On the Wave Equation of Meson.

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Introduction.

Up to the present much work on meson theory has been done by considering it as a field theory, and the equation of meson as a field equation. This situation has its origin in the fact that the meson was found originally as a field of the heavy particles. However, if we restrict ourselves to the problems of the interaction between the meson and the electromagnetic field, it seems more adequate to treat the meson equation in the form of a wave equation just like the cases of other charged particles[(1)], i.e. electron, positron and proton. Using the usual field equation for the meson as a wave equation, Laporte[(2)] has developed this stand point and calculated the elastic scattering of meson by a static electric field as a one body problem.

The authors themselves, when they were engaging in the establishment of the vector meson theory, tried to write the meson equation in the form analogous to Dirac's electron equation. But before they could finish this task, certain circumstances prevented one of the authors (M. T.) from working in this problem and in the meantime Duffin's[(3)] paper appeared.

Recently an important development of Duffin's theory was carried out by Kemmer[(4)]

The authors have investigated in this paper in I some properties of Duffin's equation and in II developed the perturbation theory analogous to the case of Dirac electron, and finally in III with some restriction they were able to reduce the representation of Duffin's β matrix ring from 10-rowed matrices to 6-rowed, and removed some

(1) Kemmer's reasoning[(4)] for a particle theory of the meson, however, seems to be inadequate from the epistemological point of view, because it is to be noted that, contrary to Kemmer's statement, the meson was found originaly as a field and later as a particle, just as, in the case of electromagnetic field, the Coulomb field was found first and then the photon.

(2) O. Laporte, Phys. Rev. **54** (1938), 905.

(3) R. J. Duffin, Phys. Rev. **54** (1938), 1114.

(4) N. Kemmer, Proc. Roy. Soc. **A 173** (1939), 91.

difficulties which were inevitable in the case of original Duffin-Kemmer's theory.

I. Some properties of Duffin's wave equation.

The meson is described by the Duffin's wave equation

$$\partial_\mu \beta_\mu \psi + \kappa \psi = 0 \tag{1}$$

together with the following commutation rules for the operator β_μ. (greek suffix runs from 1 to 4 and latin suffix from 1 to 3)

$$\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \beta_\mu \delta_{\nu\rho} + \beta_\rho \delta_{\nu\mu} \tag{2}$$

In (1)

$$\kappa = \frac{mc}{\hbar}, \quad \partial_\mu = \frac{\partial}{\partial x_\mu}, \quad x_4 = ict$$

are used, where m is the mass of the meson and the other symbols denote the usual quantities.

Kemmer introduced matrices η_μ:

$$\eta_\mu = 2\beta_\mu^2 - 1$$

obeying the following algebraic relations:—

$$\eta_\mu^2 = 1, \quad \eta_\mu \eta_\nu - \eta_\nu \eta_\mu = 0,$$

$$\eta_\mu \beta_\nu + \beta_\nu \eta_\mu = 0 \quad (\mu \neq \nu),$$

$$\beta_\mu = \eta_\mu \beta_\mu = \beta_\mu \eta_\mu. \qquad \text{(no summation)}$$

The adjoint of (1) is defined as follows:—

$$\psi^\dagger = i\psi^* \eta_4,$$

$$\partial_\mu \psi^\dagger \beta_\mu - \kappa \psi^\dagger = 0,$$

The interaction with the electromagnetic field is introduced by the substitution

$$\partial_\mu \to \partial_\mu^- = \partial_\mu - \frac{ie}{\hbar c} \Phi_\mu$$

when the differentiation applies to ψ, and

$$\partial_\mu \to \partial_\mu^+ = \partial_\mu + \frac{ie}{\hbar c} \Phi_\mu$$

when it applies to $\psi^\dagger$. Algebraic properties of β_μ matrices are investigated in Kemmer's paper.

(a) Spatial reflection:—Multiplying equation (1) on the left by η_4, we have

$$\partial_4 \beta_4 \eta_4 \psi - \partial_k \beta_k \eta_4 \psi + \kappa \eta_4 \psi = 0.$$

Therefore apart from a numerical factor, the transformation matrix for a spatial reflection through the origin must be equal to η_4:

$$\psi'(-x, -y, -z, t) = j\eta_4 \psi(x, y, z, t)$$

As by a double reflection ψ should return to itself, j^2 must be unity

$$j = \pm 1$$

$j = +1$ is pseudo case and $j = -1$ is ordinary vector and scalar case.

(b) Space-time reflection:—We put

$$\eta_5 = \eta_1\eta_2\eta_3\eta_4$$

and if we multiply equation (1) on the left by η_5, we have

$$\partial_\mu \beta_\mu \eta_5 \psi - \kappa \eta_5 \psi = 0.$$

Therefore ψ goes over

$$\psi^- = j\eta_5\psi, \qquad j^2 = 1$$

by a space time reflection. The charge density of ψ^- is

$$\frac{1}{i}(\psi^{-\dagger}\beta_4\psi^-) = -\frac{1}{i}(\psi^\dagger\beta_4\psi)$$

and Kemmer's $\theta_{\mu\nu}$ becomes

$$\theta^-_{\mu\nu} = \frac{1}{i}(\psi^{-\dagger}\beta_\nu\partial_\mu\psi^-) = -\frac{1}{i}(\psi^\dagger\beta_\nu\partial_\mu\psi) = -\theta_{\mu\nu}$$

(c) Charge conjugate[5]:—The complex conjugate of equation (1) with interaction is

$$\partial^\dagger_\mu \beta^*_\mu \psi^* + \kappa\psi^* = 0$$

After Belinfante we consider an operator Ω which obeys the following relations

$$\Omega\beta^*_4 = -\beta_4\Omega, \quad \Omega\beta^*_k = \beta_k\Omega \qquad (k = 1, 2, 3).$$

Then from the above equation it follows that

$$\partial^\dagger_\mu\beta_\mu\psi^\Omega + \kappa\psi^\Omega = 0,$$

with

$$\psi^\Omega = \Omega\psi^*,$$

and its charge density is

$$\frac{1}{i}(\psi^{\Omega\dagger}\beta_4\psi^\Omega) = -\frac{1}{i}(\psi^\dagger\beta_4\psi)$$

In the case of Proca's representation, we have

(5) F. J. Belinfante, Theory of Heavy Quanta (1939) p. 7.

$$\Omega=1, \quad \psi^{\Omega}=\psi^{*}$$

II. Perturbation theory.

The Hamiltonian for the meson is given by Kemmer as follows:—

$$H\psi=\frac{\hbar c}{\kappa}\partial_{\bar{k}}\,\frac{\beta_k\beta_4-\beta_4\beta_k}{i}\psi+mc^2\beta_4\psi-\frac{ie}{\kappa}F_{\nu\rho}(\beta_\rho\beta_4\beta_\nu-\delta_{\rho 4}\beta_\nu)\psi, \qquad (3)$$

with the initial condition

$$\partial_{\bar{k}}\beta_k\beta_4^2\psi+(1-\beta_4^2)\kappa\psi=0 \qquad (4)$$

Then energy and charge defined by Kemmer are

$$E=\frac{1}{i}\int\psi^{\dagger}\beta_4 H\psi dV$$

and

$$n=\frac{1}{i}\int\psi^{\dagger}\beta_4\psi dV$$

respectively. n is normalized to ± 1 and we put

$$\epsilon=n=\pm 1$$

Now we apply the perturbation theory in the usual way. The Hamiltonian H can be written in the form

$$H=H_0+H'$$

where H' represents a perturbing term which is small compared with H_0. H' does not contain the time explicitly. Taking the eigenvalues and eigenfunctions of the unperturbed system

$$E_n\psi_n=H_0\psi_n$$

we can develop the solution of the actual wave equation

$$-i\hbar\frac{\partial\psi}{\partial t}=(H_0+H')\psi$$

in a series of the eigenfunction of H_0:

$$\psi=\sum_n b_n(t)\psi_n\, e^{iE_nt/\hbar}$$

The coefficients $b_n(t)$ are functions only of the time and the ψ_n of the coordinates of the unperturbed system including the spin variables. We insert this ψ into above equation

$$-i\hbar\sum_n\dot{b}_n(t)\psi_n e^{iE_nt/\hbar}=\sum_n b_n(t)H'\psi_n\, e^{iE_nt/\hbar}$$

and operating by $\psi_n^{\dagger}\beta_4$ on the left and integrating over the whole coordinate space, we obtain, by applying the normalization condition

$$\frac{1}{i}\int \psi_m^+ \beta_4 \psi_n dV = \delta_{mn}\epsilon_n,$$

a system of differential equations for the $b_n(t)$

$$-i\hbar\epsilon_n \dot{b}_n(t) = \sum_n H_{nm}' b_m(t)\, e^{i(E_m - E_n)t/\hbar}$$

with

$$H_{nm}' = \frac{1}{i}\int \psi_n^+ \beta_4 H' \psi_m.$$

Putting the initial condition

$$b_n(o) = 0 \quad \text{except} \quad b_{n_0}(o) = 1$$

and using the fact that H' is small and that t is not too long, we obtain the solution

$$b_n(t) = \epsilon_n \frac{H_{nn_0}'(e^{i(E_{n_0} - E_n)t/\hbar} - 1)}{E_{n_0} - E_n}$$

and for the transition through the intermediate states n'

$$b_n(t) = \sum \epsilon_{n'}\epsilon_n \frac{H_{nn'}' H_{n'n_0}'}{E_{n_0} - E_{n'}}\left[\frac{e^{i(E_{n_0} - E_n)t/\hbar} - 1}{E_{n_0} - E_n} - \frac{e^{i(E_{n'} - E_n)t/\hbar} - 1}{E_n' - E_n}\right].$$

Now the total charge can be expressed in terms of $b_n(t)$ as follows:—

$$\int \frac{1}{i}(\psi^+ \beta_4 \psi)\, dV = \sum_n \epsilon_n |b_n(t)|^2.$$

From this expression it is possible to define the probability finding the particle in the state n, it is $|b_n(t)|^2$. In ordinary quantum mechanics with the non-relativistic Schrödinger equation, the charge density of a particle is positive definite and we can define it as the probability. In relativistic quantum mechanics, however, serious difficulty arose when the Klein-Gordon equation was used[6], because in this case the charge density is not positive definite. This difficulty was removed by Dirac's theory, in which the charge density is positive definite, and it has been considered that Dirac's theory is the only one which has both properties of being relativistic and of having real meaning. But we have here a real particle whose charge density is not positive definite, and the former difficulty has reappeared. Nevertheless it is possible to define the probability of finding the particle in the certain state by the above $|b_n(t)|^2$.

(6) W. Pauli, Handbuch der Physik, XXIV, p. 216.

For the purpose of applying Kemmer's Hamiltonian (3) to the perturbation calculation, we have to insert the initial condition (4) into it. Then the perturbing matrix element becomes

$$H_{nm}'=\frac{1}{i}\int\left[\frac{e^2}{\hbar ck}\Phi_k\Phi_r\psi_n^\dagger\beta_4^2\beta_k\beta_r\psi_m-\frac{ie}{\kappa}\Phi_r\partial_k\psi_n^\dagger\beta_4^2\beta_k\beta_r\psi_m\right.$$

$$\left.+\frac{ie}{\kappa}\Phi_k\psi_n^\dagger\beta_4^2\beta_k\beta_r\partial_r\psi_m\right]dV$$

or using the fact that ψ_n is the unperturbed wave function

$$=\frac{1}{i}\int\left[\frac{e^2}{\hbar c\kappa}\Phi_k\Phi_r\psi_n^\dagger\beta_4^2\beta_n\beta_r\psi_m-ie\Phi_k\psi_n^\dagger\beta_k\psi_m\right]dV$$

Applying Born's approximation, we have to add to above expression the term

$$+i\int\psi_n^\dagger\beta_4V\psi_m dV,$$

and to study the unperturbed system in some detail, whose Hamiltonian is

$$H_0=-i(\boldsymbol{p}\boldsymbol{\beta})\beta_4+\frac{1}{\hbar c\kappa}\beta_4(\boldsymbol{p}\boldsymbol{\beta})^2+\hbar c\kappa\beta_4,$$

we take the plane wave solution

$$\psi=ue^{i(\boldsymbol{pr})/\hbar c-iEt/\hbar}$$

where u is the wave amplitude depending upon $\boldsymbol{p}$ and the state of polarization e of this particle, then for the orthogonality relation we have

$$\frac{1}{i}\{u^\dagger(\boldsymbol{p},\epsilon',e')\beta_4u(\boldsymbol{p},\epsilon,e)\}=\epsilon\,\delta_{ee'}\delta_{\epsilon\epsilon'}\tag{5}$$

From this it is possible to deduce the converse orthogonality relations

$$\frac{1}{i}\sum_{\epsilon,e}\epsilon\{u^\dagger(\boldsymbol{p},\epsilon,e)\,\beta_4\}_l\,u_m(\boldsymbol{p},\epsilon,e)=\delta_{lm},\tag{6}$$

or

$$\frac{1}{i}\sum_{\epsilon,e}\epsilon\{u^\dagger(\boldsymbol{p},\epsilon,e)\beta_4^2\}_l\,\{\beta_4u(\boldsymbol{p},\epsilon,e\}_m=\delta_{lm},$$

$$\frac{1}{i}\sum_{\epsilon,e}\epsilon\{u^\dagger(\boldsymbol{p},\epsilon,e)\beta_4\}_l\,\{\beta_4^2u(\boldsymbol{p},\epsilon,e)\}_m=\delta_{lm},$$

$$\frac{1}{i}\sum_{\epsilon,e}\epsilon u_l^\dagger(\boldsymbol{p},\epsilon,e)\cdot\{\beta_4u(\boldsymbol{p},\epsilon,e)\}_m=\delta_{lm}.$$

Now it is possible to apply the spur calculation as in the case of Dirac's theory. From the above relations it follows that

$$\sum_{\epsilon,e} \epsilon \frac{1}{i} (u^\dagger \beta_4 O u) = \sum_{\epsilon,e} \epsilon \frac{1}{i} (u^\dagger O \beta_4 u) = \sum_{i} O_{ii};$$

$$\sum_{\epsilon',e'} \epsilon' \frac{1}{i^2} (u_0^\dagger O u') (u'^\dagger \beta_4 K u) = \sum_{\epsilon',e'} \frac{1}{i^2} (u^\dagger O \beta_4 u') (u'^\dagger K u)$$

$$= \frac{1}{i} (u_0^\dagger O K u)$$

where O and K are any products of β matrices. The conjugate complex of the expression $\frac{1}{2} (u_0^\dagger O u)$ is given by.

$$\left\{ \frac{1}{i} (u_0^\dagger O u) \right\}^* = -\frac{1}{i} (u^\dagger \eta_4 O \eta_4 u_0) = -(-1)^k \frac{1}{i} (u^\dagger O u_0)$$

where k is the sum of the number of β_k in O which lies between $u_0^\dagger$ and the first β_4 from left side and between u and the first β_4 from right side if O contains β_4, and the total number of β_k if it does not contain β_4 at all.

The above spur calculation is very much restricted compared to the case of the Dirac matrices, because we can not carry out the above calculation unless there is a β_4 adjacent to any one of the wave amplitudes in the above expressions. The difficulty is more serious if we try to carry out the following usual spur calculation.

$$\sum \epsilon\epsilon_0 \left| \frac{1}{i} (u_0^\dagger O \beta_4 u) \right|^2$$

$$= -\sum \epsilon_0 (-1)^k \frac{1}{i^2} (u_0^\dagger O \beta_4 u) (u^\dagger \beta_4 O u_0)$$

$$= -\sum \epsilon_0 (-1)^k \frac{1}{i^2} (u_0^\dagger O \beta_4 O u_0)$$

and we can not go further unless β_4 or β_4^2 and O commute with each other. Here we need β_4 on both sides of O. Thus the spur calculations do not have the generality as in the case of Dirac matrices. This comes from the fact that β_4 does not have its inverse matrix. This difficulty is removed in III.

Elastic scattering of meson:—For an example we shall apply the calculation to the case of elastic scattering of the meson which was discussed by Laporte[(2)]. The perturbing matrix element is

$$H_{nm}'=\frac{1}{i}\,(u_0^{\dagger}\beta_4 u)\int Ve^{-i(\boldsymbol{p}_0-\boldsymbol{p})\boldsymbol{r}/\hbar c}\,dV$$

where u_0 and u are the wave amplitudes of the initial and final states respectively, while $\boldsymbol{p}_0$ and $\boldsymbol{p}$ are their momentums. To obtain the scattering cross-section, the square of this matrix element is to be evaluated.

$$|H_{nm}'|^2=\epsilon^2\frac{1}{i^2}\,(u_0^{\dagger}\beta_4 u)\,(u^{\dagger}\beta_4 u_0)\,|(\boldsymbol{p}_0|\,V|\boldsymbol{p}\,)|^2$$

with

$$(\boldsymbol{p}_0|\,V\,|\,\boldsymbol{p})=\int Ve^{-i(\boldsymbol{p}_0-\boldsymbol{p})\boldsymbol{r}/\hbar c}\,dV$$

Making use of anihilation operators

$$\frac{H+E}{2E}\quad\text{and}\quad\epsilon=\frac{H}{E}$$

with $H=-i(\boldsymbol{p}\,\boldsymbol{\beta})\,\beta_4+\frac{1}{\hbar c\kappa}\,\beta_4(\boldsymbol{p}\boldsymbol{\beta})^2+\hbar c\kappa\beta_4$, (we denote the nonperturbed Hamililtonian in the initial and final state H_0 and H respectively) we carry out the summation over all polarization states of the meson after the scattering process.

$$=\sum\frac{1}{i^2}\left(u_0^{\dagger}\beta_4\,\frac{H+E}{2E}\,\frac{H}{E}\,\epsilon u\right)(u^{\dagger}\beta_4 u_0)\;|(\boldsymbol{p}_0|\,V\,|\,\boldsymbol{p}\,)|^2$$

$$=\frac{1}{i}\left(u_0^{\dagger}\beta_4\,\frac{H+E}{2E}\,\frac{H}{E}\,u_0\right)|(\boldsymbol{p}_0|\,V|\boldsymbol{p}\,)|^2$$

Further we average over the polarization states in the initial state.

$$=\sum\frac{1}{i}\left(u_0^{\dagger}\beta_4\,\frac{H+E}{2E}\,\frac{H}{E}\,\frac{H_0+E_0}{2E_0}\,\frac{H_0}{E_0}\,\epsilon_0 u_0\right)|(\boldsymbol{p}_0\,|\,V\,|\boldsymbol{p}\,)|^2$$

$$=\frac{1}{3}\,\mathrm{Spur}\left\{\frac{\overline{H}+E}{2E}\,\frac{\overline{H}}{E}\,\frac{\overline{H}_0+E_0}{2E_0}\,\frac{\overline{H}_0}{E_0}\right\}|(\boldsymbol{p}_0|\,V|\,\boldsymbol{p})|^2$$

where $\overline{H}$ is defined as follows ($\overline{H}_0$ in the same way)

$$\beta_4 H=\beta_4\overline{H}$$

$$\overline{H}=\frac{1}{\hbar c\kappa}\,\beta_4(\boldsymbol{p}\,\boldsymbol{\beta})^2+\hbar c\kappa\beta_4$$

using

$$Sp\beta_1^2\beta_2^2\beta_3^2\beta_4^2=0\ ,\ Sp\beta_\mu^2\beta_\nu^2\beta_\rho^2=1,\ \mu=\nu=\rho\quad Sp\beta_\mu^2\beta_\nu^2=3$$

and

$$Sp\beta_\mu^2=6\ .$$

We get finally

$$|H_{nm}'|^2=\left(1+\frac{2}{3}\frac{p_0^4}{4\mu^2E_0^2}\sin^2\theta\right)|(p_0|V|p)|^2$$

where
$$\mu=mc^2$$

This expression agrees with the expression obtained by Laporte.

III. Reduction of the wave equation.

To remove the difficulties which come from the fact that β_4 does not have its inverse matrix, and to simplify the theory, we shall apply Peirce's decomposition(7) of the ring theory.

We put

$$\beta_4^2=J$$

then J satisfies

$$J^2=J$$

thus J is an idempotent of this matrix-ring. Making use of this idempotent we can decompose the β matrix-ring $\mathfrak{B}$ in the following form:—

$$\mathfrak{B}=J\mathfrak{B}J+J\mathfrak{A}+\mathfrak{C}J+\mathfrak{D}$$

where $\mathfrak{A}$ and $\mathfrak{C}$ are the left and the right ideal respectively and $\mathfrak{D}$ is defined by

$$\mathfrak{D}=[\mathfrak{A},\ \mathfrak{C}]$$

of which element x satisfies

$$xJ=Jx=0$$

If we restrict ourselves the subring $J\mathfrak{B}J$, J is the unit element, because for any element b of $J\mathfrak{B}J$ we have

$$Jb=bJ=b$$

Now in any calculation, the Hamiltonian is always operated by β_4 on left, namely

$$\beta_4H=-\frac{\hbar c}{\kappa}\partial_{\bar{k}}\partial_{\bar{r}}\beta_4^2\beta_k\beta_r+\hbar c\kappa\beta_4^2 \qquad (7)$$

Here the matrices which we have to do with are only β_4 and $\beta_4^2\beta_k\beta_r$, and these matrices are all the elements of the subring $J\mathfrak{B}J$, because

$$\beta_4=J\beta_4J,$$

(7) H Weyl, The Classical Groups. p. 85.
B. L. van der Waerden, Moderne Algebra. p. 146.

$$\beta_4^2\beta_k\beta_r = J\beta_k\beta_r J.$$

Therefore if we restrict ourselves to equation (7), J becomes the unit matrix, and in this case β_4 has its inverse matrix:—

$$J=1, \quad \beta_4^{-1}=\beta_4$$

This restriction however by no means limits the solution of physical problems.

Now we introduce Duffin's spin matrices

$$S_1=\frac{1}{i}\ (\beta_2\beta_3-\beta_3\beta_2),\ S_2=\frac{1}{i}(\beta_3\beta_1-\beta_1\beta_3),\ S_3=\frac{1}{i}(\beta_1\beta_2-\beta_2\beta_1),$$

and we get the following relations:—

$$S_i^2=\frac{1}{2}(1-\eta_k\eta_r), \quad (i\neq k\neq r) \tag{8}$$

and

$$\beta_i\beta_k+\beta_k\beta_i=\eta_r(S_iS_k+S_kS_i). \quad (i\neq k\neq r)$$

From these we obtain

$$\beta_i^2=\eta_1\eta_2\eta_3\left(\frac{1}{2}-S_i^2\right)+\frac{1}{2}$$

$$\beta_k\beta_r=\frac{1}{2}\eta_1\eta_2\eta_3(1-2S_l^2)(S_kS_r+S_rS_k)\pm i\,\frac{1}{2}\,S_l \quad (l\neq k\neq r)$$

where the $\pm$ sign is to be read as follows: + when k, r is 1,2 ; 2,3 or 3,1 and − when k, r is 2,1 ; 3,2 or 1,3.

Making use of these relations and $\beta_4^2=1$, it is possible to write equation (7) in the following form:—

$$\beta_4H=-\frac{\hbar c}{\kappa}\,\partial_i^-\partial_i^-\left\{\eta_1\eta_2\eta_3\left(\frac{1}{2}-S_i^2\right)+\frac{1}{2}\right\}$$

$$-\frac{\hbar c}{\kappa}\,\partial_k^-\partial_r^-\,\frac{1}{2}\left\{\eta_1\eta_2\eta_3(1-2S_l^2)(S_kS_r+S_rS_k)\pm iS_l\right\}.$$

$$+\hbar c\kappa, \qquad (k\neq l\neq r) \tag{9}$$

Here the matrices which appear in this expression are only β_4, $\eta_1\eta_2\eta_3$ and S_i. Among these, β_4 and $\eta_1\eta_2\eta_3$ anticommute with each other, and these two commute with S_i, while S_i satisfy certain relations among themselves. Therefore according to the theory of Shur and Frobenius in Algebra[8], the former two belong to the system of linear transformation different from the system to which S_i belong, and these two systems operate on different suffices of the vector. Hence we can

(8) B. L. van der Waerden, Grúppentheoretische Methode in der Quantenmechanik (1932) § 13.

deal with and reduce these two systems separately, and the product of two reduced matrices each belonging to these different systems is a direct product.

We put
$$\eta=\eta_1\eta_2\eta_3$$
$$\zeta=-i\beta_4\eta_1\eta_2\eta_3$$
then β_4, η and ζ satisfy the following relations
$$\beta_4^2=1,\quad \eta^2=1,\quad \zeta^2=1$$
$$\beta_4\eta=-\eta\beta_4=i\zeta$$
$$\eta\zeta=-\zeta\eta=i\beta_4$$
$$\zeta\beta_4=-\beta_4\zeta=i\eta.$$
These relations are the same as those which are satisfied by Pauli's spin matrices, hence we may put
$$\left.\begin{aligned}\zeta&=\rho_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\\ \beta_4&=\rho_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\\ \eta&=\rho_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{aligned}\right\}\tag{10}$$

As to the spin matrices we obtain the following relations
$$\left.\begin{aligned}S_3S_2-S_2S_3&=iS_1,\\ S_1S_3-S_3S_1&=iS_2,\\ S_2S_1-S_1S_2&=iS_3,\end{aligned}\right\}\tag{I}$$
and further they satisfy the previously cited relation
$$S_i^2=\frac{1}{2}(1-\eta_k\eta_r),\quad (i\neq k\neq r)\tag{8}$$
From (1) and (8) we can deduce the following relation
$$S_iS_kS_r+S_rS_kS_i=S_i\,\delta_{kr}+S_r\delta_{ki}.\tag{II}$$

We can not go over from (I) to (II) nor vice versa, unless we make use of relation (8). Accordingly it is to be considered that the spin matrices S_i are defined by the two systems of relations (I) and (II).

Now we shall construct the unit matrix of the system of S_i matrices. We put
$$E=S_1^2+S_2^2+S_3^2-1$$

then E satisfies

$$E^2 = E$$

hence E, is an idempotent of the ring $\mathfrak{B}$. E satisfies moreover[9)]

$$S_i E = E S_i = S_i$$

i.e., E is the unit matrix of the subring consisted by the S_i matrices.

$$E = 1.$$

From this we obtain

$$S_1^2 + S_2^2 + S_3^2 = 2 \qquad (11)$$

On the other hand if we denote the spin eigen-value as s, we have

$$S_1^2 + S_2^2 + S_3^2 = s(s+1) \qquad (12)$$

From (11) and (12) it follows that

$$s = 1$$

i.e., this particle has the spin eigen-value 1.

Further we shall examine the algebraic properties of the S_i matrices. Although the S_i matrices satisfy both of the set of equations (I) and (II) it is interesting to compare this system to the system of algebra which satisfies only the set of equations (II). The following table gives the complete list of linearly independent elements of these two systems.

	if (II) only	Actual case of (I) and (II)
I	1	1
S_i	3	3
$S_i S_k$	6	3
$S_i S_k S_r$	3	
ξ_i	3	3
$\xi_i S_k$	6	
$\xi_i S_k S_r$	6	
$\xi_i \xi_k$	3	
$\xi_i \xi_k S_r$	3	
$\xi_i \xi_k \xi_r$	1	
Total Number	35	10

(9) The deduction of this relation is as follows:—from (I) and (II) we obtain

$$S_i S_k + S_k S_i = i\,(2S_k^2 - 1)\,S_r \qquad i \neq k \neq r$$

Using this relation and (I) we have

$$S_i S_k = i S_r S_k^2$$

This enables us to deduce the relation cited here.

Here ξ_i are defined by

$$\xi_i = 2S_i^2 - 1 = -\eta_k \eta_r, \qquad (i \neq k \neq r).$$

In the case where (II) alone is used, the following three elements commute with all others, and hence form the centrum.

I (the unit matrix), $M = \sum_i \xi_i - \sum_{i<k} \xi_i \xi_k$

$$N = \xi_1 \xi_2 \xi_3$$

The number of the elements of the centrum directly gives the number of inequivalent irreducible representations, say of degrees η and n_3 respectively, and

$$n_1^2 + n_2^2 + n_3^2 = 35$$

Their degrees are actually 5, 3, and 1 $(5^2 + 3^2 + 1^2 = 35)$.

In the actual case where both (I) and (II) are used, however, only I and M form the centrum and N becomes I itself.

$$N = I$$

Therefore in this case the number of inequivalent irreducible representations is two, and their degrees are 3 and 1 respectively, namely

$$3^2 + 1^2 = 10$$

For the representation of the degree three we may take the matrices which were found previously[10]:—

$$S_1 = S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad S_2 = S_y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$S_3 = S_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (13)$$

More detailed discussion on the algebraic properties of the β-matrix-ring will be carried out in the next paper.

Finally we can write down equation (9) as follows, (the products of ρ matrices and S_i matrices are to be understood as direct product):—

$$\rho_2 H = -\frac{\hbar c}{\kappa} \partial_i \partial_i \left\{ \rho_3 \left(\frac{1}{2} - S_i^2 \right) + \frac{1}{2} \right\}$$

(10) Yukawa, Sakata and Taketani, Proc. Phys.-Math. Soc. Japan **20** (1938), 319. Yukawa, Sakata and Kobayasi and Taketani, Proc. Phys.-Math. Soc. Japan **20** (1938), 720.

$$-\frac{\hbar c}{2\kappa}\partial_{\bar{l}}\partial_{\bar{r}}\{\rho_3(1-2S_l^2)(S_kS_r+S_rS_k)\pm iS_l\}+\hbar c\kappa,$$

$$(k\neq l\neq r)$$

or inserting the representation of the spin matrices into th's expression we have

$$-\frac{\hbar}{i}\rho_2\frac{\partial\psi}{\partial t}=mc^2\left\{1-\frac{\vec{D}^2-\frac{e}{\hbar c}(\vec{S}\vec{H})}{2\kappa^2}+\rho_2\frac{e\Phi_0}{mc^2}\right.$$

$$\left.+\rho_3\left[\frac{(\vec{S}\vec{D})^2}{\kappa^2}-\frac{\vec{D}^2-\frac{e}{\hbar}(\vec{S}\vec{H})}{2\kappa^2}\right]\right\}\psi$$

where $\vec{D}$ denotes the vector $\partial_{\bar{l}}$

The formulation of the vector meson theory using this wave equation is given by us in another paper[11] which is now in the press.

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(11) Sakata and Taketani, Scient. Pap. I. P. C. R. **37** (Sep., 1940) in the press.