ON THE WEAK HOMOLOGICAL DIMENSION OF THE GROUP ALGEBRA OF SOLVABLE GROUPS

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Let KG denote the group algebra of the group G over a field K of characteristic 0. The *weak (homological) dimension* of KG is defined as the largest integer q for which there exist KG-modules B, C such that $\mathrm{Tor}_q^{KG}(B, C) \neq 0$. By [**1**; p. 352] q is equal to the largest integer for which there exists a KG-module A such that $H_q(G, A) \neq 0$. We write w.dim $KG = q$.

The following is a corollary of the main result (Theorem 1): *For a solvable group G the weak dimension of KG is equal to the Hirsch number hG of G*, w.dim $KG = hG$.

As an application we obtain (Theorem 5):

If G is a nilpotent group of finite Hirsch number hG, then hG is equal to the largest integer q for which $H_q(G, K^+) \neq 0$, where K^+ denotes the additive group of K with trivial G-operation.

The starting point of our investigation was a result of K. W. Gruenberg [**3**], communicated privately to the author, which relates the Hirsch number of a torsion-free nilpotent group to the cohomological dimension of the group. We would like to express our sincere thanks to Gruenberg for showing us his result.

1. *The Hirsch number of G and the weak dimension of KG*

We define a class C of groups as follows: The group G is in C if and only if there exists a series of normal subgroups $\{N_i\}, i = 0, \ldots, k+1$

$$1 = N_{k+1} \subseteq N_k \subseteq \ldots \subseteq N_1 \subseteq N_0 = G$$

such that the successive quotients N_i/N_{i+1} for $i = 0, \ldots, k$ are either locally finite or abelian. Note that C contains all solvable groups.

Definition. Let G be a group in C, and $\{N_i\}$ a series of normal subgroups with the above property. We define the Hirsch number hG of G to be the sum of the ranks of those quotients N_i/N_{i+1} which are abelian. If this sum is not finite, we write $hG = \infty$.

The fact that hG does not depend on the series of subgroups is well known; also, it follows immediately from our Theorem 1. (In the literature the Hirsch number is usually defined for a slightly different class of groups, see for example [**4**; p. 150]. However, the above definition is better for our purpose.)

THEOREM 1. *For G in C we have* $hG =$ w.dim KG.

We first prove a special case:

PROPOSITION 2 [**2**]. *Let N be an abelian group of rank n. Then* w.dim $KN = n$ *and $H_n(N, K^+) = K$.*

Proof. We proceed by induction on n.

Received 23 May, 1969. Supported in part by NSF GP 7905.

[J. LONDON MATH. SOC. (2), 2 (1970), 567–570]

(*a*) Let N be an abelian group of rank zero, and let A be a KN-module. If N is finite of order $|N|$, then it is well known that $|N|.H_p(N, A) = 0$ for all $p \geqslant 1$. On the other hand $H_p(N, A)$ is a K-space. Since K is a field of characteristic 0, it follows that $H_p(N, A) = 0$ for $p \geqslant 1$. But trivially, $H_0(N, K^+) = K$.

If N is locally finite, then N is the direct limit of its finitely generated, hence finite subgroups N_α. We then have for $p \geqslant 1$

$$H_p(N, A) = \varinjlim H_p(N_\alpha, A) = 0$$

(see [**1**; p. 125]). Of course $H_0(N, K^+) = K$.

(*b*) If $n \geqslant 1$, then N contains at least one infinite cyclic subgroup C. Consider the extension $C \to N \to N/C$. Clearly, rank $(N/C) = n-1$. The result is now deduced from the Hochschild–Serre spectral sequence [**1**; p. 350]

$$H_p(N/C, H_q(C, A)) \Rightarrow H_n(N, A).$$

Since C is a free group and since $H_0(C, K^+) = K = H_1(C, K^+)$ we have w.dim $KC = 1$. By induction we have w.dim $K(N/C) = n-1$. By the "maximum principle for spectral sequences" it follows that

$$\text{w.dim}\, KN \leqslant \text{w.dim}\, K(N/C) + \text{w.dim}\, KC = n-1+1 = n.$$

But choosing $A = K^+$, we obtain $H_n(N, K^+) = H_{n-1}(N/C, H_1(C, K^+)) = K$.

Note that it follows from this proposition that $H_n(N, A) = A$ for any trivial KN-module A, for such a module is simply a K-space.

LEMMA 3. *Let N be a normal subgroup of G with quotient group G'. Suppose N is abelian of rank n. Then to any KG'-module A there exists a KG'-module B such that $H_n(N, B) \cong A$ as KG'-module.*

Proof: Let B be any trivial KN-module. By Proposition 2 we have $H_n(N, B) = H_n(N, K^+) \otimes_K B = K \otimes_K B$. In virtue of the fact that N is a normal subgroup of G, the group $H_n(N, K^+) = K$ has a G'-action which is generally non-trivial. We shall denote the *KG'-module* $H_n(N, K^+)$ by $\tilde{K}$. The G'-module structure of $\tilde{K} \otimes_K B$ is defined by $x(q \otimes b) = xq \otimes xb$ for $x \in G$, $q \in K$, $b \in B$. Denote by $\tilde{K}^{-1}$ the additive group of K together with the "inverse" G'-action, i.e., if $x \in G'$ sends $\mathbf{1} \in \tilde{K}$ into q, then x sends $\mathbf{1} \in \tilde{K}^{-1}$ into $1/q$. It is obvious that $\tilde{K}^{-1}$ is a KG'-module. Define $B = \tilde{K}^{-1} \otimes_K A$. We claim that $\tilde{K} \otimes_K (\tilde{K}^{-1} \otimes_K A) \cong A$ as KG'-modules. Clearly this is true as K-spaces; moreover

$$x(\mathbf{1} \otimes \mathbf{1} \otimes a) = q \otimes 1/q \otimes xa = xa.$$

LEMMA 4. *Let N be a normal subgroup of G with quotient group G'. Suppose N is abelian of rank n and suppose $hG' = m$, then* w.dim $KG = m+n$.

Proof. Consider the Hochschild–Serre spectral sequence

$$H_p(G', H_q(N, B)) \Rightarrow H_{p+q}(G, B).$$

Clearly w.dim $KG \leqslant n+m$. We show that there is a KG-module B such that $H_{n+m}(G, B) \neq 0$. By hypothesis there is a KG'-module A such that $H_m(G', A) \neq 0$. Take B to be the KG'-module (hence also KG-module) defined in Lemma 3. By the "maximum principle of spectral sequences" we obtain

$$H_{m+n}(G, B) = H_m(G', H_n(N, B)) = H_m(G', A) \neq 0.$$

Proof of Theorem 1. We proceed by induction on $k+h = l$ where k is the length of the given series of subgroups of G, and $h = hG$ is the Hirsch number of G.

(*a*) If $l = 0$, then G is the trivial group.

(*b*) Let $l \geqslant 1$. Divide by N the last nontrivial term N_k in the given series of subgroups, and let $G' = G/N$.

If N is locally finite, it is the direct limit of its finitely generated hence finite subgroups N_α. We therefore have $H_q(N, A) = \varinjlim H_q(N_\alpha, A) = 0$ for any KN-module A and $q \geqslant 1$. Hence by the Hochschild–Serre spectral sequence we obtain

$$H_q(G, A) = H_q(G', A_N).$$

This shows that $\text{w.dim}\, KG = \text{w.dim}\,(G') = h(G') = hG$.

If N is abelian of rank n, then we have $\text{w.dim}\, KG = \text{w.dim}\, K(G')+n$ by Lemma 2. Hence by induction $\text{w.dim}\, KG = h(G')+n = hG$.

Finally, if $hG = \infty$, then G contains subgroups of any Hirsch number; hence $\text{w.dim}\, KG$ cannot be finite.

Remark. The conjecture that hG is also equal to the cohomological dimension of KG turns out to be false: any infinite, locally finite abelian group is a counter example.

2. *Nilpotent groups.*

THEOREM 5. *Let G be a nilpotent group with $hG < \infty$. Then hG is equal to the largest integer q for which $H_q(G, K^+)$ (or equivalently $H^q(G, K^+)$) is nontrivial.*

Proof. In view of Theorem 1 we have to show that for $h = hG$, $H_h(G, K^+) \neq 0$. We prove $H_h(G, K^+) = K$ by induction on the length k of the lower central series of G.

(*a*) If G is abelian, Proposition 2 shows that

$$H_h(G, K^+) = K$$

for $h = hG$.

(*b*) If G is not abelian denote by G_k the last nonzero term of the lower central series of G. Since $[G, G_k] = 1$, the subgroup G_k must be contained in the centre of G. By the Hochschild–Serre spectral sequence

$$H_p(G/G_k, H_q(G_k, K^+)) \Rightarrow H_n(G, K^+)$$

we obtain for $p = h(G/G_k), q = hG_k$, and $h = hG$

$$H_h(G, K^+) = H_p(G/G_k, H_q(G_k, K^+)) = H_p(G/G_k, K^+) = K,$$

using the fact that G_k is central, Proposition 2 and the induction assumption, respectively.

The cohomology part now follows from the remark that

$$H^q(G, K^+) = \text{Hom}_K(H_q(G, K^+), K^+).$$

References

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