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ON THE VALUE DISTRIBUTION OF ENTIRE FUNCTIONS OF ORDER LESS THAN ONE

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§1. Tsuzuki [4] proved the following;

THEOREM A. *Let $f(z)$ be an entire function of order less than one and $\{w_n\}_{n=1}^{\infty}$ be a sequence such that $|w_n|\to\infty$ as $n\to\infty$. Suppose that there exists ω such that $0<\omega<\pi/2$ and all the roots of the equations*

$$f(z)=w_n \qquad (n=1, 2, \cdots)$$

lie in the angle $A(\omega)=\{z;\ |\arg z-\pi|<\omega\}$. Then $f(z)$ is linear.

The purpose of this note is to extend Theorem A and to prove the following.

THEOREM. *Let $f(z)$ be an entire function of order less than one and $\{w_n\}_{n=1}^{\infty}$ be a sequence such that $|w_n|\to\infty$ as $n\to\infty$. Suppose that all the roots of the equations*

$$f(z)=w_n \qquad (n=1, 2, \cdots)$$

lie in the upper half plane $\mathrm{Im}\, z\geqq 0$. Then $f(z)$ is a polynomial of degree not greater than two.

§2. **Proof of Theorem.** Suppose that $f(z)$ satisfies the conditions of Theorem and that $f(z)$ is transcendental. Without loss of generality, we may suppose that $w_1=0$, $f(0)\neq 0$ and we have

$$f(z)=\lambda\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)$$

where λ $(\neq 0)$ is a constant. Choose ω and η such that $0<\omega<\pi/2$, $\eta=\pi/2-\omega$. Then we have

$$f(z)=\lambda f_1(z)f_2(z)$$

where

$$f_1(z)=\prod_{j_1=1}^{\infty}\left(1-\frac{z}{z_{j_2}}\right) \quad (\eta<\arg z_{j_1}<\pi-\eta),$$

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$$f_2(z)=\prod_{j_2=1}^{\infty}\left(1-\frac{z}{z_{j_2}}\right) \quad (0\leqq \arg z_{j_2}\leqq \eta, \text{ or } \pi-\eta\leqq \arg z_{j_2}\leqq \pi).$$

Then we have

$$|f_1(z)| \longrightarrow +\infty \qquad \text{as } z\to\infty$$

in $\{z;\, |\arg z+\pi/2|<\eta\}$ [4]. Since

$$f_2(z)f_2(-z)=\prod_{j_2=1}^{\infty}\left(1-\frac{z^2}{z_{j_2}^2}\right)$$

is a function of z^2, we put

$$F(\zeta)=F(z^2)=f_2(z)f_2(-z)$$

with $\zeta=z^2$. Then the order of $F(\zeta)$ is less than 1/2 and the zeros of $F(\zeta)$ lies in $\{\zeta;\, |\arg\zeta|\leqq 2\eta\}$. Choosing δ such that $2\delta<\pi/2-2\eta$ $(\delta<\eta)$, we find that

$$|F(\zeta)| \longrightarrow +\infty \qquad \text{as } \zeta\to\infty$$

in $\{\zeta;\, |\arg\zeta-\pi|\leqq 2\delta\}$. Hence $f_2(z)$ is unbounded either on the ray $\arg z=+\pi/2-\varepsilon$ or on the ray $\arg z=-\pi/2-\varepsilon$ $(|\varepsilon|\leqq\delta)$. On the other hand by the location of the zeros of $f_2(z)$

$$|f_2(z)|\leqq|f_2(\bar{z})| \qquad \text{for } z\in\{z;\, |\arg z-\pi/2|\leqq\delta\}.$$

Thus $f_2(z)$ is unbounded either on the ray $\arg z=-\pi/2+\varepsilon$ or on the ray $\arg z=-\pi/2-\varepsilon(|\varepsilon|\leqq\delta)$.

Now we use the similar arguments to those used in the proof of Baker's theorem [1]. We consider

$$D=\frac{z\cdot f'(z)}{f(z)}=z\cdot\sum_{j=1}^{\infty}\frac{1}{z-z_j}$$

in $\{z;\, |\arg z+\pi/2|\leqq\delta\}$. Let K be a positive number such that $K\delta\geqq 2\pi$. If we set $z_j=r_je^{i\theta_j}$ $(0\leqq\theta_j\leqq\pi)$ and $z=re^{i(-\pi/2+\theta)}$ $(|\theta|\leqq\delta)$, then we have

$$\operatorname{Im}\frac{1}{z-z_j}=\frac{r\sin\left(\frac{\pi}{2}-\theta\right)+r_j\sin\theta_j}{r^2+r_j^2-2rr_j\cos\left(-\frac{\pi}{2}+\theta-\theta_j\right)}>0$$

and for each j

$$|z|\cdot\operatorname{Im}\frac{1}{z-z_j}\longrightarrow\sin\left(\frac{\pi}{2}-\theta\right) \qquad (z=re^{i\left(-\frac{\pi}{2}+\theta\right)},\ r\to\infty);$$

Thus there exists a positive number $r_1=r_1(K)$ such that

$$|D|\geqq|z|\cdot\operatorname{Im}\frac{f'(z)}{f(z)}>K$$

in $\{z;\, |\arg z+\pi/2|\leqq\delta,\ |z|>r_1\}$. We choose w_n such that $|f(z)|<|w_n|$ for $|z|\leqq r_1$. Let Ω be the region $\{w;\, |w|>|w_n|\}$. We consider the component $\sigma(\Omega)$ of $f^{-1}(\Omega)$ containing $\{z;\, \arg z=-\pi/2,\ |z|\geqq r_0\}$ where r_0 is a sufficiently large number. If

$\partial\sigma(\Omega)\frown\{z\,;\arg z=-\pi/2\pm\delta\}=\phi$, then we can define at least two asymptotic spots of $f(z)$ over ∞. In fact $f(z)$ is unbounded either on the ray $\arg z=-\pi/2+\varepsilon$ or on the ray $\arg z=-\pi/2-\varepsilon(|\varepsilon|\leqq\delta)$. Thus in view of Heins' main theorem [3] we can see that the order of $f(z)$ is not less than one. This contradicts the assumption of Theorem. Therefore we may assume that $\partial\sigma(\Omega)\frown\{z\,;|\arg z+\pi/2|\leqq\delta\}$ contains an arc of a level curve γ of $f(z)$ which joins a point of the ray $\arg z=-\pi/2-\delta$ to a point of the ray $\arg z=-\pi/2$ and lies in $|z|>r_1$. If an increment δz on γ corresponds to an increment δw on $|w|=|w_n|$ under $w=f(z)$, then we have

$$\frac{\delta w}{w}=\frac{\delta z}{z}\cdot\frac{z\cdot f'(z)}{f(z)}\{1+o(\delta z)\}\,.$$

Putting $z=re^{i\theta}$ and $w=|w_n|e^{i\varphi}$, we have

$$\left|\frac{\partial\varphi}{\partial\theta}\right|\geqq\left|\frac{z\cdot f'(z)}{f(z)}\right|\geqq K \qquad \text{on } \gamma\,.$$

In view of $f'(z)\neq 0$ on γ, as z traverses γ in the fixed direction, w traverses the circle Γ; $|w|=|w_n|$ in the fixed direction and φ increases or decreases at least $K\delta$. Thus w traverses the whole of Γ and in particular $f(z)=w_n$ for some point $z\in\gamma$. But this contradicts the assumption of Theorem. Hence if $f(z)$ satisfies the conditions of Theorem, $f(z)$ must be a polynomial. Then it is easy to show that the degree of $f(z)$ is at most two.

§ **3.** Edrei [**2**] proved the following;

THEOREM B. *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{w_n\}_{n=1}^{\infty}$ such that all the roots of the equations*

$$f(z)=w_n \qquad (n=1,2,\cdots)$$

be real. Then $f(z)$ is a polynomial of degree not greater than two.

In this section we shall prove Theorem B by the similar arguments to those used in the proof of our theorem instead of the main part of Edrei's proof.

The order of $f(z)$ is not greater than one (Corollary in [2]). We may assume that the order of $f(z)$ is one in view of our theorem and that $w_n\to\infty$ $(n\to\infty)$, $w_1=0$ and $f(0)\neq 0$. Then $f(z)$ may be expressed

$$f(z)=\lambda e^{az}\cdot\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)e^{\frac{z}{z_j}}$$

where λ $(\neq 0)$ is a constant.

Consider

$$D=\frac{zf'(z)}{f(z)}=z\cdot\sum_{j=1}^{\infty}\left(\frac{1}{z-z_j}+\frac{1}{z_j}\right)+az$$

in $\{z\,;|\arg z+\pi/2|\leqq\delta\}$ and in $\{z\,;|\arg z-\pi/2|\leqq\delta\}$. By the similar arguments to those used in the proof of our theorem we have

$$|D| \geqq |z| \cdot \left| \operatorname{Im} \frac{f'(z)}{f(z)} \right| > K$$

in $\{z ; |\arg z+\pi/2| \leqq \delta, |z|>r_1\}$ (or in $\{z ; |\arg z-\pi/2| \leqq \delta, |z|>r_1\}$), if $\operatorname{Im} a>0$ (or if $\operatorname{Im} a<0$).

Since

$$f(z)f(-z)=\lambda^2 \prod_{j=1}^{\infty}\left(1-\frac{z^2}{z_j^2}\right)$$

is a function of z^2, we have

$$F(\zeta)=F(z^2)=f(z)f(-z)$$

with $\zeta=z^2$. Then $F(\zeta)$ has only positive zeros and the order of $F(\zeta)$ is not greater than 1/2. If we choose δ sufficiently small $(2\delta<\pi/2)$, then we find that

$$|F(\zeta)| \longrightarrow +\infty \qquad \text{as } \zeta \to \infty$$

in $\{\zeta ; |\arg \zeta-\pi| \leqq 2\delta\}$. Therefore $f(z)$ is unbounded either on the ray $\arg z= -\pi/2-\varepsilon$ or on the ray $\arg z=\pi/2-\varepsilon (|\varepsilon| \leqq \delta)$.

Case 1. a is real.

Since $|e^{az}|=1$ on the rays $\arg z=\pm\pi/2$, we have

$$|f(z)|=\left|\lambda \prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)e^{\frac{z}{z_j}}\right|.$$

Hence we have $|f(z)| \to +\infty$ $(z\to\infty, \arg z=\pm\pi/2)$. Since $|f(z)|=|f(\bar{z})|$ and $|F(\zeta)| =|f(z)f(-z)| \to +\infty$ as $\zeta\to\infty$ in $|\zeta ; |\arg \zeta-\pi| \leqq 2\delta\}$, $f(z)$ is unbounded either on the ray $\arg z=-\pi/2-\delta$ or on the ray $\arg z=-\pi/2+\delta$. We choose w_n such that $|f(z)|<|w_n|$ for $|z| \leqq r_1$. Let Ω be the region $\{w ; |w|>|w_n|\}$. We consider the component $\sigma(\Omega)$ of $f^{-1}(\Omega)$ containing $\{z ; \arg z=-\pi/2, |z| \geqq r_0\}$ where r_0 is a sufficiently large number. If $\partial\sigma(\Omega) \cap \{z ; \arg z=-\pi/2 \pm \delta\}=\phi$, then we can define at least three asymptotic spots of $f(z)$ over ∞. In fact $f(z)$ is unbounded either on the ray $\arg z=-\pi/2+\delta$ or on the ray $\arg z=-\pi/2-\delta$ and $|f(z)| \to +\infty$ as $z\to\infty$ on the rays $\arg z=\pm\pi/2$. Therefore we have a contradiction by the similar reasonings to those used in the proof of our theorem.

Case 2. a is not real.

If $\operatorname{Im} a>0$, then we have $|e^{az}| \geqq 1$ in $\{z ; |\arg z+\pi/2| \leqq \delta\}$ for a sufficiently small number δ. Let

$$g(z)=\lambda \prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)e^{\frac{z}{z_j}} \qquad \left(=\frac{f(z)}{e^{az}}\right).$$

Since $|g(z)|=|g(\bar{z})|$ and $|G(\zeta)|=|g(z)g(-z)| \to +\infty$ as $\zeta\to\infty$ in $\{\zeta ; |\arg \zeta-\pi| \leqq 2\delta\}$ $(\zeta=z^2)$, $g(z)$ is unbounded either on the ray $\arg z=-\pi/2-\delta$ or on the ray $\arg z=-\pi/2+\delta$. We choose w_n such that $|f(z)|<|w_n|$ for $|z| \leqq r_1$. Let Ω be the region $\{w ; |w|>|w_n|\}$. We consider the component $\sigma(\Omega)$ of $f^{-1}(\Omega)$ containing $\{z ; \arg z=-\pi/2, |z| \geqq r_0\}$ where r_0 is a sufficiently large number. If

$\sigma(\Omega)\cap\{z; \arg z=-\pi/2\pm\delta\}=\phi$, then we can define at least three asymptotic spots of $g(z)$ over ∞. In fact $g(z)$ is unbounded either on the ray $\arg z=-\pi/2+\delta$ or on the ray $\arg z=-\pi/2-\delta$ and $|g(z)|\to+\infty$ as $z\to\infty$ on the rays $\arg z=\pm\pi/2$. Therefore we have a contradiction by the similar reasonings to those used in the proof of our theorem. If $\mathrm{Im}\, a<0$, then we have a contradiction by considering the region $\{z; |\arg z-\pi/2|\leqq\delta\}$ instead of $\{z; |\arg z+\pi/2|\leqq\delta\}$.

Therefore if $f(z)$ satisfies the conditions of Theorem B, then $f(z)$ must be a polynomial. Then it is easy to show that the degree of $f(z)$ is at most two.

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