The Annals of Mathematical Statistics
1972, Vol. 43, No. 3, 977–982

ON THE VARIANCE OF THE NUMBER OF ZEROS OF A STATIONARY GAUSSIAN PROCESS

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For a real, stationary Gaussian process $X(t)$, it is well known that the mean number of zeros of $X(t)$ in a bounded interval is finite exactly when the covariance function $r(t)$ is twice differentiable. Cramér and Leadbetter have shown that the variance of the number of zeros of $X(t)$ in a bounded interval is finite if $(r''(t) - r''(0))/t$ is integrable around the origin. We show that this condition is also necessary. Applying this result, we then answer the question raised by several authors regarding the connection, if any, between the existence of the variance and the existence of continuously differentiable sample paths. We exhibit counterexamples in both directions.

1. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $X(t, \omega)$, or just $X(t)$, a real, separable, stationary Gaussian process (SGP) on $(-\infty, +\infty) \times \Omega$ with covariance $r(t)$ and spectrum $F(d\lambda)$. The mean number of zeros of $X(t)$ in a finite interval is finite if and only if

$$\lambda_2 = \int \lambda^2 F(d\lambda) < \infty ; \tag{1}$$

equivalently, $r(t) \in C^{(2)}$ where $C^{(k)}$ denotes the class of real functions with a kth continuous derivative. In [2] it is shown that a sufficient condition for the existence of the variance of the number of zeros of $X(t)$ in a finite interval is

$$\int_0^\delta \frac{r''(t) - r''(0)}{t}\, dt < \infty \qquad \text{for some} \quad \delta > 0 . \tag{2}$$

Also, the question of necessity is raised. In Section 2 we show that (2) is, in fact, necessary. Section 3 deals with the *mean* number of zeros of $X(t)$ "given $X(0) = 0$."

Applying the result of Section 2, we then consider the relationship of the number of zeros of $X(t)$ to the smoothness of its trajectories. In the absence of (1) the trajectories are non-differentiable a.s. On the other hand, it is also known that if

$$r''(t) - r''(0) = 0(|\log t|^{-\alpha}) \qquad \text{as} \quad t \to 0^+ , \quad \text{some} \quad \alpha > 1 , \tag{3}$$

then $X(t) \in C^{(1)}$ a.s. Clearly, (2) and (3) are close. Consequently, Qualls [7] and Ylvisaker [9] have raised the question of whether the existence of the variance is either necessary or sufficient for $X(t) \in C^{(1)}$ a.s. In Section 4 we provide counterexamples in both directions.

2. A necessary condition for the variance. For convenience take $EX(t) = 0$, $r(0) = EX^2(t) = 1$, and concentrate $F(d\lambda)$ on $(0, \infty)$ so that

Received May 10, 1971; revised November 3, 1971.

$$r(t) = \int_0^\infty \cos \lambda t F(d\lambda) .$$

As we are assuming $\lambda_2 = -r''(0) < \infty$, it follows that $X(t)$ is differentiable in quadratic mean:

$$E\left[\frac{X(t+h) - X(t)}{h} - X'(t)\right]^2 \to 0 \qquad \text{as} \quad h \to 0$$

for every $t \in (-\infty, +\infty)$ where $X'(t)$ is a SGP with covariance function $-r''(t)$. Of course, E denotes expectation.

Throughout this section we will assume that $F(d\lambda)$ is not purely discrete. This insures that the joint normal distribution of $(X(0), X(t), X'(0), X'(t))$ is nonsingular (see [2] pages 203–204). In fact, the only available expression for the second moment of the number of zeros of $X(t)$ involves the mixed distribution of $X(t)$ and $X'(t)$. Specifically (see [1]),

$$EN^2(0, T) = EN(0, T) + \int_0^T \int_0^T dt_1\, dt_2 \int_{-\infty}^\infty \int_{-\infty}^\infty |xy| p_t(0, 0, x, y)\, dx\, dy , \tag{4}$$

where $N(0, T)$ denotes the number of zeros of $X(t)$, $t \in (0, T)$, p_t denotes the joint density of $(X(t_1), X(t_2), X'(t_1), X'(t_2))$, $t = |t_1 - t_2| > 0$, and the equality in (4) persists if either side is infinite.

THEOREM. *With $X(t)$ as above*

$$EN^2(0, T) < \infty \quad \textit{if and only if} \quad \int_0^\delta \frac{\lambda_2 + r''(t)}{t}\, dt < \infty$$

for some $\delta > 0$.

PROOF. The "if" part appears in [2], provided that $F(d\lambda)$ is not purely discrete.

For the converse, we first outline a standard reduction of the integral in (4). The joint density of $(X(t_1), X(t_2), X'(t_1), X'(t_2))$ depends only on the difference $|t_1 - t_2|$ so that, changing variables, the integral in (4) can be rewritten

$$2 \int_0^T (T - t)\Psi(t)\, dt , \qquad \Psi(t) = \int_{-\infty}^\infty \int_{-\infty}^\infty |xy| p_t(0, 0, x, y)\, dx\, dy ,$$

in which p_t is thus taken as the joint density of $(X(0), X(t), X'(0), X'(t))$. Moreover, it is easily seen that $\Psi(t)$ is integrable away from the origin; hence the problem reduces to showing that if $(\lambda_2 + r''(t))/t$ diverges on $(0, \delta)$ then so does $\Psi(t)$. (Notice that $\lambda_2 + r''(t) \geqq 0 \forall t$.) Now let $\Lambda = \Lambda(t)$ denote the covariance matrix corresponding to p_t. Writing p_t in terms of Λ and its cofactors, then changing variables we have

$$\Psi(t) = \frac{|\Lambda|^{\frac{3}{2}}}{4\pi^2 M_{33}^2} \int_{-\infty}^\infty \int_{-\infty}^\infty |xy| \exp - \tfrac{1}{2}(x^2 + y^2 - 2\rho xy)\, dx\, dy ,$$

where $|\Lambda| = \det \Lambda$, M_{ij} is the cofactor of Λ_{ij}, and $\rho = -M_{34}/M_{33}$. Change variables once again to obtain

$$\Psi(t) = \frac{|\Lambda|^{\frac{3}{2}}}{2\pi M_{33}^2 (1 - \rho^2)^{\frac{3}{2}}} \int_{-\infty}^\infty \int_{-\infty}^\infty |xy| \phi(x, y, \rho) dx\, dy ,$$

wherein $\phi(x, y, \rho)$ is the bivariate normal density with zero mean, unit variances, and covariance ρ. Finally, $|\Lambda|^{\frac{3}{2}}/((1-\rho^2)^{\frac{3}{2}}M_{33}^{\frac{3}{2}})$ reduces to $M_{33}/(1-r^2(t))^{\frac{3}{2}}$.

We can now make some estimates. Viewed as a function of ρ, the integral above, call it $\xi(\rho)$, is continuous at $\rho = 0$ and obviously $\xi(0) = 2\pi^{-1}$. So choose $\beta > 0$ such that $\xi(\rho) \geqq \pi^{-1}$ whenever $|\rho(t)| < \beta$. Observing that $\xi(\rho) > |\rho|$ for every t, and letting $S = \{|\rho(t)| < \beta\}$, $S^c = \{|\rho(t)| \geqq \beta\}$.

$$\begin{aligned}\int_0^\delta \Psi(t)\,dt &= \int_{(0,\delta)\cap S} \xi(\rho(t)) \frac{M_{33}(t)}{(1-r^2(t))^{\frac{3}{2}}}\,dt \\ &\quad + \int_{(0,\delta)\cap S^c} \xi(\rho(t)) \frac{M_{33}(t)}{(1-r^2(t))^{\frac{3}{2}}}\,dt \\ &\geqq \min(\pi^{-1}, \beta) \int_0^\delta \frac{M_{33}(t)}{(1-r^2(t))^{\frac{3}{2}}}\,dt\,.\end{aligned}$$

(Actually, it can be shown that $d\xi/d\rho = (2 \arcsin \rho)/\pi$ and hence $\xi(\rho) = 2[\rho \arcsin \rho + (1-\rho^2)^{\frac{1}{2}}]/\pi \geqq 2/\pi$. But our estimate will suffice.) Noting that $(1-r^2(t))^{\frac{3}{2}} \sim \lambda_2^{\frac{3}{2}} t^3$ we have shown that

$$\int_0^\delta \Psi(t)\,dt < \infty \Rightarrow \int_0^\delta \frac{M_{33}(t)}{t^3}\,dt < \infty\,.$$

To conclude the proof, observe that $M_{33}(t) = \lambda_2(1-r^2(t)) - (r'(t))^2 \in C^{(1)}$. Integrating by parts

$$\int_0^\delta \frac{M_{33}(t)}{t^3}\,dt = -\frac{M_{33}(t)}{2t^2}\Big|_0^\delta - \int_0^\delta \left(\frac{r'(t)}{t}\right)\left(\frac{\lambda_2 r(t) + r''(t)}{t}\right)dt\,.$$

But $\lambda_2(1-r^2(t))/t^2 \to \lambda_2^2$ as $t \to 0^+$ and $r'(t)/t \to -\lambda_2$ as $t \to 0^+$. It follows that $M_{33}(t)/t^3$ is integrable over $(0, \delta)$ exactly when $(\lambda_2 + r''(t))/t$ is integrable over $(0, \delta)$.

3. Conditional processes. We can view condition (2) in another light. Since $\lambda_2 < \infty$, we can condition $X(t)$ on the null set $(X(0) = 0)$ in the "horizontal-window" sense of Kac and Slepian [4]. The conditional "Palm" distribution may be viewed as the extension of the (consistent) set of finite-dimensional distributions

$$\lim_{\varepsilon \downarrow 0} P(\textstyle\bigcap_{i=1}^n X(t_i) \leqq \beta_i \,|\, N(-\varepsilon, 0) \geqq 1)\,, \qquad t_i \neq t_j,\ j = 1, \cdots, n\,.$$

The conditional process, call it $X_0(t)$, is neither stationary ($X_0(0) = 0$ a.s.) nor Gaussian but admits the decomposition ([8]):

$$X_0(t) = Z(t) + \lambda_2^{-\frac{1}{2}} r'(t)\xi \tag{5}$$

where $Z(t)$ is a Gaussian process, $EZ(t) \equiv 0$, $EZ(t)Z(s) = r(t-s) - r(t)r(s) - \lambda_2^{-1}r'(t)r'(s)$, and ξ is a single variable, independent of $Z(t)\forall t$, with the Rayleigh density $\frac{1}{2}|x| \exp -\frac{1}{2}x^2$ on $(-\infty, \infty)$. It can be shown (see [3]) that the variance of the number of zeros of $X(t)$, $0 \leqq t \leqq T$, is finite exactly when the *mean* number of zeros of $X_0(t)$ is finite in a neighborhood of the origin.

Now let $\hat{X}_0(t)$ denote $X(t)$ conditioned on $(X(0) = 0)$ in the ordinary sense so that the finite-dimensional distributions of $\hat{X}_0(t)$ are ratios of normal densities. A short calculation yields $E\hat{X}_0(t) \equiv 0$, $E\hat{X}_0(t)\hat{X}_0(s) = r(t-s) - r(t)r(s)$. Hence,

$$\hat{X}_0(t) = Z(t) + \lambda_2^{-\frac{1}{2}} r'(t)\hat{\xi}\,, \tag{6}$$

where $Z(t)$ is as in (5) and $\hat{\xi}$ is independent of $Z(t)\forall t$ but is now standard normal instead of Rayleigh.

Comparing (5) and (6), in particular the distribution of ξ and $\hat{\xi}$, conditions for the existence of the mean number of zeros of $\hat{X}_0(t)$, $t \in (0, T)$, ought to be similar, though more restrictive, than (2). From the proof in Section 1 we know that (2) holds precisely when

$$\int_0^\delta \frac{EZ^2(t)}{t^3}\, dt < \infty \qquad \text{for some} \quad \delta > 0 (EZ^2(t) = \lambda_2 M_{33}(t))\,.$$

After an easy calculation (applying the formula in [2] page 285) we find that the mean number of zeros of $\hat{X}_0(t)$ exists in a neighborhood of the origin if and only if

$$\int_0^\delta \frac{(EZ^2(t))^{\frac{1}{2}}}{t^2}\, dt < \infty \qquad \text{for some} \quad \delta > 0\,.$$

4. Two counterexamples. In view of the theorem, it is not difficult to specify a class of processes for which $X(t) \in C^{(1)}$ a.s. but $EN^2(0, T) = +\infty$. Our example will rely on the fact (see [7]) that (2) is equivalent to

$$\int_0^\infty [\log(1 + \lambda)]\lambda^2 F(d\lambda) < \infty\,. \tag{7}$$

To begin with, let $\{\alpha_n\}_{n\geqq 1}$ and $\{\beta_n\}_{n\geqq 1}$ be nonnegative sequences of real numbers, the former square summable. If $\{\eta_n\}_{n\geqq 1}$ and $\{\hat{\eta}_n\}_{n\geqq 1}$ are all independent and standard normal, then

$$X(t) = \sum_{n=1}^\infty \alpha_n(\eta_n \cos\beta_n t + \hat{\eta}_n \sin\beta_n t)$$

represents a SGP with zero mean and covariance

$$r(t) = EX(t)X(0) = \sum_{n=1}^\infty \alpha_n^2 \cos\beta_n t\,.$$

Obviously, $F(d\lambda)$ is purely discrete with jumps of magnitude α_n^2 at the points β_n, $n \geqq 1$. It is immediate that $\lambda_2 = -r''(0) < \infty$ whenever $\{\gamma_n\}_{n\geqq 1}$ is square summable, $\gamma_n = \alpha_n\beta_n$, $n \geqq 1$.
Assuming this,

$$-r''(t) = \sum_{n=1}^\infty \gamma_n^2 \cos\beta_n t\,.$$

Defer, momentarily, the problem of the discrete spectrum. The idea is to choose the sequences so that (7) fails but $X(t) \in C^{(1)}$ a.s. First, we provide for differentiability by strengthening our assumption on $\{\gamma_n\}_{n\geqq 1}$ from square summability to summability. To see that $X'(t)$ is continuous a.s., observe that

$$\sum_{n=1}^\infty \gamma_n E\{|\eta_n| + |\hat{\eta}_n|\} < \infty$$

so that

$$\sum_{n=1}^{\infty} \gamma_n\{\eta_n \cos \beta_n t + \hat{\eta}_n \sin \beta_n t\}$$

converges uniformly a.s. Now choose $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ such that

$$\int_0^\infty [\log (1 + \lambda)]\lambda^2 F(d\lambda)) = \sum_{n=1}^{\infty} \gamma_n^2 \log (1 + \beta_n) = +\infty .$$

(For example, take $\beta_n = 2^{2^n}$, $\alpha_n = (n^2 2^{2^n})^{-1}$.) Finally, let $Y(t)$ be any SGP independent of $X(t)$ such that $Y(t) \in C^{(1)}$ a.s. and the spectrum $G(d\lambda)$ of $Y(t)$ is continuous. Our example, then, is the SGP $X(t) + Y(t)$ with spectrum $F(d\lambda) + G(d\lambda)$.

Our second example, that of an SGP with $EN^2(0, T) < \infty$ but $X(t) \notin C^{(1)}$ a.s., could also be developed by way of random trigonometric series. However, applying the following recent result [5] on the continuity of Gaussian processes, it is no more difficult to find an example with an absolutely continuous spectrum.

THEOREM (Marcus and Shepp). *Let $X(t)$ be an* SGP *with covariance $\phi(\eta)$. Let $\Psi(\eta)$ be any non-decreasing function such that for some $\delta > 0$, $(\phi(0) - \phi(\eta))^{\frac{1}{2}} \geqq \Psi(\eta) \geqq 0$, $0 \leqq \eta \leqq \delta$. If the integral*

(8) $$\int_0^\delta \frac{\Psi(n)\, d\eta}{\eta|\log \eta|^{\frac{1}{2}}} = \infty ,$$

then the trajectories of $X(t)$ are discontinuous a.s.

Our plan, simply, is to locate a characteristic function $\phi(t) \in C^{(2)}$ for which (2) holds with $\phi''(t) - \phi''(0)$ in place of $\lambda_2 + r''(t)$ and for which (8) holds for some monotonic minorant $\Psi(t)$ of $[\phi''(t) - \phi''(0)]^{\frac{1}{2}}$.

Let $0 < \varepsilon < 1$ and choose $\xi > 0$ so that $(\log x)^{-1}(\log_2 x)^{-1-\varepsilon}(\log_2 x \equiv \log (\log x))$ decreases for $x \geqq \xi$. Now define

$$g_1(x) = -C_1 H'(|x|)/2 , \qquad |x| \geqq \xi$$
$$= 0 , \qquad |x| < \xi$$

where $H(x) = (\log x)^{-1}(\log_2 x)^{-1-\varepsilon}$, $x \geqq \xi$, and $C_1 = 1/H(\xi)$. $H(x)$ is slowly varying as $x \to \infty$, $g_1(x)$ is a probability density, and if F and ϕ are the corresponding distribution function and characteristic function, respectively, it follows that

$$1 - F(x) + F(-x) = C_1 H(x) , \qquad x \geqq \xi$$

which implies that

(9) $$1 - \phi_1(x) \sim C_1 H(1/x) \qquad \text{as} \quad x \downarrow 0 .$$

(See, e.g., [6].)

Now consider the probability density $g_2(x) = C_2 x^{-2} g_1(x)$ (where $g_2(x) = 0$, $|x| < \xi$, and C_2 is the normalizing constant), and corresponding characteristic function ϕ, so that

(10) $$-\phi_2''(x) = C_2 \phi_1(x) \forall x .$$

We may now apply the the theorem above. Regard ϕ_2 as the covariance of a SGP. Since $-\phi''(0) = \lambda_2 = C_2 < \infty$, $-\phi_2''$ is the covariance of the quadratic mean derivative $X'(t)$. By (9) and (10) there exists a $0 < \delta < 1/\xi$ such that

$$E(X'(t+\eta) - X'(t)]^2 = 2(C_2 + \phi_2''(\eta)) \geqq C_1 C_2 H(1/\eta) \equiv \Psi^2(\eta), \qquad 0 \leqq \eta \leqq \delta .$$

Obviously, $\Psi(\eta)$ is nonnegative and increasing. Moreover,

$$\int_0^\delta \frac{\Psi(\eta)\, d\eta}{\eta|\log \eta|^{\frac{1}{2}}} = (C_1 C_2)^{\frac{1}{2}} \int_0^\delta [\eta|\log \eta||\log_2 \eta|^{(1+\varepsilon)/2}]^{-1}\, d\eta = \infty .$$

Consequently, $X'(t)$ is discontinuous a.s.

On the other hand, by (9) and (10) again

$$\int_0^\delta \frac{C_2 + \phi_2''(\eta)}{\eta}\, d\eta \leqq 2C_1 C_2 \int_0^\delta [\eta|\log \eta||\log_2 \eta|^{1+\varepsilon}]^{-1}\, d\eta < \infty$$

for small $\delta > 0$. Hence $EN^2(0, T) < \infty$.

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