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On the Volume of the Union of Balls*

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Abstract. We prove that if some balls in the Euclidean space move continuously in such a way that the distances between their centers decrease, then the volume of their union cannot increase. The proof is based on a formula expressing the derivative of the volume of the union as a linear combination of the derivatives of the distances between the centers with nonnegative coefficients.

1. Introduction

We denote by $\mathbf{R}^n$ the n-dimensional Euclidean space with the standard scalar product, norm, and distance $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$, $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, and $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$.

For $\mathbf{x} \in \mathbf{R}^n$ and $r \in \mathbf{R}_+$, where $\mathbf{R}_+$ denotes the set of positive real numbers, let $B(\mathbf{x}, r)$ denote the closed ball of radius r centered at $\mathbf{x}$, and let $S(\mathbf{x}, r)$ denote its boundary sphere. A system of N balls in $\mathbf{R}^n$ can be given by the system of centers $\mathbf{X} = (\mathbf{x_1}, \ldots, \mathbf{x}_N) \in \mathbf{R}^{nN}$ and that of radii $\mathbf{r} = (r_1, \ldots, r_N) \in \mathbf{R}_+^N$. Given $\mathbf{X}$ and $\mathbf{r}$, we denote by $B(\mathbf{X}, \mathbf{r})$ the union of the balls $B(\mathbf{x_i}, r_i)$ for $1 \le i \le N$, by $S(\mathbf{X}, \mathbf{r})$ the boundary of the domain $B(\mathbf{X}, \mathbf{r})$, and by $V(\mathbf{X}, \mathbf{r})$ the n-dimensional volume of $B(\mathbf{X}, \mathbf{r})$.

In 1954–56 Poulsen [9], Kneser [8], and Hadwiger [7] formulated the conjecture that if $\mathbf{X}, \mathbf{Y} \in \mathbf{R}^{2N}$ are such that $d(\mathbf{x}_i, \mathbf{x}_j) \le d(\mathbf{y}_i, \mathbf{y}_j)$ for each choice of i and j, then we have $V(\mathbf{X}, \mathbf{r}) \le V(\mathbf{Y}, \mathbf{r})$.

Though the conjecture was formulated originally for congruent disks in the plane, it seems to be true also for higher-dimensional balls having different radii.

The conjecture is still open even for $n = 2$. Partial results have been obtained by Bollobás [3], Alexander [1], Sudakov [11], Capoyleas and Pach [5], Bern and Sahai [2],

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and Csikós [6]. The present work is a continuation of the investigation started in [3], [2], and [6].

Bollobás [3] proved that the planar Kneser conjecture for congruent disks is true if we can continuously move the centers $\mathbf{x}_i$ to the centers $\mathbf{y}_i$ in such a way that the distances between the centers do not increase during the motion. In [2] and [6] Bern, Sahai, and the author generalized Bollobás' theorem for disks with different radii. The approaches of [2] and [6] are completely different. Reference [6] gives a suitable modification of the ideas of Bollobás, while [2] introduces the Dirichlet–Voronoi decomposition into the study of the question. Though the proof in [6] seems to be simpler, methods of [2] turn out to be more powerful. For example, Bern and Sahai could prove that the area of holes enclosed by the moving disks does not increase during the motion.

Our goal is to generalize these results to n-dimensional balls (Theorems 4.2 and 5.4). First we consider the volume of the union of some balls moving smoothly in $\mathbf{R}^n$. The main result of the paper is formula (3) expressing the derivative of this volume (with respect to time) as a linear combination of the derivatives of the distances between the centers. This formula is an effective generalization of Lemma 3 of [2], easily implying Theorem 4.2. In Section 5 we drop the differentiability assumption on the motion of the centers and generalize Theorem 4.2 for continuous motions.

We note that some special cases of Theorem 5.4 have also been obtained in [5] and [2].

2. A Formula for the Derivative of $V(\mathbf{X}, \mathbf{r})$

Suppose that we are given $\mathbf{r} \in \mathbf{R}_+^N$ and a smooth (i.e., infinitely many times differentiable) curve $\mathbf{X}\colon (a, b) \to \mathbf{R}^{nN}$. $\mathbf{X}(t)$ describes the motion of the centers of N balls in $\mathbf{R}^n$. Our aim now is to prove a formula for the derivative of the function $V(t) = V(\mathbf{X}(t), \mathbf{r})$. We start with some technical lemmas.

Lemma 2.1. *We define on $\mathbf{R}^{nN}$ the norm $||\mathbf{X}|| = \max_i |\mathbf{x}_i|$. Then, for any fixed $\mathbf{r} \in \mathbf{R}_+^N$, we have*

$$|V(\mathbf{X}, \mathbf{r}) - V(\mathbf{Y}, \mathbf{r})| = O(||\mathbf{X} - \mathbf{Y}||)$$

as $||\mathbf{X} - \mathbf{Y}||$ tends to zero.

Proof. Denoting by ε the distance $||\mathbf{X} - \mathbf{Y}||$, we have the following simple estimation:

$$\begin{aligned}
&|V(\mathbf{X}, \mathbf{r}) - V(\mathbf{Y}, \mathbf{r})| \\
&\quad \le \max\left\{ V\left(\bigcup_{i=1}^{N} B(\mathbf{x}_i, r_i + \varepsilon)\setminus B(\mathbf{x}_i, r_i)\right), V\left(\bigcup_{i=1}^{N} B(\mathbf{y}_i, r_i + \varepsilon)\setminus B(\mathbf{y}_i, r_i)\right)\right\} \\
&\quad \le V(B(\mathbf{0}, 1)) \sum_{i=1}^{N} ((r_i + \varepsilon)^n - r_i^n) = O(\varepsilon).
\end{aligned}$$

□

Corollary 2.2. *Let $\mathbf{X}\colon (a, b) \to \mathbf{R}^{nN}$ be a smooth curve, let $t_0 \in (a, b)$, and let $\mathbf{Y}(t) = \mathbf{X}(t_0) + (t - t_0)\mathbf{X}'(t_0)$ be the linear part of the Taylor expansion of $\mathbf{X}$ at t_0.*

Then, for any fixed $\mathbf{r} \in \mathbf{R}_+^N$*, the function* $V(t) = V(\mathbf{X}(t), \mathbf{r})$ *is differentiable at* t_0 *if and only if the function* $t \mapsto V(\mathbf{Y}(t), \mathbf{r})$ *is differentiable at* t_0 *and if the two functions are differentiable at* t_0*, then their derivatives at* t_0 *are equal.*

Proof. Since $||\mathbf{X}(t) - \mathbf{Y}(t)|| = O((t - t_0)^2)$, we have

$$|V(\mathbf{X}(t), \mathbf{r}) - V(\mathbf{Y}(t), \mathbf{r})| = O((t - t_0)^2),$$

and this implies the statement. □

Theorem 2.3. *Suppose* $n \geq 2$ *and let* $\mathbf{X}\colon (a, b) \to \mathbf{R}^{nN}$ *be a smooth curve,* $\mathbf{r} \in \mathbf{R}_+^N$*. Assume that* $t_0 \in (a, b)$ *is such that the balls* $B(\mathbf{x}_i(t_0), r_i)$ *are distinct. Denote by* F_i *the intersection* $S(\mathbf{x}_i(t_0), r_i) \cap S(\mathbf{X}(t_0), \mathbf{r})$*, by* $\mathbf{n}_i$ *the outer unit normal vector field on the sphere* $S(\mathbf{x}_i(t_0), r_i)$ *and by* μ_i *the* $(n-1)$*-dimensional volume measure on* $S(\mathbf{x}_i(t_0), r_i)$ *induced by the Riemannian metric. Then the function* $V(t) = V(\mathbf{X}(t), \mathbf{r})$ *is differentiable at* t_0 *and*

$$V'(t_0) = \sum_{i=1}^{N} \int_{F_i} \langle \mathbf{n}_i, \mathbf{x}_i'(t_0) \rangle \, d\mu_i. \tag{1}$$

Proof. By Corollary 2.2, we may replace $\mathbf{X}$ with its linear approximation $\mathbf{Y}(t) = \mathbf{X}(t_0) + (t - t_0)\mathbf{X}'(t_0)$. Think of $\mathbf{R}^{n+1}$ as the product $\mathbf{R}^n \times \mathbf{R}$ and write a typical element of it in the form $(\mathbf{x}, t)$, where $\mathbf{x} \in \mathbf{R}^n$, $t \in \mathbf{R}$. We fix $t \in (a, b)$ and apply the Gauss–Ostrogradskii formula (also known as “the divergence theorem”) for the union of the solid cylinders

$$\mathbb{B}_i(t) = \{(\mathbf{x}, \tau) \in \mathbf{R}^{n+1} \mid a \leq \tau \leq t, \mathbf{x} \in B(\mathbf{y}_i(\tau), r_i)\}$$

and the constant vector field

$$\xi \equiv (\mathbf{0}, 1).$$

Recall that, according to the Gauss–Ostrogradskii formula, if Ω is a compact domain in $\mathbf{R}^m$ with piecewise smooth boundary $\partial\Omega$, ξ is a smooth vector field on a neighborhood of Ω, then

$$\int_{\Omega} \operatorname{div} \xi \, d\lambda^m = \int_{\partial\Omega} \langle \mathbf{n}, \xi \rangle \, d\mu,$$

where $\operatorname{div} \xi$ denotes the divergence of ξ, λ^m is the m-dimensional Lebesgue measure, $\mathbf{n}$ is the outer unit normal vector field on $\partial\Omega$ (defined almost everywhere, on all smooth components of $\partial\Omega$), and μ is the $(m-1)$-dimensional volume measure on $\partial\Omega$.

The boundary of $\Omega(t) = \bigcup_{i=1}^{N} \mathbb{B}_i(t)$ is piecewise smooth. (We replaced the motion $\mathbf{X}(t)$ with the linear motion $\mathbf{Y}(t)$ for the sake of this statement. In general, the “life-tubes” of two smoothly moving spheres can intersect one another in a way difficult to handle.)

$\partial\Omega(t)$ has two flat components: a bottom one and a top one. The outer unit normal vector field on the top $B(\mathbf{Y}(t), \mathbf{r}) \times \{t\}$ coincides with ξ, while on the bottom $B(\mathbf{Y}(a), \mathbf{r}) \times \{a\}$ it is $-\xi$.

The lateral part of the boundary of $\Omega(t)$ is the closure of

$$\mathbb{S}(t) = \{(\mathbf{x}, \tau) \in \mathbf{R}^{n+1} \mid a \le \tau \le t, \mathbf{x} \in S(\mathbf{Y}(\tau), \mathbf{r})\}$$

and is covered by the boundaries of the tubes $\mathbb{B}_i(t)$. Let $\mathbb{F}_i(t)$ be the intersection of $\mathbb{S}(t)$ with the tube

$$\mathbb{S}_i(t) = \{(\mathbf{x}, \tau) \in \mathbf{R}^{n+1} \mid a \le \tau \le t, \mathbf{x} \in S(\mathbf{y}_i(\tau), r_i)\},$$

and, for $a \le \tau \le t$, set $F_i(\tau) = \mathbb{F}_i(t) \cap (\mathbb{R}^n \times \{\tau\})$. The outer unit normal vector field of the hypersurface $\mathbb{F}_i$ at $(\mathbf{x}, \tau) \in \mathbb{F}_i$ is

$$\frac{(\mathbf{n}_i(\mathbf{x}, \tau), -\langle \mathbf{n}_i(\mathbf{x}, \tau), \mathbf{x}_i'(t_0)\rangle)}{\sqrt{1 + \langle \mathbf{n}_i(\mathbf{x}, \tau), \mathbf{x}_i'(t_0)\rangle^2}},$$

where $\mathbf{n}_i(\mathbf{x}, \tau)$ is the outer unit normal of the sphere $S(\mathbf{y}_i(\tau), r_i)$ at $\mathbf{x}$.

The tube $S(\mathbf{x}_i(t_0), r_i) \times [a, b]$ is diffeomorphic to $\mathbb{S}_i(t)$. If we identify the two spaces by the diffeomorphism

$$(\mathbf{x}, \tau) \leftrightarrow (\mathbf{x} + (\tau - t_0)\mathbf{x}'(t_0), \tau) \in \mathbb{S}_i(t),$$

for all $\mathbf{x} \in S(\mathbf{x}_i(t_0), r_i)$, $\tau \in [a, b]$, then the n-dimensional volume measure can be expressed with μ_i and the one-dimensional Lebesgue measure λ on $[a, b]$ as follows:

$$\sqrt{1 + \langle \mathbf{n}_i(\mathbf{x}, \tau), \mathbf{x}_i'(t_0)\rangle^2}\mu_i \times \lambda.$$

Since the vector field $\xi = (\mathbf{0}, 1)$ is divergence free, the Gauss–Ostrogradskii formula gives that

$$\begin{aligned} 0 &= \int_{\Omega(t)} \operatorname{div} \xi \, d\lambda^{n+1} = \int_{\partial\Omega(t)} \langle \mathbf{n}, \xi \rangle \, d\mu \\ &= V(\mathbf{Y}(t), \mathbf{r}) - V(\mathbf{Y}(t_0), \mathbf{r}) - \sum_{i=1}^{N} \int_{\mathbb{F}_i(t)} \frac{\langle \mathbf{n}_i, \mathbf{x}_i'(t_0)\rangle}{\sqrt{1 + \langle \mathbf{n}_i, \mathbf{x}_i'(t_0)\rangle^2}} d\mu \\ &= V(\mathbf{Y}(t), \mathbf{r}) - V(\mathbf{Y}(t_0), \mathbf{r}) - \sum_{i=1}^{N} \int_a^t \left(\int_{F_i(\tau)} \langle \mathbf{n_i}, \mathbf{x}_i'(t_0)\rangle \, d\mu_i \right) d\tau. \end{aligned} \tag{2}$$

When $n \ge 2$, the integrals

$$\int_{F_i(\tau)} \langle \mathbf{n_i}, \mathbf{x}_i'(t_0)\rangle \, d\mu_i$$

depend continuously on τ in the neighborhood of any τ_0 where the balls $B(\mathbf{x}_i(\tau_0), r_i)$ are different, in particular, the dependence is continuous in a neighborhood of t_0. Differentiating (2) with respect to t at t_0, we obtain (1). $\square$

3. The Dirichlet–Voronoi Decomposition

The integrals in formula (1) can be computed with the help of the Gauss–Ostrogradskii formula applied to the cells of the Dirichlet–Voronoi decomposition of $B(\mathbf{X}, \mathbf{r})$. We recall how this decomposition is defined.

We define the power of a point $\mathbf{p}$ with respect to a ball $B(\mathbf{x}, r)$ to be the real number $|\mathbf{x} - \mathbf{p}|^2 - r^2$. Now consider N spheres given by the centers $\mathbf{X}$ and radii $\mathbf{r}$ as above and denote by $K_i(\mathbf{p}) = |\mathbf{x_i} - \mathbf{p}|^2 - r_i^2$ the power of $\mathbf{p}$ with respect to the ith ball. For any pair $1 \le i < j \le N$, if $\mathbf{x_i} \ne \mathbf{x}_j$, then the inequality $K_i(\mathbf{p}) \le K_j(\mathbf{p})$ defines a half-space bounded by the hyperplane

$$2\langle(\mathbf{x_i} - \mathbf{x}_j), \mathbf{p}\rangle = \mathbf{x}_i^2 - \mathbf{x}_j^2 + r_j^2 - r_i^2.$$

If two spheres have the same center $\mathbf{x}_i = \mathbf{x}_j$, then $K_i(\mathbf{p}) - K_j(\mathbf{p}) = r_j^2 - r_i^2$ is constant, consequently, its sign does not depend on $\mathbf{p}$.

If the balls are distinct, then the balls determine a decomposition of the space into N possibly unbounded or empty polyhedral domains. The ith cell of the decomposition is given as the closure of the open domain

$$C_i = \{\mathbf{p} \in \mathbf{R}^n \mid K_i(\mathbf{p}) < K_j(\mathbf{p}) \text{ for all } 1 \le j \le N,\ j \ne i\}.$$

This decomposition is generally called the Dirichlet–Voronoi decomposition. The Dirichlet–Voronoi decomposition induces a decomposition of $B(\mathbf{X}, \mathbf{r})$. The ith cell of this induced decomposition is

$$\tilde{C}_i = \bar{C}_i \cap B(\mathbf{X}, \mathbf{r}) = \bar{C}_i \cap B(\mathbf{x}_i, r_i),$$

where $\bar{C}$ denotes the closure of the set C.

We define the *wall* W_{ij} between the cells $\tilde{C}_i$ and $\tilde{C}_j$ as the intersection $W_{ij} = \tilde{C}_i \cap \tilde{C}_j$. When W_{ij} is not empty it is the intersection of a polyhedral domain lying in the hyperplane $K_i(\mathbf{p}) = K_j(\mathbf{p})$ with the ball $B(\mathbf{x}_i, r_i)$.

When we want to express dependence of $\tilde{C}_i$ and W_{ij} on $(\mathbf{X}, \mathbf{r})$, we write $\tilde{C}_i(\mathbf{X}, \mathbf{r})$ and $W_{ij}(\mathbf{X}, \mathbf{r})$.

4. Main Formula for the Derivative of $V(\mathbf{X}, \mathbf{r})$

Theorem 4.1. *Let $n \ge 2$ and let $\mathbf{X}\colon (a, b) \to \mathbf{R}^{nN}$ be a smooth curve, $\mathbf{r} \in \mathbf{R}_+^N$. Suppose that $t_0 \in (a, b)$ is such that the centers $\mathbf{x}_i(t_0)$ are different. Then the function $V(t) = V(\mathbf{X}(t), \mathbf{r})$ is differentiable at t_0 and its derivative is equal to*

$$V'(t_0) = \sum_{1 \le i < j \le N} d'_{ij}(t_0) V_{n-1}(W_{ij}(\mathbf{X}(t_0), \mathbf{r})), \tag{3}$$

where $d_{ij}(t) = d(\mathbf{x}_i(t), \mathbf{x}_j(t))$, and V_{n-1} denotes the $(n-1)$-dimensional volume.

Proof. By Theorem 2.3, V is differentiable at t_0 and $V'(t_0)$ can be given by formula (1). We compute the integral

$$\int_{F_i} \langle \mathbf{n}_i, \mathbf{x}'_i(t_0)\rangle \, d\mu_i$$

applying the Gauss–Ostrogradskii formula to the domain $\tilde{C}_i$ and the constant vector field $\xi_i \equiv \mathbf{x}_i'(t_0)$.

The cell $\tilde{C}_i$ is bounded by the spherical domain F_i and the walls W_{ij}. The outer unit normal of a nonempty wall W_{ij} is

$$\frac{\mathbf{x}_j(t_0) - \mathbf{x}_i(t_0)}{d_{ij}(t_0)}.$$

Since the constant vector field ξ_i is divergence free, we get

$$0 = \int_{\tilde{C}_i} \operatorname{div} \xi_i \, d\lambda^n = \int_{\partial \tilde{C}_i} \langle \mathbf{n}, \xi_i \rangle \, d\mu,$$

where $\mathbf{n}$ and μ here mean the outer unit normal field and the $(n-1)$-dimensional volume measure on $\tilde{C}_i$. Furthermore, we have

$$\int_{\partial \tilde{C}_i} \langle \mathbf{n}, \xi_i \rangle \, d\mu = \int_{F_i} \langle \mathbf{n}_i, \mathbf{x}_i'(t_0) \rangle \, d\mu_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \left\langle \frac{\mathbf{x}_j(t_0) - \mathbf{x}_i(t_0)}{d_{ij}(t_0)}, \mathbf{x}_i'(t_0) \right\rangle V_{n-1}(W_{ij}),$$

therefore

$$\int_{F_i} \langle \mathbf{n}_i, \mathbf{x}_i'(t_0) \rangle \, d\mu_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \left\langle \frac{\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)}{d_{ij}(t_0)}, \mathbf{x}_i'(t_0) \right\rangle V_{n-1}(W_{ij}). \tag{4}$$

Summing (4) for all i, we obtain

$$\begin{aligned} V'(t_0) &= \sum_{i=1}^{N} \int_{F_i} \langle \mathbf{n}_i, \mathbf{x}_i'(t_0) \rangle \, d\mu_i \\ &= \sum_{1 \le i < j \le N} \left\langle \frac{\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)}{d_{ij}(t_0)}, \mathbf{x}_i'(t_0) - \mathbf{x}_j'(t_0) \right\rangle V_{n-1}(W_{ij}) \\ &= \sum_{1 \le i < j \le N} d_{ij}'(t_0) V_{n-1}(W_{ij}), \end{aligned}$$

as was to be proved. □

Theorem 4.2. *If* $\mathbf{X}\colon [a,b] \to \mathbf{R}^{nN}$ *is a piecewise smooth continuous curve such that the distance functions* $d_{ij}(t) = |\mathbf{x}_i(t) - \mathbf{x}_j(t)|$ *are decreasing on* $[a,b]$*, then, for any* $\mathbf{r} \in \mathbf{R}_+^N$*, we have*

$$V(B(\mathbf{X}(a), \mathbf{r})) \ge V(B(\mathbf{X}(b), \mathbf{r})). \tag{5}$$

Proof. It is enough to deal with the smooth case. If two of the centers $\mathbf{x}_i(b)$ are equal, say $\mathbf{x}_i(b) = \mathbf{x}_j(b)$ while $r_i \le r_j$, then, omitting the ith ball from the consideration, an obvious induction on the number of spheres gives (5).

Suppose that the centers $\mathbf{x}_i(a)$ are different, then $\mathbf{x}_i(t) \neq \mathbf{x}_j(t)$ for any $t \in [a, b]$, $i \neq j$, therefore, by Theorem 4.1, the function $V(t) = V(\mathbf{X}(t), \mathbf{r})$ is differentiable and since $d'_{ij} \leq 0$, $V_{n-1}(W_{ij}) \geq 0$ for all i, j, we have

$$V' = \sum_{1 \leq i < j \leq N} d'_{ij} V_{n-1}(W_{ij}) \leq 0,$$

thus $V(a) \geq V(b)$. □

Remark. It is easy to modify Theorem 4.1 and its proof to get a similar formula for the derivative of the volume of holes enclosed by the balls. This modified formula shows that if some balls move smoothly in such a way that the distances between the centers decrease, then the volume of any hole enclosed by the balls (weakly) decreases.

5. Continuous Motions of Balls

The aim of this section is to show that the differentiability condition on $\mathbf{X}$ in Theorem 4.2 can be replaced by continuity. The sketch of the proof is the following. First we show that for short periods of time the increase in volume can be bounded from above by an expression quadratic in the change in center positions (Lemma 5.2). An application of this estimation to subintervals of fine subdivisions of the time interval implies inequality (5) if we assume that the curve $\mathbf{X}$ is rectifiable (the reader is recommended to check this). However, rectifiability of $\mathbf{X}$ does not follow from our assumptions. Indeed, one can "shake" the centers without changing the distances between them in such a way that the centers move along nonrectifiable curves. Thus, change in center positions can be large even if the distances between the centers change little or not at all. We overcome this difficulty with the help of Lemma 5.3. which claims roughly that if for two systems of N points the distances between the corresponding points are close to one another, then a congruent copy of the first system will be sufficiently close to the second system. This lemma allows us to calm down a nonrectifiable "shivering" motion of the centers by replacing $\mathbf{X}(t)$ with $\varphi(t)\mathbf{X}(t)$, where $\varphi(t)$ is a suitable isometry of the space.

Lemma 5.1. *Let $\mathbf{p}, \mathbf{q} \in \mathbf{R}^n$ be two vectors, such that $|\mathbf{p}| \geq |\mathbf{q}|$. Choose the points O, P, Q in such a way that $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$, and let T be the orthogonal projection of O on PQ. Set $\omega = 0$ if T is not on the segment (P, Q) and $\omega = (1/OT^2)\langle \mathbf{q} - \mathbf{p}, \mathbf{q} \rangle$ if $T \in (P, Q)$. Then the function*

$$d(\tau) = |\mathbf{p} + \tau(\mathbf{q} - \mathbf{p})| e^{-\omega\tau}$$

is decreasing in the interval $[0, 1]$. If PQ is the smallest side of the triangle OPQ, that is $PQ < OQ$, then

$$\omega \leq \frac{2\, PQ^2}{3\, OQ^2}. \tag{6}$$

Proof. Denoting by $\mathbf{v}$ the vector $\mathbf{q} - \mathbf{p}$, we can write

$$e^{2\omega\tau} d'(\tau)\, d(\tau) = \langle \mathbf{v}, \mathbf{p} + \tau\mathbf{v} \rangle - \omega |\mathbf{p} + \tau\mathbf{v}|^2.$$

The maximum of $\langle \mathbf{v}, \mathbf{p} + \tau\mathbf{v}\rangle$ on $[0, 1]$ is always attained at $\tau = 1$. The minimum of $|\mathbf{p} + \tau\mathbf{v}|^2$ on the interval $[0, 1]$ is OQ^2 if T is not in the interval (P, Q) and it is OT^2 otherwise. Therefore, if T is not in the interval (P, Q), we have

$$d'(\tau)\, d(\tau) = \langle \mathbf{v}, \mathbf{p} + \tau\mathbf{v}\rangle \leq \langle \mathbf{v}, \mathbf{q}\rangle \leq 0.$$

When $T \in (P, Q)$, we have

$$e^{2\omega\tau} d'(\tau)\, d(\tau) \leq \langle \mathbf{v}, \mathbf{p} + \mathbf{v}\rangle - \omega OT^2 = 0.$$

Inequality (6) is obviously true when $\omega = 0$. Assuming $PQ < OQ$ and $T \in (P, Q)$, we have from $OP > OQ$ that $TQ < TP$, in particular, $TQ < PQ/2 < OQ/2$. Furthermore,

$$\omega = \frac{PQ \cdot OQ}{OT^2} \cos(PQO) = \frac{PQ \cdot TQ}{OQ^2 - TQ^2} < \frac{PQ^2}{2(OQ^2 - OQ^2/4)} = \frac{2\,PQ^2}{3\,OQ^2}. \qquad \square$$

Lemma 5.2. *Let $\mathbf{P} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ and $\mathbf{Q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathbf{R}^{nN}$ be two systems of N points in $\mathbf{R}^n$, such that*

$$\Delta \geq d(\mathbf{p}_i, \mathbf{p}_j) \geq d(\mathbf{q}_i, \mathbf{q}_j) \geq \delta > 0$$

and

$$||\mathbf{P} - \mathbf{Q}|| < \min\left\{1, \frac{\delta}{2}\right\}$$

for all i, j, where Δ and δ are given constants. Then for any system of radii $\mathbf{r} \in \mathbf{R}_+^N$ there exists a constant c depending only on $\mathbf{r}$, Δ, and δ such that

$$V(\mathbf{Q}, \mathbf{r}) - V(\mathbf{P}, \mathbf{r}) \leq c||\mathbf{P} - \mathbf{Q}||^2.$$

Proof. We connect $\mathbf{P}$ and $\mathbf{Q}$ with the linear constant speed motion $\mathbf{X}(t) = t\mathbf{Q} + (1-t)\mathbf{P}$, $t \in [0, 1]$. By Lemma 5.1, if we set

$$\omega = \frac{2||\mathbf{P} - \mathbf{Q}||^2}{3\delta^2},$$

then the scaled distance functions

$$|\mathbf{x}_i(t) - \mathbf{x}_j(t)|e^{-\omega t}$$

will be decreasing on $[0, 1]$. Applying Theorem 4.2 to the curve $\mathbf{X}(t)e^{-\omega t}$, we obtain

$$V(e^{-\omega}\mathbf{Q}, \mathbf{r}) \leq V(\mathbf{P}, \mathbf{r}). \tag{7}$$

Let $\tilde{\mathbf{Q}}$ be the translate of $\mathbf{Q}$ with the vector $-\mathbf{q}_1$, that is, set $\tilde{\mathbf{q}}_i = \mathbf{q}_i - \mathbf{q}_1$. Clearly, we have $V(\mathbf{Q}, \mathbf{r}) = V(\tilde{\mathbf{Q}}, \mathbf{r})$, $V(e^{-\omega}\mathbf{Q}, \mathbf{r}) = V(e^{-\omega}\tilde{\mathbf{Q}}, \mathbf{r})$, and

$$||\tilde{\mathbf{Q}} - e^{-\omega}\tilde{\mathbf{Q}}|| = (1 - e^{-\omega}) \max_i |\mathbf{q}_i - \mathbf{q}_1| < \omega\Delta.$$

Combining this inequality with the estimation obtained in the proof of Lemma 2.1 and using $\omega < 2/3\delta^2$, we get

$$\begin{aligned}|V(\mathbf{Q},\mathbf{r}) - V(e^{-\omega}\mathbf{Q},\mathbf{r})| \;&\leq\; V(B(\mathbf{0},1))\sum_{i=1}^{N}((r_i+\omega\Delta)^n - r_i^n)\\ &<\; \omega V(B(\mathbf{0},1))\sum_{i=1}^{N}\sum_{j=1}^{n}\binom{n}{j} r_i^{n-j}\left(\frac{2}{3\delta^2}\right)^{j-1}\Delta^j.\end{aligned}$$

Setting

$$c = V(B(\mathbf{0},1))\sum_{i=1}^{N}\sum_{j=1}^{n}\binom{n}{j} r_i^{n-j}\left(\frac{2\Delta}{3\delta^2}\right)^{j} \tag{8}$$

we get

$$V(\mathbf{Q},\mathbf{r}) - V(\mathbf{P},\mathbf{r}) \leq |V(\mathbf{Q},\mathbf{r}) - V(e^{-\omega}\mathbf{Q},\mathbf{r})| \leq c||\mathbf{P}-\mathbf{Q}||^2,$$

and this proves the lemma. □

For $\mathbf{X} \in \mathbf{R}^{nN}$, we define the *rank* rk $\mathbf{X}$ *of* $\mathbf{X}$ as the dimension of the affine subspace spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_N$.

Lemma 5.3. *For any* $\mathbf{X} \in \mathbf{R}^{nN}$ *of rank* k *we can find two positive constants* L *and* ε *in such a way that if* $\mathbf{Y}, \mathbf{Y}' \in \mathbf{R}^{nN}$ *are arbitrary systems of* N *points of rank* k *satisfying*

$$\begin{aligned}|d(\mathbf{x}_i,\mathbf{x}_j) - d(\mathbf{y}_i,\mathbf{y}_j)| &< \varepsilon,\\ |d(\mathbf{x}_i,\mathbf{x}_j) - d(\mathbf{y}'_i,\mathbf{y}'_j)| &< \varepsilon\end{aligned} \tag{9}$$

for all i, j, *then we can find an isometry* $\varphi \in \mathrm{Iso}(\mathbf{R}^n)$ *such that*

$$d(\mathbf{y}_i,\varphi(\mathbf{y}'_i)) < L\sum_{1\leq i<j\leq N}|d(\mathbf{y}_i,\mathbf{y}_j) - d(\mathbf{y}'_i,\mathbf{y}'_j)|$$

for all i.

Proof. As the isometry group acts transitively on k-dimensional affine subspaces of $\mathbf{R}^n$ and any isometry of a k-dimensional subspace into itself can be extended to an isometry of the whole space, it is enough to consider the case $k = n$.

The isometry group of the Euclidean space is the semidirect product of the group of translations and the orthogonal group

$$\mathrm{Iso}(\mathbf{R}^n) = \mathbf{R}^n \rtimes O_n.$$

The isometry group and the orthogonal group act on $\mathbf{R}^{nN}$, $\mathbf{R}^{n(N-1)} \times \{\mathbf{0}\} \subset \mathbf{R}^{nN}$ is an invariant subspace of the restricted action of the orthogonal group. Clearly, the intersection of an $\mathrm{Iso}(\mathbf{R}^n)$-orbit in $\mathbf{R}^{nN}$ with $\mathbf{R}^{n(N-1)} \times \{\mathbf{0}\}$ is an O_n-orbit, therefore we may assume that $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Y}'$ come from $\mathbf{R}^{n(N-1)} \times \{\mathbf{0}\}$ and with this assumption we will be able to choose φ from O_n.

We consider the O_n-action on $\mathbf{R}^{n(N-1)} = \mathbf{R}^{n(N-1)} \times \{\mathbf{0}\}$. We apply some standard facts on compact Lie group actions to this special action. For reference see [4].

The stabilizer of a system of points $\mathbf{X} \in \mathbf{R}^{n(N-1)}$ is isomorphic to O_{n-k} if rk $\mathbf{X} = k$. In particular, the stabilizer is minimal (and trivial) if and only if rk $\mathbf{X} = n$. Orbits of systems of maximal rank are the principal orbits of the action. The union of principal orbits is an open dense subset of $\mathbf{R}^{n(N-1)}$. Principal orbits are diffeomorphic to O_n.

By the equivariant tubular neighborhood theorem (see Theorem 5.4. in Chapter II of [4]), every principal orbit Ω has an O_n-invariant open neighborhood, equivariantly diffeomorphic to $\Omega \times \nu$, where ν is a linear space on which O_n acts trivially.

Since we have assumed rk $\mathbf{X} = k = n$, we can apply this theorem to the O_n-orbit $\Omega_{\mathbf{X}}$ of $\mathbf{X}$ and get an O_n-equivariant diffeomorphism

$$i\colon \Omega_{\mathbf{X}} \times \nu \to U,$$

where U is an open neighborhood of $\Omega_{\mathbf{X}}$.

Consider the polynomial mapping

$$\sigma = (d_{12}^2, d_{13}^2, \ldots, d_{(N-1)N}^2)\colon \mathbf{R}^{n(N-1)} \to \mathbf{R}^{\binom{N}{2}},$$

whose coordinate functions are the squares of the distance functions $d_{ij} = |\mathbf{x_i} - \mathbf{x}_j|$, $1 \le i < j \le N$. The polynomials d_{ij}^2, $1 \le i < j \le N$, generate the ring of invariant polynomials of the O_n-action, therefore σ induces a homeomorphism

$$\sigma_*\colon \mathbf{R}^{n(N-1)}/O_n \to \sigma(\mathbf{R}^{n(N-1)}).$$

In particular, we can find an $\varepsilon_0 > 0$ such that $|\sigma(\mathbf{X}) - \sigma(\mathbf{Y})| < 4\varepsilon_0$ implies $\mathbf{Y} \in U$.

Set $V = \sigma^{-1}(B(\sigma(\mathbf{X}), 2\varepsilon_0))$ and $W = \sigma^{-1}(B(\sigma(\mathbf{X}), 3\varepsilon_0))$. V and W are compact tubular neighborhoods of $\Omega_{\mathbf{X}}$ lying in U. Choose a smooth function $g\colon \mathbf{R}^{n(N-1)} \to \mathbf{R}$ such that $g|_V \equiv 1$, and g is constant zero outside W. Take the averaged function

$$f(\mathbf{Z}) = \int_{O_n} g(\varphi(\mathbf{Z}))\, d\varphi,$$

where the integral is taken with respect to the Haar measure $d\varphi$ normalized by the condition $\int_{O_n} d\varphi = 1$. Define the map $F\colon \mathbf{R}^{n(N-1)} \to \nu$

$$F(\mathbf{Z}) = \begin{cases} f(\mathbf{Z})\, pr_2 \circ i^{-1}(\mathbf{Z}) & \text{if} \quad \mathbf{Z} \in U, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

F is a smooth function, invariant under the action of the orthogonal group, thus, by a theorem of Schwarz [10], there is a smooth map $\Phi\colon \mathbf{R}^{\binom{N}{2}} \to \nu$ such that $F = \Phi \circ \sigma$.

Since Φ is smooth, it has the Lipschitz property on any compact subset. In particular, we can find $L_0 > 0$ such that, for any $\sigma_1, \sigma_2 \in B(\sigma(\mathbf{X}), \varepsilon_0)$, we have

$$||\Phi(\sigma_1) - \Phi(\sigma_2)|| < L_0 ||\sigma_1 - \sigma_2||_1,$$

where $|| \cdot ||_1$ denotes the l_1-norm in $\mathbf{R}^{\binom{N}{2}}$.

Now suppose that $\mathbf{Y}, \mathbf{Y}' \in \mathbf{R}^{n(N-1)}$ satisfy (9) with $\varepsilon = \varepsilon_0/N$. Then $\sigma(\mathbf{Y})$ and $\sigma(\mathbf{Y}')$ are in $B(\sigma(\mathbf{X}), \varepsilon_0)$ and therefore

$$||\Phi(\sigma(\mathbf{Y})) - \Phi(\sigma(\mathbf{Y}'))|| < L_0||\sigma(\mathbf{Y}) - \sigma(\mathbf{Y}')||_1 = L_0 \sum_{1\le i<j\le N} |d^2(\mathbf{y}_i, \mathbf{y}_j) - d^2(\mathbf{y}'_i, \mathbf{y}'_j)|.$$

Using the estimation

$$\begin{aligned} |d^2(\mathbf{y}_i, \mathbf{y}_j) - d^2(\mathbf{y}'_i, \mathbf{y}'_j)| &= |d(\mathbf{y}_i, \mathbf{y}_j) - d(\mathbf{y}'_i, \mathbf{y}'_j)| \cdot |d(\mathbf{y}_i, \mathbf{y}_j) + d(\mathbf{y}'_i, \mathbf{y}'_j)| \\ &< 2(\operatorname{diam} \mathbf{X} + \varepsilon)|d(\mathbf{y}_i, \mathbf{y}_j) - d(\mathbf{y}'_i, \mathbf{y}'_j)| \end{aligned}$$

we obtain

$$||\Phi(\sigma(\mathbf{Y})) - \Phi(\sigma(\mathbf{Y}'))|| < L \sum_{1\le i<j\le N} |d(\mathbf{y}_i, \mathbf{y}_j) - d(\mathbf{y}'_i, \mathbf{y}'_j)|$$

with $L = 2(\operatorname{diam} \mathbf{X} + \varepsilon)L_0$. Now it remains to observe that ε and L depends only on $\mathbf{X}$ and that $\mathbf{Y}$ and $\Phi(\sigma(\mathbf{Y}))$ as well as $\mathbf{Y}'$ and $\Phi(\sigma(\mathbf{Y}'))$ belong to the same O_n-orbit. Therefore, we can find $\varphi_1, \varphi_2 \in O_n$ such that $\Phi(\sigma(\mathbf{Y})) = \varphi_1\mathbf{Y}$ and $\Phi(\sigma(\mathbf{Y}')) = \varphi_2\mathbf{Y}'$. Setting $\varphi = \varphi_1^{-1} \circ \varphi_2$ we obtain

$$\begin{aligned} d(\mathbf{y}_i, \varphi(\mathbf{y}'_i)) &\le ||\mathbf{Y} - \varphi(\mathbf{Y}')|| = ||\Phi(\sigma(\mathbf{Y})) - \Phi(\sigma(\mathbf{Y}'))|| \\ &< L \sum_{1\le i<j\le N} |d(\mathbf{y}_i, \mathbf{y}_j) - d(\mathbf{y}'_i, \mathbf{y}'_j)|, \end{aligned}$$

and this proves the lemma. □

Theorem 5.4. *If* $\mathbf{X}\colon [a, b] \to \mathbf{R}^{nN}$ *is a continuous curve such that the distance functions* $d_{ij}(t) = |\mathbf{x}_i(t) - \mathbf{x}_j(t)|$ *are decreasing on* $[a, b]$, *then, for any* $\mathbf{r} \in \mathbf{R}^N_+$, *we have*

$$V(B(\mathbf{X}(a), \mathbf{r})) \ge V(B(\mathbf{X}(b), \mathbf{r})).$$

Proof. We define $k = \min\{\operatorname{rk} \mathbf{X}(t) \mid t \in [a, b]\}$ and use induction on $i = n - k$. Since $n - k \ge 0$ automatically, the initial case $i = -1$ is empty, hence trivial.

Suppose that the theorem is true for continuous motions of balls during which the rank of the system of centers does not fall below $k + 1$ and consider a continuous contraction $\mathbf{X}\colon [a, b] \to \mathbf{R}^{nN}$ with $k = \min\{\operatorname{rk} \mathbf{X}(t) \mid t \in [a, b]\}$.

Just as in the smooth case, we may assume that $\mathbf{x}_i(t)$ and $\mathbf{x}_j(t)$ are different for any $i \ne j$ and $t \in [a, b]$. Thus, we can find $\Delta > 0$ and $\delta > 0$ such that

$$\Delta \ge d_{ij}(t) \ge \delta > 0 \qquad \text{for all} \quad t \in [a, b].$$

Fix a system of radii $\mathbf{r}$ and define c by formula (8) of Lemma 5.2.

The set

$$K = \{t \in [a, b] \mid \operatorname{rk} \mathbf{X}(t) = k\}$$

is a compact subset of $[a, b]$. Using standard compactness arguments and Lemma 5.3, we can find $\varepsilon > 0$ and $L > 0$ such that for any two $\tau_1, \tau_2 \in K$ satisfying $|\tau_1 - \tau_2| < \varepsilon$ we can find an isometry φ such that

$$||\mathbf{X}(\tau_1) - \varphi(\mathbf{X}(\tau_2))|| < L \sum_{1\le i<j\le N} |d_{ij}(\tau_1) - d_{ij}(\tau_2)|. \tag{10}$$

Now fix an arbitrary positive number η. Choose a positive number $\zeta > 0$ with the following properties:

(i) $\varepsilon > \zeta$;
(ii) for any two $\tau_1, \tau_2 \in [a, b]$ with $|\tau_1 - \tau_2| < \zeta$ we have

$$|d_{ij}(\tau_1) - d_{ij}(\tau_2)| < \min\left\{1, \frac{\delta}{2}, \eta\right\}.$$

Let $a = t_0 < t_1 < \ldots < t_m = b$ be a subdivision of the interval $[a, b]$ such that $t_i - t_{i-1} < \zeta$, $1 \leq i \leq m$, and estimate the difference $V(\mathbf{X}(t_i), \mathbf{r}) - V(\mathbf{X}(t_{i-1}), \mathbf{r})$ from above for each subinterval. If the interval $[t_{i-1}, t_i]$ intersects K, then set

$$\tau_1 = \inf(K \cap [t_{i-1}, t_i]) \quad \text{and} \quad \tau_2 = \sup(K \cap [t_{i-1}, t_i]).$$

By the induction hypothesis, $V(\mathbf{X}(t), \mathbf{r})$ is decreasing on the intervals $[t_{i-1}, \tau_1]$ and $[\tau_2, t_i]$, thus

$$V(\mathbf{X}(t_i), \mathbf{r}) - V(\mathbf{X}(t_{i-1}), \mathbf{r}) \leq V(\mathbf{X}(\tau_2), \mathbf{r}) - V(\mathbf{X}(\tau_1), \mathbf{r}).$$

Since $|\tau_1 - \tau_2| < \varepsilon$ we can find an isometry φ such that (10) is fulfilled.

Applying the estimation of Lemma 5.2 to $\mathbf{X}(\tau_1)$ and $\varphi(\mathbf{X}(\tau_2))$ and the inequality (10), we obtain

$$\begin{aligned} V(\mathbf{X}(\tau_2), \mathbf{r}) - V(\mathbf{X}(\tau_1), \mathbf{r}) &= V(\varphi(\mathbf{X}(\tau_2)), \mathbf{r}) - V(\mathbf{X}(\tau_1), \mathbf{r}) \\ &\leq c||\mathbf{X}(\tau_1) - \varphi(\mathbf{X}(\tau_2))||^2 \\ &< cL^2 \left(\sum_{1 \leq i < j \leq N} |d_{ij}(\tau_1) - d_{ij}(\tau_2)|\right)^2 \\ &< cL^2 \binom{N}{2} \eta \sum_{1 \leq i < j \leq N} (d_{ij}(\tau_1) - d_{ij}(\tau_2)). \end{aligned}$$

We could drop the absolute values in the last inequality since the functions d_{ij}, $1 \leq i < j \leq N$, are decreasing. Combining these inequalities we get

$$V(\mathbf{X}(t_i), \mathbf{r}) - V(\mathbf{X}(t_{i-1}), \mathbf{r}) \leq cL^2 \binom{N}{2} \eta \sum_{1 \leq i < j \leq N} (d_{ij}(t_{i-1}) - d_{ij}(t_i)). \tag{11}$$

We proved (11) with the assumption that $[t_{i-1}, t_i]$ intersects K, however, it is obviously true also for subintervals disjoint from K, since on such intervals $V(\mathbf{X}(t), \mathbf{r})$ is decreasing by the induction hypothesis. Summing up inequality (11) for $i = 1, \ldots, m$, we conclude

$$V(\mathbf{X}(b), \mathbf{r}) - V(\mathbf{X}(a), \mathbf{r}) \leq cL^2 \binom{N}{2} \eta \sum_{1 \leq i < j \leq N} (d_{ij}(a) - d_{ij}(b)). \tag{12}$$

Since η can be arbitrarily small, inequality (12) provides the stronger inequality

$$V(\mathbf{X}(b), \mathbf{r}) - V(\mathbf{X}(a), \mathbf{r}) \leq 0,$$

as we wanted to prove. □

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