satisfy certain growth restrictions. In the papers cited, however, it has been applied only in the case of entire functions. One of the purposes of this paper is to show that Steinmetz's theorem can be used to obtain results for meromorphic functions as well, although some additional difficulties are encountered.

We note that if f is entire, then the conclusion of our theorem remains valid if R is constant, cf. [10] and [20]. It is easy to show that this is not the case if f is meromorphic.

We shall assume that the reader is familiar with the basic results of Nevanlinna theory (cf. [16, 17]) and we shall freely use standard notations of this theory such as $T(r, f)$ and $N(r, f)$.

2 A result of Steinmetz

The result of Steinmetz referred to in the introduction is the following.

Lemma (Steinmetz [21, Satz 1, Korollar 1]) *Let $F_0, F_1, \ldots, F_m$ be not identically vanishing meromorphic functions and let $h_0, h_1, \ldots, h_m$ be meromorphic functions that do not all vanish identically. Let g be a nonconstant entire function and suppose that there exists a positive constant K such that*

$$\sum_{j=0}^{m} T(r, h_j) \leq KT(r, g) + S(r, g),$$

where $S(r, g) = o(T(r, g))$ as $r \to \infty$ outside some exceptional set of finite measure. Suppose also that

$$F_0(g)h_0 + F_1(g)h_1 + \ldots F_m(g)h_m = 0.$$

Then there exist polynomials $P_0, P_1, \ldots, P_m$ that do not all vanish identically such that

$$P_0(g)h_0 + P_1(g)h_1 + \ldots P_m(g)h_m = 0$$

and there exist polynomials $Q_0, Q_1, \ldots, Q_m$ that do not all vanish identically such that

$$Q_0F_0 + Q_1F_1 + \ldots Q_mF_m = 0.$$

Generalizations and different proofs of this result have been given by Brownawell [5] and Gross and Osgood [13, 14].

3 Proof of the theorem

First we note that f has order zero and g has finite order by a result of Edrei and Fuchs [7, Corollary 1.1]. By f_1 and f_2 we denote the canonical products of the zeros and poles of f. Then f_1 and f_2 have order zero. We may assume without loss of generality that $f(0) = 1$, that is, $f = f_1/f_2$.

Suppose now that $f(g(z)) - R(z)$ has only finitely many zeros. Then $f_1(g(z)) - R(z)f_2(g(z))$ has only finitely many zeros and poles so that

$$f_1(g(z)) - R(z)f_2(g(z)) = q(z)e^{\alpha(z)} \tag{1}$$

for a rational function q and an entire function α.

First we show that the left side of (1) has finite lower order. To do this, we note that because g has finite order, we can apply a result of Edrei and Fuchs [7, Theorem 1] and Valiron [22] and deduce that

$$n\left(M(r,g), \frac{1}{f}\right) \leq n\left(r^{1+\varepsilon}, \frac{1}{f(g)}\right) + O(1)$$

and

$$n\left(M(r,g), f\right) \leq n\left(r^{1+\varepsilon}, f(g)\right) + O(1)$$

if $\varepsilon > 0$ (cf. [17, p. 148]). We define $N(r) = N(r, 1/f_1) + N(r, 1/f_2)$, that is, $N(r) = N(r, 1/f) + N(r, f)$. Since f has order zero, there exist arbitrary large r such that

$$\frac{N(t)}{\sqrt{t}} \leq \frac{N(M(r,g))}{\sqrt{M(r,g)}}$$

for $t \geq M(r,g)$. Well-known estimates of canonical products (cf. [16, p. 102]) show that

$$\begin{aligned}
& \log M(M(r,g), f_1) + \log M(M(r,g), f_2) \\
\leq\ & M(r,g) \int_{M(r,g)}^{\infty} \frac{N(t)}{t^2} dt \\
\leq\ & N(M(r,g)) \sqrt{M(r,g)} \int_{M(r,g)}^{\infty} \frac{1}{t^{3/2}} dt \\
=\ & 2N(M(r,g))
\end{aligned}$$

for these r. Since $N(r,1/f_j) \le n(r,1/f_j)\log r + O(1)$ and $M(r,f_j(g)) \le M(M(r,g),f_j)$ for $j=1,2$, we obtain

$$\begin{aligned} & \log M(r,f_1(g)) + \log M(r,f_2(g)) \\ \le\ & 2\left(n\left(M(r,g),\frac{1}{f}\right) + n(M(r,g),f)\right)\log M(r,g) + O(1) \\ \le\ & 2\left(n\left(r^{1+\varepsilon},\frac{1}{f(g)}\right) + n\left(r^{1+\varepsilon},f(g)\right)\right)\log M(r,g) + O(\log M(r,g)) \\ \le\ & r^{(1+\varepsilon)(\rho(f(g))+\varepsilon)+\rho(g)+\varepsilon} \end{aligned}$$

for arbitrary large r, where $\rho(f(g))$ and $\rho(g)$ denote the order of $f(g)$ and g, respectively. We deduce that the left side of (1) has finite lower order.

It follows that e^{α} has finite lower order and this implies that α is a polynomial. Differentiating (1) we find that

$$f_1'(g)g' - R'f_2(g) - f_2'(g)g'R = (q' + q\alpha')e^{\alpha} = \left(\frac{q'}{q} + \alpha'\right)(f_1(g) - Rf_2(g)),$$

that is,

$$f_1'(g)g' - f_2'(g)g'R - f_1(g)\left(\frac{q'}{q} + \alpha'\right) + f_2(g)\left(\left(\frac{q'}{q} + \alpha'\right)R - R'\right) = 0.$$

Steinmetz's result shows that there are polynomials P_0, P_1, P_2, and P_3 that do not all vanish identically such that

$$P_0(g)g' - P_1(g)g'R - P_2(g)\left(\frac{q'}{q} + \alpha'\right) + P_3(g)\left(\left(\frac{q'}{q} + \alpha'\right)R - R'\right) = 0.$$

We deduce that $g'(z) = P(g(z),z)$ for some function P which is rational in $g(z)$ and z. (Here we have used the hypothesis that R is nonconstant.) A theorem of Malmquist (cf. [17, p. 170]) implies that this differential equation for g is actually a Riccati equation. Because g is entire, we can deduce that P has the form $P(g(z),z) = a(z)g(z) + b(z)$ where a and b are rational functions. It follows that if A is an antiderivative of a, then

$$g(z) = e^{A(z)}\left(\int_{z_0}^{z} b(t)e^{-A(t)}dt + c\right) \tag{2}$$

for suitable constants c and z_0. Suppose that $A(z) = A_0 e^{i\theta} z^d + o(|z|^d)$ as $|z| \to \infty$ where $A_0 > 0$. Then g has order d, in fact, we have $T(r,g) \sim \gamma r^d$ for some positive γ as $r \to \infty$. Moreover, we deduce from (2) that if $\cos(\theta + d\varphi) < 0$, then $|g(re^{i\varphi})| < r^\beta$ for some positive β and all sufficiently large r. Since f_1 and f_2 have order zero, we deduce that if $\cos(\theta + d\varphi) < 0$ and $\varepsilon > 0$, then

$$\begin{aligned}\operatorname{Re}\alpha(re^{i\varphi}) &= \log|f_1(g(re^{i\varphi}) - R(re^{i\varphi})f_2(g(re^{i\varphi})| \\ &\leq \log M(r^\beta, f_1) + \log M(r^\beta, f_2) + O(\log r) \\ &\leq r^\varepsilon\end{aligned}$$

for sufficiently large r. Choosing $\varepsilon < 1$ we find that α is either constant or $\alpha(z) = \alpha_0 e^{i\theta} z^d + o(|z|^d)$ as $|z| \to \infty$ where $\alpha_0 > 0$. In the latter case, we find $T(r, e^\alpha) \sim \alpha_0 r^d/\pi$ as $r \to \infty$. In any case, we have $T(r, e^\alpha) = O(T(r, g))$ as $r \to \infty$. Hence Steinmetz's theorem can be applied to the equation (1). It follows that there exist polynomials Q_1, Q_2, and Q_3 that do not all vanish identically such that

$$Q_1 f_1 + Q_2 f_2 = Q_3. \tag{3}$$

Suppose first that Q_2 does not vanish identically. Solving (3) for f_2, substituting in (1), and solving the resulting equation for $f_1(g)$ we find that

$$f_1(g) = \frac{qe^\alpha Q_2(g) + RQ_3(g)}{Q_2(g) + RQ_1(g)}.$$

But if Q_2 does vanish identically, then this equation follows directly from (3). Hence

$$T(r, f_1(g)) = T\left(r, \frac{e^\alpha Q_2(g) + RQ_3(g)}{Q_2(g) + RQ_1(g)}\right) = O(r^d) = O(T(r, g))$$

in any case. Combining this with (1), we find that

$$T(r, f_2(g)) = O(T(r, g)).$$

From a result of Clunie [6, Theorem 1] we can deduce now that f_1 and f_2 are polynomials. It follows that $f = f_1/f_2$ is rational, contradicting the hypothesis. This completes the proof.

References

[1] W. Bergweiler, Fixpoints of meromorphic functions and iterated entire functions, J. Math. Anal. Appl. 151 (1990), 261-274.

[2] W. Bergweiler, Proof of a conjecture of Gross concerning fix-points, Math. Z. 204 (1990), 381-390.

[3] W. Bergweiler, On the existence of fixpoints of composite meromorphic functions, Proc. Amer. Math. Soc., to appear.

[4] W. Bergweiler, On factorization of certain entire functions, in "The Mathematical Heritage of Carl Friedrich Gauß", World Scientific, Singapore, to appear.

[5] W. D. Brownawell, On the factorization of partial differential equations, Can. J. Math. 39 (1987), 825-834.

[6] J. Clunie, The composition of entire and meromorphic functions, in "Mathematical essays dedicated to A. J. Macintyre", Ohio University Press, Athens, Ohio, 1970, 75-92.

[7] A. Edrei and W. H. J. Fuchs, On the zeros of f(g(z)) where f and g are entire functions. J. Analyse Math. 12 (1964), 243-255.

[8] A. A. Gol'dberg and G. S. Prokopovich, On the simplicity of certain entire functions, Ukrain. Math. J. 22 (1970), 701-704 (translation from Ukrain. Mat. Zh. 22, 813-817).

[9] R. Goldstein, On factorisation of certain entire functions, II, Proc. London Math. Soc. 22 (1971), 483-506.

[10] F. Gross, On factorization of meromorphic functions, Trans. Amer. Math. Soc. 131 (1968), 215-222.

[11] F. Gross, Factorization of meromorphic functions, U. S. Government Printing Office, Washington, D. C., 1972.

[12] F. Gross and C. F. Osgood, On fixed points of composite entire functions, J. London Math. Soc. (2) 28 (1983), 57-61.

[13] F. Gross and C. F. Osgood, A simpler proof of a theorem of Steinmetz, J. Math. Anal. Appl. 143 (1989), 490-496.

[14] F. Gross and C. F. Osgood, An extension of a theorem of Steinmetz, J. Math. Anal. Appl. 156 (1991), 290-294.

[15] F. Gross and C. C. Yang, Further results on prime entire functions, Trans. Amer. Math. Soc. 142 (1974), 347-355.

[16] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford 1964.

[17] G. Jank and L. Volkmann, Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, Birkhäuser, Basel - Boston - Stuttgart 1985.

[18] K. Katajamäki, L. Kinnunen, and I. Laine, On the value distribution of composite entire functions, preprint.

[19] E. Mues, Über faktorisierbare Lösungen Riccatischer Differentialgleichungen, Math. Z. 121 (1971), 145-156.

[20] M. Ozawa, On the solution of the functional equation $f \circ g(z) = F(z)$, I, Kodai Math. Sem. Rep. 20 (1968), 159-162.

[21] N. Steinmetz, Über faktorisierbare Lösungen gewöhnlicher Differentialgleichungen, Math. Z. 170 (1980), 169-180.

[22] G. Valiron, Sur un théorème de M. Fatou, Bull. Sci. Math. 46 (1922), 200-208.

[23] C. C. Yang, Further results on the fix-points of composite transcendental functions, J. Math. Anal. Appl. 90 (1982), 259-269.

[24] C. C. Yang, On the fix-points of composite transcendental entire functions, J. Math. Anal. Appl. 108 (1985), 366-370.

Hong Kong University of Science & Technology
Department of Mathematics
Clear Water Bay
Kowloon
Hong Kong

Present address of W. B.:

Lehrstuhl II für Mathematik
RWTH Aachen
Templergraben 55
5100 Aachen
Germany

Email addresses: sf010be@dacth11.bitnet (W. B.)
mayang@usthk.bitnet (C.-C. Y.)