On the WDVV-equation in quantum K-theory

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0. Introduction. Quantum cohomology theory can be described in general words as intersection theory in spaces of holomorphic curves in a given Kähler or almost Kähler manifold X. By quantum K-theory we may similarly understand a study of complex vector bundles over the spaces of holomorphic curves in X. In these notes, we will introduce a K-theoretic version of the Witten-Dijkgraaf-Verlinde-Verlinde equation which expresses the associativity constraint of the "quantum multiplication" operation on $K^*(X)$.

Intersection indices of cohomology theory,

$$\int_{[\text{ space of curves }]} \omega_1 \wedge \ldots \wedge \omega_k$$

obtained by evaluation on the fundamental cycle of cup-products of cohomology classes, are to be replaced in K-theory by Euler characteristics

$$\chi(\text{ space of curves }; V_1 \otimes \ldots \otimes V_k)$$

of tensor products of vector bundles. The hypotheses needed in the definitions of the intersection indices and Euler characteristics — that the spaces of curves are compact and non-singular, or that the bundles are holomorphic — are rarely satisfied. We handle this foundational problem by restricting ourselves throughout the notes to the setting where the problem disappears. Namely, we will deal with the so called *moduli spaces $X_{n,d}$ of degree d genus 0 stable maps to X with n marked points* **assuming that X is a homogeneous Kähler space**. Under the hypothesis, the moduli spaces $X_{n,d}$ (we will review their definition and properties when needed) are known to be compact complex orbifolds (see [9, 1]). We use their fundamental cycle $[X_{n,d}]$, well-defined over $\mathbb{Q}$, in the definition of intersection indices, and we use sheaf cohomology in the definition of the Euler characteristic of a holomorphic *orbi-bundle* V:

$$\chi(X_{n,d};V) := \sum (-1)^k \dim H^k(X_{n,d};\Gamma(V)).$$

1. Correlators. The WDVV-equation is usually formulated in terms of the following generating function for *correlators*:

$$F(t,Q) = \sum_d \sum_{n=0}^{\infty} \frac{Q^d}{n!} (t,...,t)_{n,d}.$$

Here $d \in H_2(X,\mathbb{Z})$ runs the Mori cone of *degrees*, that is homology classes represented by fundamental cycles of rational holomorphic curves in X, and the correlators $(\phi_1,...,\phi_n)_{n,d}$ are defined using the *evaluation maps* at the marked points:

$$\mathrm{ev}_1 \times ... \times \mathrm{ev}_n : X_{n,d} \to X \times ... \times X.$$

In cohomology theory, we pull-back to the moduli space $X_{n,d}$ the n cohomology classes $\phi_1,...,\phi_n \in H^*(X,\mathbb{Q})$ of X and define the correlator among them by

$$(\phi_1,...,\phi_n)_{n,d} := \int_{[X_{n,d}]} \mathrm{ev}_1^*(\phi_1) \wedge ... \wedge \mathrm{ev}_n^*(\phi_n).$$

In K-theory, we pull-back n elements $\phi_1,...,\phi_n \in K^*(X)$ (representable under our restriction on X by holomorphic vector bundles or their formal differences) and put

$$(\phi_1,...,\phi_n)_{n,d} := \chi(X_{n,d};\mathrm{ev}_1^*(\phi_1) \otimes ... \otimes \mathrm{ev}_n^*(\phi_n)).$$

We will treat the series F as a formal function of $t \in H$ depending on formal parameters $Q = (Q_1,...,Q_{\mathrm{Betti}_2(X)})$, where $H = H^*(X,\mathbb{Q})$ or $H = K^*(X)$.

Let $\{\phi_\alpha\}$ be a graded basis in $H^*(X,\mathbb{Q})$, and

$$g_{\alpha\beta} := \langle \phi_\alpha, \phi_\beta \rangle = \int_{[X]} \phi_\alpha \wedge \phi_\beta$$

denote the intersection matrix. Let $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ be the inverse matrix (so that $\sum (\phi_\alpha \otimes 1) g^{\alpha\beta} (1 \otimes \phi_\beta)$ is Poincare-dual to the diagonal in $X \times X$).

In quantum cohomology theory, one defines the *quantum cup-product* $\bullet$ on the tangent space T_tH by

$$\langle \phi_\alpha \bullet \phi_\beta, \phi_\gamma \rangle := F_{\alpha\beta\gamma}(t)$$

(where the subscripts on the RHS mean partial derivatives in the basis $\{\phi_\alpha\}$). In the above notation the associativity of the quantum cup-product is equivalent to the WDVV-identity:

$\sum_{\varepsilon,\varepsilon'} F_{\alpha\beta\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta}$ *is totally symmetric in* $\alpha, \beta, \gamma, \delta$.

2. Stable maps, gluing and contraction. In order to explain the proof of the WDVV-identity we have to discuss some properties of the moduli spaces $X_{n,d}$ (see [9, 1, 4] for more details).

We consider prestable marked curves $(C, \mathbf{z})$, that is compact connected complex curves C with at most double singular points and with n marked points $\mathbf{z} = (z_1, ..., z_n)$ which are non-singular and distinct. Two holomorphic maps, $f : (C, \mathbf{z}) \to X$ and $f' : (C', \mathbf{z}') \to X$, are called *equivalent* if they are identified by an isomorphism $(C, \mathbf{z}) \to (C', \mathbf{z}')$ of the curves. This definition induces the concept of *automorphism* of a map $f : (C, \mathbf{z}) \to X$, and one calls f *stable* if it has no non-trivial infinitesimal automorphisms. The moduli spaces $X_{n,d}$ consist of equivalence classes of stable maps with fixed number n of marked points, degree d and arithmetical genus 0 (it is defined as $g = \dim H^1(C, \mathcal{O}_C)$).

In plain words, the space of degree d holomorphic spheres in X with n marked points is compactified by prestable curves which are trees of $\mathbb{C}P^1$'s and satisfy the stability condition: each irreducible component $\mathbb{C}P^1$ mapped to a point in X must carry at least 3 marked or singular points. Under the hypothesis that X is a homogeneous Kähler space, the moduli space $X_{n,d}$ has the structure of a compact complex orbifold of dimension $\dim_{\mathbb{C}} X + \int_d c_1(T_X) + n - 3$.

In the case when X is a point the moduli spaces coincide with the Deligne-Mumford compactifications $\bar{\mathcal{M}}_{0,n}$ of moduli spaces of configurations of marked points on $\mathbb{C}P^1$. For instance, $\mathcal{M}_{0,4}$ is the set $\mathbb{C}P^1 - \{0, 1, \infty\}$ of legitimate values of the cross-ratio of 4 marked points on $\mathbb{C}P^1$. The compactification $\bar{\mathcal{M}}_{0,4} = \mathbb{C}P^1$ fills-in the fibidden values of the cross-ratio by equivalence classes of the reducible curves $\mathbb{C}P^1 \cup \mathbb{C}P^1$ with one double point and two marked point on each irreducible component.

For $n \geq 3$, there is a natural *contraction* map $X_{n,d} \to \bar{\mathcal{M}}_{0,n}$ defined by composing the map $f : (C, \mathbf{z}) \to X$ with $X \to pt$ (so that the components of C carrying < 3 special points become unstable) and contracting the unstable components. Similarly, one can define the *forgetting* maps $\mathrm{ft}_i : X_{n+1,d} \to X_{n,d}$ by disregarding the i-th marked point and contracting the component if it has become unstable.

In particular, we will make use of the contraction map

$$\mathrm{ct} : X_{n+4,d} \to \bar{\mathcal{M}}_{0,4}$$

defined by forgetting the map $f : (C, \mathbf{z}) \to X$ and all the marked points except the first four. A legitimate value $\lambda = \mathrm{ct}[f]$ of the cross-ratio means the following: the curve C has a component $C_0 = \mathbb{C}P^1$ carrying 4 special points with the cross-ratio λ, and the first 4 marked point are situated on the branches of the tree connected to C_0 at those 4 special points. A forbidden value $\mathrm{ct}[f] = 0$,1 or ∞ means that C containing a *chain* $C_0, ..., C_k$ of $k > 0$ of $\mathbb{C}P^1$'s such that 2 of the 4 branches of the tree carrying the marked points are connected to the chain via C_0, and the other two — via C_k. Such stable maps form a stratum of codimension k in the moduli space $X_{n,d}$. We will refer to them as strata (or stable maps) *of depth* k.

A stable map of depth 1 is glued from 2 stable maps obtained by disconnecting C_0 from C_1. This gives rise to the *gluing map*

$$X_{n_0+3,d_0} \times_\Delta X_{n_1+3,d_1} \to X_{n_0+n_1+4,d_0+d_1}$$

as follows. Consider the map from $X_{n_0+3,d_0} \times X_{n_1+3,d_1}$ to $X \times X$ defined by evaluation at the 3-rd marked points. The source of the gluing map is the preimage of the diagonal $\Delta \subset X \times X$. [1] It consists of pairs of stable maps which have the same image of the third marked point and which therefore can be glued at this point into a single stable map of degree $d_0 + d_1$ with $n_0 + 2 + n_1 + 2$ marked points.

Similarly, gluing stable maps of depth k from $k + 1$ stable maps subject to k diagonal constraints at the double points of the chain $C_0, ..., C_k$ defines appropriate gluing maps parameterizing the strata of depth k.

3. Proof of the WDVV-identity. All points in $\bar{\mathcal{M}}_{0,4}$ represent the same (co)homology class. Thus the analytic fundamental cycles of the fibers

[1] Note that for a homogeneous Kähler X, the evaluation map is conveniently transverse to the diagonal in $X \times X$.

$\mathrm{ct}^{-1}(\lambda)$ are homologous in $X_{n+4,d}$. The cohomological WDVV-identity follows from the fact that for $\lambda = 0$, 1 or ∞ the fiber $\mathrm{ct}^{-1}(\lambda)$ consists of strata of depth > 0, and moreover — the corresponding gluing maps (for all splittings $d = d_0 + d_1$ of the degree and all splittings of the $n = n_0 + n_1$ marked points), being isomorphisms at generic points, identify the analytic fundamental cycle of the fiber with the sum of the fundamental cycles of $X_{n_0+3,d_1} \times_\Delta X_{n_1+3,d_2}$. This allows one to equate 3 quadratic expression of the correlators which differ by the order of the indices $\alpha, \beta, \gamma, \delta$ associated with the first 4 marked points.

We leave the reader to work out some standard combinatorial details which are needed in order to translate this argument into the WDVV-identity for the generating function F and note only that the contraction with the intersection tensor $(g^{\varepsilon\varepsilon'})$ in the WDVV-equation takes care of the diagonal constraint $\Delta \subset X \times X$ for the evaluation maps.

In K-theory, similarly, the push-forward to $X \times X$ of the structural sheaf $\mathcal{O}_\Delta$ of the diagonal is expressed as

$$\sum (\phi_\varepsilon \otimes 1) g^{\varepsilon\varepsilon'} (1 \otimes \phi_{\varepsilon'})$$

via $(g^{\varepsilon,\varepsilon'})$ inverse to the "intersection matrix"

$$g_{\alpha\beta} := \langle \phi_\alpha, \phi_\beta \rangle = \chi(X; \phi_\alpha \otimes \phi_\beta).$$

The argument justifying the WDVV-equation fails, however, since the above gluing map to $\mathrm{ct}^{-1}(\lambda)$ is one-to-one only at the points of depth 1 and does not identify the corresponding structural sheaves. Indeed, a stable map of depth k can be glued from two stable maps in k different ways and thus belongs to the k-fold self-intersection in the image of the gluing map.

Let us examine the variety $\mathrm{ct}^{-1}(\lambda)$ at a point of depth $k > 1$. One of the properties of Kontsevich's compactifications $X_{m,d}$ is that *after passing to the local non-singular covers* (defined by the orbifold structure of the moduli spaces) *the compactifying strata form a divisor with normal crossings* [9, 1]. Moreover, analyzing (inductively in k) the local structure of the contraction map $\mathrm{ct} : X_{n+4,d} \to \bar{\mathcal{M}}_{0,4}$ near a depth-k point, one easily finds the local model $\lambda(x_1, ..., x_k, ...) = x_1 ... x_k$ for the map ct in a suitable local coordinate system. In this model, the components $x_1 = 0, ..., x_k = 0$ of the divisor with normal crossings represent the strata of depth 1, their intersections $x_{i_1} = x_{i_2} = 0$ — the strata of depth 2, etc. Denote by $\mathcal{O}$ the algebra of functions on our local chart, so that $\mathcal{O}/(x_{i_1},...,x_{i_l})$, $i_1 < ... < i_l$, are the algebras of functions on the depth-l strata. We have the following exact sequence of $\mathcal{O}$-modules:

$$0 \to \mathcal{O}/(x_1...x_k) \to \oplus\mathcal{O}/(x_i) \to \oplus\mathcal{O}/(x_{i_1},x_{i_2}) \to \oplus\mathcal{O}/(x_{i_1},x_{i_2},x_{i_3}) \to ....$$

Notice that the $\oplus$-terms in the sequence are the algebras of functions on the normalized strata of depth 1, depth 2, etc. Translating this local formula to a global K-theoretic statement about gluing maps, we conclude that in the Grothendieck group of orbi-sheaves on $X_{n+4,d}$), the element represented by the structural sheaf of $\mathrm{ct}^{-1}(\lambda)$ for $\lambda = 0{,}1$ or ∞ is identified with the structural sheaf of the corresponding alternated disjoint sum over positive depth strata:

$$\sum X_{n_0+3,d_0} \times_\Delta X_{n_1+3,d_1} - \sum X_{n_0+3,d_0} \times_\Delta X_{n_1+2,d_1} \times_\Delta X_{n_2+3,d_2} + ...$$

4. Formulation and consequences. Now we can apply the above K-theoretic statement about the moduli spaces to our generating functions.

Introduce

$$G(t,Q) := \frac{1}{2}\sum_{\alpha,\beta} g_{\alpha\beta}t_\alpha t_\beta \; + \; F(t,Q).$$

Let $(G^{\alpha\beta})$ be the matrix inverse to $(G_{\alpha\beta}) = (\partial_\alpha\partial_\beta G)$.

Theorem.

$$\sum_{\varepsilon,\varepsilon'} G_{\alpha\beta\varepsilon}G^{\varepsilon\varepsilon'}G_{\varepsilon\gamma\delta} \text{ is totally symmetric in } \alpha,\beta,\gamma,\delta.$$

Proof. We have rewritten

$$F_{\alpha\beta\varepsilon}g^{\varepsilon\varepsilon'}F_{\varepsilon'\gamma\delta} - F_{\alpha\beta\varepsilon}g^{\varepsilon\mu}F_{\mu\mu'}g^{\mu'\varepsilon'}F_{\varepsilon'\gamma\delta} + ...$$

using the famous matrix identity $1 - F + F^2 - ... = (1+F)^{-1}$. $\square$

Introduce the *quantum tensor product* on T_tH (with $H = K^*(X)$) by

$$(\phi_\alpha \bullet \phi_\beta, \phi_\gamma) := G_{\alpha\beta\gamma}(t),$$

and the metric $(,)$ on TH is defined by $(\phi_\mu,\phi_\nu) := G_{\mu\nu}(t)$.

Corollary 1. *The operations* $(,)$ *and* $\bullet$ *define on the tangent bundle the structure of a formal commutative associative Frobenius algebra with the unity* 1. [2]

Proof. As in the cohomology theory, it is a formal corollary of the Theorem, except that the statement about the unity 1 means that $G_{\alpha,1,\beta} = G_{\alpha\beta}$ and follows from the simplest instance of the *string equation* in the K-theory: $(1, t, ..., t)_{n+1,d} = (t, ..., t)_{n,d}$. The last equality is obvious. Indeed, the push-forward of the constant sheaf 1 along the map ft : $X_{n+1,d} \to X_{n,d}$ forgetting the first marked point is the constant sheaf 1 on $X_{n,d}$ since the fibers are curves C of zero arithmetic genus, $g = \dim H^1(C, \mathcal{O}_C) = 0$, while $H^0(C, \mathcal{O}_C) = \mathbb{C}$ by Liouville's theorem. □

We introduce on T^*H the 1-parametric family of connection operators

$$\nabla_q := (1 - q)d - \sum_\alpha (\phi_\alpha \bullet)^* \ dt_\alpha \wedge \ \ .$$

Corollary 2. *The connections* ∇_q *are flat for any* $q \neq 1$.

Proof. This follows from $\phi_\alpha \bullet \phi_\beta = \phi_\beta \bullet \phi_\alpha$, $d^2 = 0$, and $\partial_\alpha(\phi_\beta \bullet) = \partial_\beta(\phi_\alpha \bullet)$:

$$\partial_\alpha(\phi_\beta \bullet)^\nu_\mu = G_{\mu\alpha\beta\varepsilon} G^{\varepsilon\nu} - G_{\mu\beta\varepsilon} G^{\varepsilon\varepsilon'} G_{\varepsilon'\alpha\varepsilon''} G^{\varepsilon''\nu}$$

is symmetric with respect to α and β due to the WDVV-identity. □

Proposition. *The operator* ∇_{-1} *is twice the Levi-Civita connection of the metric* $(G^{\alpha\beta})$ *on* T^*H.

Proof. For a metric of the form $G_{\alpha\beta} = \partial_\alpha \partial_\beta G$ the famous explicit formulas for the Christoffel symbols yield

$$2\Gamma^\gamma_{\alpha\beta} = [G_{\alpha\varepsilon,\beta} + G_{\beta\varepsilon,\alpha} - G_{\alpha\beta,\varepsilon}] G^{\varepsilon\gamma} = G_{\alpha\beta\varepsilon} G^{\varepsilon\gamma} = (\phi_\beta \bullet)^\gamma_\alpha.$$

Corollary 3. *The metric* $(,)$ *on* TH *is flat.*

We complete this section with a description of flat sections of the connection operator ∇_q in terms of K-theoretic "gravitational descendents". Let us introduce the generating functions

$$S_{\alpha\beta}(t, Q) := g_{\alpha\beta} + \sum_{n,d} \frac{Q^d}{n!} (\phi_\alpha, t, ..., t, \frac{\phi_\beta}{1 - qL})_{n+2,d},$$

[2] At $t = 0, Q = 0$ it turns into the usual multiplicative structure on $K^*(X)$.

where the correlators are defined by

$$(\psi_1, ..., \psi_n L^k)_{m,d} := \chi(X_{m,d}; \mathrm{ev}_1^*(\psi_1) \otimes ... \otimes \mathrm{ev}_m^*(\psi_m) \otimes L^{\otimes k}).$$

Here L is the line *orbi*bundle over the moduli space $X_{m,d}$ of stable maps $(C, \mathbf{z}) \to X$ formed by the cotangent lines to C at the *last* marked point (as specified by the position of the geometrical series $1 + qL + q^2L^2 + ... = (1 - qL)^{-1}$ in the correlator).

Theorem. *The matrix* $S := (S_{\mu\nu})$ *is a fundamental solution to the linear PDE system:*

$$(1 - q)\partial_\alpha S = (\phi_\alpha \bullet)S \ .$$

Proof. Taking $\phi_\mu, \phi_\alpha, \phi_\beta$ and $\phi_\nu/(1 - qL)$ for the content of the four distinguished marked points in the proof of the WDVV-identity, we obtain its generalization in the form:

$$G_{\mu\alpha\varepsilon} G^{\varepsilon\varepsilon'} \partial_\beta S_{\varepsilon'\nu} = G_{\mu\beta\varepsilon} G^{\varepsilon\varepsilon'} \partial_\alpha S_{\varepsilon'\nu},$$

or $(\phi_\alpha \bullet)\partial_\beta S = (\phi_\beta \bullet)\partial_\alpha S$. Now it remains to put $\phi_\beta = 1$ and use $(1-q)\partial_1 S = S$, which is another instance of the string equation:

$$(1, t, ..., t, \phi L^k)_{n+2,d} = (t, ..., t, \phi(1 + L + ... + L^k))_{n+1,d}.$$

The last relation is obtained by computing the push forward of $L^{\otimes k}$ along $\mathrm{ft}_1 : X_{n+2,d} \to X_{n+1,d}$. [3]

5. Some open questions.

(a) Definitions. It is natural to expect that the above results extend from the case of homogeneous Kähler spaces X to general compact Kähler and, even more generally, almost Kähler target manifolds.

In the Kähler case, the moduli of stable degree d genus g maps with n marked points form compact complex orbi-spaces $X_{g,n,d}$ equipped with the

[3] Some details can be found in [15, 14, 11, 5]. Briefly, one identifies the fibers of ft_1 with the curves underlying the stable maps $f : (C, \mathbf{z}) \to X$ with $n+1$ marked points. It is important to realize that the pull-back $L' := \mathrm{ft}_1^*(L)$ of the line bundle named L on $X_{n+1,d}$ differs from the line bundle named L on $X_{n+2,d}$. In fact, there is a holomorphic section of $Hom(L', L)$ with the divisor D defined by the last marked point $z_{n+1} \in C$, and the bundle L restricted to D is trivial (while $L'|_D$ is therefore conormal to D). Since L' is trivial along the fibers C, we find that $H^1(C, L^k) = 0$ and $H^0(C, L^k) = (L')^k \otimes H^0(C, \mathcal{O}_C(kD)) \simeq (L')^k(1 + (L')^{-1} + ... + (L')^{-k})$.

intrinsic normal cone [13]. The cone gives rise [3] to an element in K-group of $X_{g,n,d}$ which should be used in the definition of K-theoretic correlators in the same manner as the virtual fundamental cycle $[X_{g,n,d}]$ is used in quantum cohomology theory.

The moduli space $X_{g,n,d}$ can be also described as the zero locus of a section of a bundle $E \to B$ over a non-singular space. Due to the famous "deformation to the normal cone" [3], the virtual fundamental cycle represents the Euler class of the bundle. This description survives in the almost Kähler case and yields a topological definition and symplectic invariance of the cohomological correlators. In K-theory, there exists a topological construction of the push forward from B to the point based on Whitney embedding theorem and Thom isomorphisms. We don't know however how to adjust the construction to our actual setting where B is non-singular only in the *orbi*fold sense.

One (somewhat awkward) option is to define K-theoretic correlators topologically by the RHS of the Kawasaki-Riemann-Roch-Hirzebruch formula [8] for orbi-bundles over B. This proposal deserves further study even in the Kähler case since it may lead to a "quantum Riemann-Roch formula".

(b) Frobenius-like structures. Our results in Section 4 show that K-theoretic Gromov-Witten invariants of genus 0 define on the space $H = K^*(X)$ a geometrical structure very similar to the *Frobenius structure* [2] of cohomology theory, but not identical to it.

One of the lessons is that the metric tensor on H, which can be in both cases described as $F_{\alpha,1,\beta}$, is constant in cohomology theory and equal to $g_{\alpha\beta}$ only by an "accident", but remains flat in K-theory even though it is not constant anymore.

The translation $t \mapsto t + \tau 1$ in the direction of $1 \in H$ leaves the structure invariant in cohomology theory, but causes multiplication by e^{τ} in K-theory — because of a new form of the string equation. Also, the $\mathbb{Z}$-grading missing in K-theory makes an important difference. It would be interesting to study the axiomatic structure that emerges here and to compare it with the structure implicitly encoded by K-theory on Deligne-Mumford spaces.

(c) Deligne-Mumford spaces. When the target space X is the point, the moduli spaces $X_{g,n,0}$ are Deligne-Mumford compactifications of the moduli spaces of genus g Riemann surfaces with n marked points. The parallel between cohomology and K-theory suggest several problems.

Holomorphic Euler characteristics of universal cotangent line bundles and their tensor products satisfy the string and dilation equations. [4] K-theoretic generalization of the rest of Witten – Kontsevich intersection theory [15, 10] is unclear.

The case of genus 0 and 1 has been studied in [14, 11] and [12]. The formula

$$\chi(\bar{\mathcal{M}}_{0,n}; \frac{1}{(1-q_1L_1)...(1-q_nL_n)}) = \frac{(1+q_1/(1-q_1)+...+q_n/(1-q_n))^{n-3}}{(1-q_1)...(1-q_n)}$$

found by Y.-P. Lee [11] is analogous to the famous intersection theory result [15, 9]

$$\int_{[\bar{\mathcal{M}}_{0,n}]} \frac{1}{(1-x_1c_1(L_1))...(1-x_nc_1(L_n))} = (x_1+...+x_n)^{n-3}.$$

The latter formula is a basis for fixed point computations [9, 5] in equivariant cohomology of the moduli spaces $X_{n,d}$ for toric X. As it was notices by Y.-P. Lee, the former formula is not sufficient for similar fixed point computation in K-theory: it requires Euler characteristics accountable for *invariants with respect to permutations of the marked points*. Finding an S_n-equivariant version of Lee's formula is an important open problem.

(d) Computations. The quantum K-ring is unknown even for $X = \mathbb{C}P^1$. It turns out that the WDVV-equation is not powerful enough in the absence of grading constraints and *divisor equation* (see, for instance, [5]).

On the other hand, for $X = \mathbb{C}P^n$, it is not hard to compute the generating functions $G(t,Q)$ and even $S_{\alpha\beta}(t,Q,q)$ *at* $t=0$ (see [12]). In cohomology theory, this would determine the *small* quantum cohomology ring due to the divisor equation which, roughly speaking, identifies the Q-deformation at $t=0$ with the t-deformation at $Q=1$ along the subspace $H^2(X,\mathbb{Q}) \subset H$. No replacement for the divisor equation seems to be possible in K-theory.

At the same time, the heuristic study [6] of S^1-equivariant geometry on the loop space LX suggests that the generating functions $S = S_{1,\beta}(0,Q,q)$ should satisfy certain linear q-difference equations (instead of similar linear

[4] The same is true not only for $X = pt$ (see [12]). By the way, the push forward $\mathrm{ft}_*(L)$ along $\mathrm{ft}: X_{g,n+1,d} \to X_{g,n,d}$, described by the dilation equation, equals $\mathcal{H}+\mathcal{H}^*-2+n$. Here $\mathcal{H}$ is the g-dimensional *Hodge bundle* with the fiber $H^1(C,\mathcal{O}_C)$. This answer replaces a similar factor $2g-2+n$ in the cohomological dilation equation, but also shows that tensor powers of $\mathcal{H}$ must be included to close up the list of "observables".

differential equations of quantum cohomology theory). This expectation is supported by the example of $X = \mathbb{C}P^n$: Y.-P. Lee [12] finds that the generating functions are solutions to the q-difference equation $D^{n+1}S = QS$ (where $(DS)(Q) := S(Q) - S(qQ)$).

In the case of the flag manifold X the generating functions S have been identified with the so called *Whittaker functions* — common eigen-functions of commuting operators of the q-difference Toda system. This result and its conjectural generalization [7] to the flag manifolds $X = G/B$ of complex simple Lie algebras links quantum K-theory to representation theory and quantum groups. Originally this conjecture served as a motivation for developing the basics of quantum K-theory.

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