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On the volume growth and the topology of complete minimal submanifolds of a Euclidean space

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Abstract. Let M be a n-dimensional complete properly immersed minimal submanifold of a Euclidean space. We show that the number of the ends of M is bounded above by $k = \sup \frac{\text{volume}(M \cap B(t))}{\omega_n t^n}$, where $B(t)$ is the ball of the Euclidean space of center 0 and radius t, ω_n is the volume of n-dimensional unit Euclidean ball. Moreover, we prove that the number of ends of M is equal to k under some curvature decay condition.

1. Introduction

The study of complete minimal surfaces in Euclidean space has flourished for a long time and many results of their global geometric and topological structure have been obtained (for instance, [C.O], [O], [J.M]). Recently several author have begun the study of higher dimensional complete minimal submanifolds in Euclidean space. For example, Schoen([Sc]) studied the behaviour of minimal hypersurfaces at infinity and has proved a uniqueness theorem. Anderson ([A]) defined the generalized total scalar curvature of minimal submanifolds in Euclidean space and gave a generalization of Chern-Osserman theorem ([C.O]) to such submanifolds with finite total scalar curvature. The behaviour of minimal submanifolds at infinity is also investigated in [J.M]. At the same time, Kasue considered the *"gap phenomena"* for minimal submanifolds, and several uniqueness theorems are obtained in [K].[K.S] under the curvature decay conditions.

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Let M be a proper immersed complete minimal submanifold of a Euclidean space. The number of the ends of M is defined to be the supremum of the number of different components with non-compact closure of $M-C$, for every compact subset $C \subset M$. It is known the number of ends of M is the simplest topological invariant of M since M is non-compact. In some case, M can be uniquely determined by the behavior of its ends, for example, see [Sc] and [K] and [K,S].

The aim of this paper is to discuss some relation between the volume growth of a minimal submanifold of a Euclidean space and the number of its ends. By studying the tangent cone of a minimal submanifold at infinity, we find the number of its ends is controlled by its volume growth (Theorem 2.1). More precisely, the volume growth of a minimal submanifold determines the number of its ends if it looks like flat n-planes from infinity (Theorem 2.2).

We shall state the theorems in the next section and the proof of the theorems will be given in Section 3. At the end of the paper we discuss some related results.

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2. Curvature estimate and main theorems

Let M be a submanifold of an m-dimensional Euclidean space R^m equipped with the standard flat metric $\langle\ ,\ \rangle$. The covariant derivatives of R^m and M are denoted by D and ∇ respectively. And the second fundamental form of M is defined by

$$\begin{aligned} A :\ TM \otimes TM &\to\ T^{\perp}M \\ A(X,Y) &= D_X Y - \nabla_X Y\ . \end{aligned} \tag{2.1}$$

Let r be the distance function of R^m with respect to the origin, and $B(t)$ denote the open distance ball of radius t of R^m. Our main theorems are following:

Theorem 2.1. *Let M be an n-dimensional complete properly immersed minimal submanifold in R^m. Suppose*

$$\sup \frac{Vol(M \cap B(t))}{\omega_n t^n} < +\infty. \tag{2.2}$$

Then the number of ends of M, say $k(M)$, satisfies

$$k(M) \leq \sup \frac{Vol(M \cap B(t))}{\omega_n t^n}, \tag{2.3}$$

where ω_n is the volume of the unit ball in R^n.

THEOREM 2.2. *Let M be an n-dimensional complete properly immersed minimal submanifold in R^m which satisfies*

$$\lim_{t\to\infty} \sup_{\substack{x\in M \\ r(x)\geq t}} r(x)|A|(x) = 0. \tag{2.4}$$

Then

$$\lim_{t\to\infty} \frac{Vol(M \cap B(t))}{\omega_n t^n} = k(M) < +\infty \tag{2.5}$$

provided either of the following two conditions is satisfied:

(1) $n = 2,\ m = 3$ *and each end of M is embedded.*
(2) $n \geq 3$.

REMARK 2.3. It was showed in [B. deG. Gi] that for $n \geq 8$, there exist nonlinear minimal graphs $f : R^n \to R$ which have one end and satisfy

$$1 < \sup \frac{\mathrm{Vol}(M \cap B(t))}{\omega_n t^n} < +\infty.$$

So the curvature decay assumption (2.4) is necessary. For $n = 2,\ m = 3$, the embeddedness of each end of M is also necessary(a counter example is Enneper's surface).

The proof of the theorems will be given in the next section. First we show some lemmas.

Denote

$$f(t) = \sup_{\substack{x\in M \\ r(x)\geq t}} r(x)|A|(x), \tag{2.6}$$

and also write r for $r|_M$. We have the following estimate of the Hessian of r^2 on M. For any $X \in TM$, one has

$$
\begin{aligned}
\frac{1}{2}\nabla^2 r^2(X,X) &= \frac{1}{2}D^2 r^2(X,X) + \frac{1}{2}\langle A(X,X), \nabla^\perp r^2\rangle \\
&\geq |X|^2 - r|A(X,X)| \\
&\geq |X|^2(1-f(r)),
\end{aligned}
\tag{2.7}
$$

where $\nabla^\perp$ is the normal connection of M.

LEMMA 2.4. *Suppose M is as in Theorem 2.2. Then*
(1): M has finitely many ends, and each of its ends is of finite topological type.
(2): $\lim_{t\to\infty} \inf_{r\geq t} |\nabla r|^2 = 1$.

PROOF. (1): By assumption, there exists a real number $t_0 > 0$ such that $f(t) \leq f(t_0) < 1$ for all $t \geq t_0$. From (2.7) we know r^2 is a strictly convex function on $M - B(t_0)$, in particular ∇r never vanishes on $M - B(t_0)$. Let

$$\phi \ : \partial(M - B(t_0)) \times [t_0, +\infty) \to \ M - B(t_0)$$

be the integral flow of a vector field $\frac{\nabla r}{|\nabla r|^2}$ with

$$\phi(p, t_0) = p \ \in \partial(M - B(t_0)).$$

It is obvious that $r(\phi(p,t)) = t$ and

$$\phi(\cdot, t): \ \partial(M \cap B(t_0)) \to \partial B(M \cap B(t))$$

is a diffeomorphism. So (1) is true.

(2): Along any curve $\phi(p,t)$ ($t \in [t_0, +\infty)$) as above, we have

$$
\begin{aligned}
\frac{d}{dt} r|\nabla r|^2 &= |\nabla r|^2 + r\frac{d}{dt}|\nabla r|^2 \\
&= |\nabla r|^2 + 2r|\nabla r|^{-2}\langle \nabla_{\nabla r}\nabla r, \nabla r\rangle \\
&= |\nabla r|^2 + 2r|\nabla r|^{-2}(\nabla^2 r)(\nabla r, \nabla r) \\
&= |\nabla r|^2 + 2r\langle A(\nabla r, \nabla r), \nabla^\perp r\rangle |\nabla r|^{-2} + \\
&\qquad 2|\nabla r|^{-2}(|\nabla r|^2 - |\nabla r|^4) \\
&\geq 1 - 2f(r).
\end{aligned}
\tag{2.8}
$$

Therefore (2) follows from (2.8). □

Using Lemma 2.4, we shall work on each end separately, so denote by V one of the ends of M.

Let $V(t) = V \cap B(t)$ for each $t > t_0$, and $S(t) = \partial B(t)$. Then $\partial V(t)$ is an immersed closed submanifold in M (also in $S(t)$). The second fundamental form A_t of $\partial V(t)$ in M is given by

$$A_t(X,Y) = \langle \nabla_X Y, \frac{\nabla r}{|\nabla r|}\rangle \, \frac{\nabla r}{|\nabla r|} = -|\nabla r|^{-2}(\nabla^2 r(X,Y)) \, \nabla r. \tag{2.9}$$

The following lemma is essentially due to [K.S] and [A].

Lemma 2.5. *Suppose M is as in Theorem 2.2.*

(1): The second fundamental form of $\partial V(t)$ in $S(t)$, denoted by C_t, satisfies

$$|C_t| \leq \frac{\epsilon_1(t)}{t}$$

and $\lim_{t\to\infty} \epsilon_1(t) = 0$.

(2): $n \geq 3$, the sectional curvature of $\partial V(t)$ is bounded below by

$$\frac{1}{t^2}(1 - cf(t))$$

for some constant c.

Proof. (1): Observe that the second fundamental form of $S(t)$ in R^m is

$$\langle D_X Y, Dr\rangle Dr = -(D^2 r(X,Y))Dr$$

for any $X, Y \in TS(t)$, so that for $X, Y \in T(\partial V(t))$

$$\begin{aligned} C_t(X,Y) &= A(X,Y) + D^2 r(X,Y) Dr - \nabla^2 r(X,Y)\frac{\nabla r}{|\nabla r|^2} \\ &= A(X,Y) + D^2 r(X,Y) Dr - (D^2 r(X,Y) + \frac{o(1)}{t})\frac{\nabla r}{|\nabla r|^2}. \end{aligned}$$

Then (1) is follows from Lemma 2.4 (2).

(2): For any 2-plane $\pi(e_i, e_j) \subset T(\partial V(t))$, by (2.9), the sectional curvature of π is given by

$$K(e_i, e_j) = \langle A(e_i, e_i), A(e_j, e_j)\rangle - |A(e_i, e_j)|^2 + |\nabla r|^{-2}(\nabla^2 r(e_i, e_i)\nabla^2 r(e_j, e_j) - |\nabla^2 r(e_i, e_j)|^2).$$

Hence we have

$$\begin{aligned} K(e_i, e_j) &\geq \frac{-2f^2(t)}{t^2} + |\nabla r|^{-2}\{\frac{1}{t^2} + \frac{1}{t}\langle A(e_i, e_i) + A(e_j, e_j), \nabla^{\perp} r\rangle + \\ &\quad \langle A(e_i, e_i), \nabla^{\perp} r\rangle\langle A(e_j, e_j), \nabla^{\perp} r\rangle - \langle A(e_i, e_j), \nabla^{\perp} r\rangle^2\} \\ &\geq \frac{1}{t^2}(1 - 2f(t) - 2f^2(t)) - \frac{2f^2(t)}{t^2} \\ &\geq \frac{1}{t^2}(1 - cf(t)). \ \square \end{aligned}$$

Finally we recall the following well known fact:

Lemma 2.6.

$$Vol(M \cap B(t)) \leq \frac{t}{n} Vol\ (\partial(M \cap B(t))). \tag{2.10}$$

3. The proof of theorems

To prove Theorem 2.1 we need some facts from Geometric Measure Theory. For the definitions, the terminology and proofs see [Al] or [Si](Chapter 8).

Suppose V is one of the ends of M, and consider the rescaling of V, by a sequence $\{r_i\}$ of real numbers tending to infinity,

$$V_i = \frac{V}{r_i} = \{\ \frac{x}{r_i} :\ x \in V \subset R^m\}.$$

By (2.2) we know for any $0 < \delta < 1$

$$\sup_i \mathrm{Vol}(V_i \cap (B(1) - \overline{B(\delta)})) < +\infty.$$

Then the compactness theorem of integral varifolds (see, [Al] or [Si] Th 42.7) implies that there exists a subsequence of $\{V_i\}$, denoted again by $\{V_i\}$, converging to a stationary integral varifold T in $B(1) - \overline{B(\delta)}$.

LEMMA 3.1. *T is a cone.*

PROOF. For $\delta' < \delta$ we see $\{V_i\}$ converges (at least for a subsequence) in $B(1) - \overline{B(\delta')}$. Thus we may assume V_i converges to T in $B(1) - \{0\}$.

Without lose of generality, we suppose $\partial V \subset B(1)$ and denote

$$v(s) = \mathrm{Vol}(V \cap (B(s) - B(1))).$$

Since $\partial(V \cap (B(s) - B(1))) \subset \partial B(s) \cup \partial B(1)$, integrate $\frac{1}{2}\Delta r^2 = n$ over $V \cap (B(s) - B(1))$, by Green's formula and the co-area formula, we obtain

$$\begin{aligned} nv(s) &= s \int_{V\cap\partial B(s)} \langle \nabla r, \frac{\nabla r}{|\nabla r|}\rangle - \int_{V\cap\partial B(1)} \langle \nabla r, \frac{\nabla r}{|\nabla r|}\rangle \\ &\leq sv'(s) \qquad (v'(s) = \int_{V\cap\partial B(t)} \frac{1}{|\nabla r|}). \end{aligned}$$

This implies

$$(\frac{v(s)}{s^n})' \geq 0, \text{ i.e. } \frac{v(s_1)}{s_1^n} \leq \frac{v(s_2)}{s_2^n} \tag{3.1}$$

for $s_2 \geq s_1 \geq 1$.

Put $\mathcal{M}(T) = \lim \mathrm{Vol}(\frac{1}{r_i} V \cap (B(1) - B(\frac{1}{r_i})))$, where $\mathcal{M}$ is the mass norm of varifolds, then for given $t \in (0,1)$, by the monotone formula of stationary varifolds (see [Si]) we have

$$\frac{\mathcal{M}(T \cap B(t))}{t^n} \leq \mathcal{M}(T).$$

On the other hand, suppose $r_i t > r_j$ for some j ($i > j$ and $j \to \infty$ as $i \to \infty$), then by (3.1)

$$\frac{\mathcal{M}(T \cap B(t))}{t^n} = \lim_{i\to\infty} \frac{\mathrm{Vol}(\frac{V}{r_i} \cap (B(t) - B(\frac{1}{r_i})))}{t^n}$$

$$
\begin{aligned}
&= \lim_{i\to\infty} \frac{v(r_i t)}{(r_i t)^n} \\
&\geq \lim_{j\to\infty} \frac{v(r_j)}{r_j^n} \\
&= \lim_{j\to\infty} \frac{\mathrm{Vol}(V \cap (B(r_j) - B(1)))}{r_j^n} = \mathcal{M}(T).
\end{aligned}
$$

Thus we obtain

$$
\frac{\mathcal{M}(T \cap B(t))}{t^n} \equiv \mathcal{M}(T) \text{ for all } t \in (0,1).
$$

Then the proof that T is a cone follows from the well-known methods in the varifold theory. See the proof of Theorem 19.3 in [Si]. □

Proof of Theorem 2.1.

Since T is stationary in $B(1) - \overline{B(\delta)}$, and we know the multiplicity (density) function of T is everywhere great than or equal to 1 by the upper-semicontinuity of the multiplicity function and almost everywhere regularity of T (also cf.[L]). Thus we obtain

$$
\begin{aligned}
\lim_{i\to\infty} \mathrm{Vol}(V_i \cap (B(1) - \overline{B(\delta)})) &= \mathcal{M}(T \cap (B(1) - \overline{B(\delta)})) \\
&\geq \omega_n (1 - \delta^n)
\end{aligned} \tag{3.2}
$$

by means of the convergence and the fact that T is a cone.

By (3.2) we can deduce M has only finite many ends. Indeed, let $\{V^j\}_{j=1}^{k(M)}$ be the collection of the ends of M. We can assume the sequences of all ends rescaled by $\{r_i\}$ converge in $B(1) - \overline{B(\delta)}$. By the monotonicity formula of the volume growth of minimal submanifolds in R^m, we have

$$
k := \lim_{t\to\infty} \frac{\mathrm{Vol}(M \cap B(t))}{\omega_n t^n} < \infty.
$$

Note

$$
\frac{M}{r_i} \cap (B(1) - \overline{B(\delta)}) = \bigcup_{j=1}^{k(M)} \frac{V^j}{r_i} \cap (B(1) - \overline{B(\delta)}) \tag{3.3}
$$

when i is big enough. Then (3.2) implies

$$\begin{aligned} k &= \lim_{i\to\infty} \frac{\mathrm{Vol}(M\cap B(r_i))}{\omega_n r_i^n} \\ &= \lim_{i\to\infty} \omega_n^{-1}\mathrm{Vol}(\frac{M}{r_i}\cap(B(1)-\overline{B(\delta)})) + \lim_{i\to\infty}\frac{\mathrm{Vol}(M\cap B(\delta r_i^n))}{\omega_n r_i^n} \\ &= \lim_{i\to\infty} \omega_n^{-1}\sum_{j=1}^{k(M)} \mathrm{Vol}(\frac{V^j}{r_i}\cap(B(1)-\overline{B(\delta)})) + k\delta^n \\ &\geq k(M)(1-\delta^n)+k\delta^n. \end{aligned}$$

This proves the theorem. □

Proof of Theorem 2.2.

First we have for one of the ends (say V) of M,

$$\text{(3.4)} \quad \limsup_{t\to\infty} \frac{\mathrm{Vol}(\partial V(t))}{t^{n-1}} \leq \omega_{n-1} \text{ (the volume of the unit sphere in } R^n).$$

This can be shown by Lemma 2.5 (2) and Bishop's volume comparison theorem for $n\geq 3$. When $n=2$ and $m=3$, since $\partial(\frac{V(t)}{t})$ is an embedded closed curve in $S^2(1)$, Lemma 2.5 (1) implies

$$\lim_{t\to\infty}\frac{\mathrm{Vol}(\partial V(t))}{2\pi t} = \lim_{t\to\infty}(2\pi)^{-1}\mathrm{Vol}(\partial(\frac{V(t)}{t})) = 1.$$

By Lemma 2.6 and Lemma 2.4 (1), for $t>t_0$

$$\begin{aligned} \mathrm{Vol}(M\cap B(t)) &\leq \frac{t}{n}\ \mathrm{Vol}\ (\partial(M\cap B(t))) \\ &= \frac{t}{n}\sum_{j=1}^{k(M)} \mathrm{Vol}\ (\partial(V^j(t))). \end{aligned}$$

Thus we obtain

$$\text{(3.5)} \qquad \limsup_{t\to\infty}\frac{\mathrm{Vol}\ (M\cap B(t))}{\omega_n t^n} \leq k(M).$$

Then the theorem follows by (3.5) and Theorem 2.1. □

4. Minimal submanifolds of finite total scalar curvature

In this section we consider M is a complete minimal submanifold with total scalar curvature. Following Anderson [A] the total scalar curvature of M is defined to be

$$\int_M |A|^n = \int_M (-R)^{\frac{n}{2}}$$

where R is the scalar curvature of M. In [A] Anderson showed the immersion of M in R^m is proper if the total scalar curvature of M is finite. The following two facts were also proved in [A].

Lemma 4.1. *If* $\int_M |A|^n < +\infty$, *then*

(1) $|A|(x) \leq \frac{c_1}{r(x)^n}$, *for some constant* c_1;

(2) $\sup_{x \in \partial B_\rho(r)} |A|^2(x) < \frac{\mu(r)}{r^2}$ *when* r *is big enough,*

where $\lim_{r\to\infty} \mu(r) = 0$, *and* $B_\rho(r)$ *is the geodesic ball of* M *of radius* r *around a fixed point of* M.

Thus we have

Corollary 4.2. *Let* M *be an* n*-dimensional complete minimal submanifold in* R^m *with finite total scalar curvature. Then*

$$\lim_{t\to\infty} \frac{Vol\ (M \cap B(t))}{\omega_n t^n} = k(M)$$

provided either of the following two conditions is satisfied:

(1) $n = 2$, $m = 3$ *and each end of* M *is embedded.*

(2) $n \geq 3$.

The next theorem states that Theorem 2.2 is also true for the volume growth related to the intrinsic distance of M.

Theorem 4.3. *Let* M *be an* n*-dimensional complete minimal submanifold in* R^m *with finite total scalar curvature. Then*

$$\lim_{t\to\infty} \frac{Vol\ (B_\rho(t))}{\omega_n t^n} = k(M)$$

if either of the following two conditions is satisfied:

(1) $n = 2$, $m = 3$ *and each end of* M *is embedded.*

(2) $n \geq 3$.

Here $B_\rho(t)$ *is the geodesic ball of* M *with radius* t *around a fixed point of* M.

Proof. Without lose of generality, suppose $0 \in M$. Let $\rho(x) = \mathrm{dist}_M(0, x)$ and $\gamma : [0, l] \to M$ be a distance minimizing geodesic of M starting from the origin. Let x be the position vector of M in R^m. Along $\gamma(t)$, by Lemma 4.1(2)

$$\begin{aligned}\frac{d}{dt}\langle x(\gamma(t)), \gamma'(t)\rangle &= 1 + \langle x, A(\gamma'(t), \gamma'(t))\rangle \\ &\geq 1 - \mu(t).\end{aligned}$$

Thus for any $\epsilon > 0$, there exists a $t_0 > 0$ such that if $l > t_0$,

$$\frac{d}{dt}\langle x(\gamma(t)), \gamma'(t)\rangle \geq 1 - \epsilon \qquad \text{for all } t \in (t_0, l]. \tag{2.12}$$

Since $\langle x(\gamma(t)), \gamma'(t)\rangle \leq r(\gamma(t))$, integrating of (2.12) from t_0 to t, we have

$$r(\gamma(t)) \geq (1 - \epsilon)t + c(t_0) = (1 - \epsilon)\rho(\gamma(t)) + c(t_0)$$

for some constant $c(t_0)$, which means for $t > t_0$ (notice $r(x) \leq \rho(x)$)

$$M \cap B((1 - \epsilon)t + c(t_0)) \subset B_\rho(t) \subset M \cap B(t).$$

Hence we get

$$(1 - \epsilon)^n k(M) \leq \liminf_{t\to\infty} \frac{\mathrm{Vol}(B_\rho(t))}{\omega_n t^n} \leq \limsup_{t\to\infty} \frac{\mathrm{Vol}(B_\rho(t))}{\omega_n t^n} \leq k(M).$$

Letting $\epsilon \to 0$, we have

$$\lim_{t\to\infty} \frac{\mathrm{Vol}(B_\rho(t))}{\omega_n t^n} = k(M).$$

This completes the proof of the theorem. $\square$

A well known Cohn-Vossen theorem reads that if N is a 2-dimensional complete Riemannian manifold with the total Gaussian curvature $\int_N K_N$ being absolutely convergence, then $(1/2\pi)\int_N K_N \le \chi(N)$, where $\chi(N)$ is Euler characteristic of N. The difference $\chi(N)-(1/2\pi)\int_N K_N$ is interpreted by several authors and recently Shiohama([Shi]) proved

$$\lim_{t\to\infty}\frac{A(t)}{\pi t^2}=\chi(N)-\frac{1}{2\pi}\int_N K_N$$

where $A(t)$ is the area of the geodesic ball of N of radius t around a fixed point of M.

We can generalize this result to the class of minimal submanifolds in Euclidean space with finite total curvature.

Corollary 4.4. *Let M be as in Theorem 4.3 and $n \ge 3$, then*

$$\chi(M)-\int_M \Omega=\lim_{t\to\infty}\frac{Vol\ (M_\rho(t))}{\omega_n t^n},$$

where Ω is the Gauss-Bonnet-Chern curvature form of M.

Proof. It is a direct consequence of Theorem 4.3 and Theorem 5.1 of [A]. □

References

[Al] Allard, W. K., On the first variation of a varifold, Ann. math. **(2) 95** (1972), 417–491.

[An] Anderson, M., The compactification of a minimal submanifold in Euclidean space by the Gauss map, I.H.E.S. preprint (1984).

[Ch.O] Chern, S. S. and R. Osserman, Complete minimal surface in E^N, J. d'Analyse Math. **19** (1967), 15–34.

[J.M] Jorge, L. P. de Melo and W. H. Meeks III, The topology of complete minimal surface of finite total Gaussian curvature, Topology **(2) 22** (1983), 203–221.

[K] Kasue, K., Gap theorems for minimal submanifolds of Euclidean space, J. Math. Soc. Japan **38(3)** (1986), 473–492.

[K.S] Kasue, A. and K. Sugahara, Gap theorems for certain submanifolds of Euclidean space and Hyperbolic space form, Osaka J. Math. **24** (1987), 690–704.

[L] Lawson, H. B., Jr., Minimal variates, Proc. Symp. Pure. Math. **27** (1975), A.M.S. Publ.

[O] Osserman, R., A survey of minimal surfaces, Van Norstrand Rienhold, New York, 1969.

[Shi] Shiohama, K., Total curvatures and minimal areas of complete open surface, Proc. A.M.S. **(2) 94** (1985), 310–316.

[Sc] Schoen, R., Uniqueness, symmetry and embeddedness of minimal surfaces, J. Diff. Geom. **18** (1983), 771–789.

[Si] Simon, L., Lectures on Geometric Measure Theory, 1983, C.M.A. Australian National University Vol. 3.

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