International Mathematical Forum, 4, 2009, no. 12, 569 - 579

On the Weak Type of Meromorphic Functions

Sanjib Kumar Datta

Department of Mathematics
University of North Bengal
PIN: 734013, West Bengal, India
sanjib_kr_datta@yahoo.co.in

Arindam Jha

Tarangapur N.K. High School
Post: Tarangapur, Uttar Dinajpur
PIN: 733129, West Bengal, India
d_ajha@yahoo.co.in

Abstract

In this paper we introduce the definition of weak type of meromorphic functions and establish its integral representation. We also investigate some growth properties related to the weak type of meromorphic functions.

Mathematics Subject Classification: 30D35, 30D30

Keywords: Order, lower order, type, weak type, integral representation, growth

1 Introduction, Definitions and Notations.

Let f be a meromorphic function of finite positive order ρ_f defined in the open complex plane $\mathbb{C}$. The type σ_f of f is defined as follows :

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}.$$

When f is entire one can easily verify that

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}},\ 0 < \rho_f < \infty.$$

In the paper we introduce the following two definitions :

Definition 1 *The weak type* τ_f *of a meromorphic function* f *of finite positive lower order* λ_f *is defined by*

$$\tau_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}}.$$

For entire f,

$$\tau_f = \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda_f}},\ 0 < \lambda_f < \infty.$$

Definition 2 *A meromorphic function* f *of finite non zero lower order* λ_f *is said to be of weak type* τ_f *if the integral* $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr\ (r_0 > 0)$ *converges for* $k > \tau_f$ *and diverges for* $k < \tau_f$.

In this paper we establish the equivalence of Definition 1 and Definition 2. We also study some growth properties related to the weak type of meromorphic functions. We do not explain the standard definitions and notations of the theory of entire and meromorphic functions as those are available in [4] and [3].

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([1]) *If* f *is meromorphic and* g *is entire then for all sufficiently large values of* r,

$$T(f\circ g) \le \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 2 *Let the integral* $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr\ (r_0 > 0)$ *converges where* $0 < \lambda_f < \infty$. *Then*

$$\liminf_{r\to\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k}} = 0.$$

Proof. Since the integral $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr\ (r_0 > 0)$ converges, then

$$\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr < \varepsilon,\ \text{if } r_0 > R(\varepsilon).$$

Therefore,

$$\int_{r_0}^{\exp\left\{(r_o)^{\lambda_f}\right\}+r_0} \frac{\exp\left\{T\left(r,f\right)\right\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}}dr < \varepsilon.$$

Since $\exp\left\{T\left(r,f\right)\right\}$ increases with r, so

$$\int_{r_0}^{\exp\left\{(r_o)^{\lambda_f}\right\}+r_0} \frac{\exp\left\{T\left(r,f\right)\right\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}}dr \geq \frac{\exp\left\{T\left(r_0,f\right)\right\}}{\left[\exp\left(r_0^{\lambda_f}\right)\right]^{k+1}}.\left[\exp\left(r_o^{\lambda_f}\right)\right].$$

i.e., for all large values of r,

$$\int_{r_0}^{\exp\left\{(r_o)^{\lambda_f}\right\}+r_0} \frac{\exp\left\{T\left(r,f\right)\right\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}}dr \geq \frac{\exp\left\{T\left(r_0,f\right)\right\}}{\left[\exp\left(r_0^{\lambda_f}\right)\right]^{k}},$$

so that

$$\frac{\exp\left\{T\left(r_0,f\right)\right\}}{\left[\exp\left(r_0^{\lambda_f}\right)\right]^{k}} < \varepsilon \text{ if } r_0 > R\left(\varepsilon\right).$$

$$i.e., \liminf_{r\to\infty} \frac{\exp\left\{T\left(r,f\right)\right\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k}} = 0.$$

This proves the lemma. ■

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 *Let f be a meromorphic function of lower order λ_f and of weak type τ_f. Also let $0 < \lambda_f < \infty$. Then Definition 1 and Definition 2 are equivalent.*

Proof.
Case 1. $\tau_f = \infty$.
Definition 1 $\Rightarrow$ **Definition** 2.

As $\tau_f = \infty$, from Definition 1 we obtain for arbitrary G and for all sufficiently large values of r,

$$\begin{aligned} T(r,f) &> G.\left(r^{\lambda_f}\right) \\ i.e., \ \exp\{T(r,f)\} &> \left[\exp\left(r^{\lambda_f}\right)\right]^G . \end{aligned} \tag{1}$$

If possible let the integral $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{G+1}} dr$ $(r_0 > 0)$ be converge.
Then by Lemma 2,

$$\liminf_{r\to\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^G} = 0.$$

So for a sequence of values of r tending to infinity

$$\exp\{T(r,f)\} < \left[\exp\left(r^{\lambda_f}\right)\right]^G . \tag{2}$$

Therefore from (1) and (2) we arrive at a contradiction.
Hence $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{G+1}} dr$ $(r_0 > 0)$ diverges whenever G is finite, which is the Definition 2.

Definition 2 $\Rightarrow$ **Definition** 1**.**

Let G be any positive number. Since $\tau_f = \infty$, from Definition 2, the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{G+1}} dr$ $(r_0 > 0)$ gives for arbitrary positive ε and for all sufficiently large values of r that

$$\begin{aligned} \exp\{T(r,f)\} &> \left[\exp\left(r^{\lambda_f}\right)\right]^{G-\varepsilon} \\ i.e., \ T(r,f) &> (G-\varepsilon)\, r^{\lambda_f}, \end{aligned}$$

which implies that

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} \geq G - \varepsilon.$$

Since G is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} = \infty.$$

Thus Definition 1 follows.

Case 2. $0 \leq \tau_f < \infty$.

Subcase (a). $0 < \tau_f < \infty$.

Let f be of type τ_f, where $0 < \tau_f < \infty$. Then according to the Definition 1, for arbitrary positive ε and for a sequence of values of r tending to infinity we obtain that

$$\begin{aligned} T(r,f) &< (\tau_f+\varepsilon) r^{\lambda_f} \\ i.e., \ \exp\{T(r,f)\} &< \left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f+\varepsilon} \\ i.e., \ \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k}} &< \frac{\left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f+\varepsilon}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k}} \\ i.e., \ \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k}} &< \frac{1}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k-(\tau_f+\varepsilon)}}. \end{aligned} \tag{3}$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr \ (r_0 > 0)$ converges for $k > \tau_f$.
Again by Definition 1, we obtain for all large values of r that

$$\begin{aligned} T(r,f) &> (\tau_f-\varepsilon) r^{\lambda_f} \\ i.e., \ \exp\{T(r,f)\} &> \left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f-\varepsilon}. \end{aligned} \tag{4}$$

So for $k < \tau_f$, we get from (4) that

$$\frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k}} > \frac{1}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k-(\tau_f-\varepsilon)}}.$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr \ (r_0 > 0)$ diverges for $k < \tau_f$.
Hence $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr \ (r_0 > 0)$ converges for $k > \tau_f$ and diverges for $k < \tau_f$.

Subcase (b)**.** $\tau_f = 0$.

When f is of weak type $\tau_f = 0$, Definition 1 gives for a sequence of values of r tending to infinity that

$$\frac{T(r,f)}{r^{\lambda_f}} < \varepsilon.$$

Then as before we obtain that $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr \ (r_0 > 0)$ converges for $k > 0$ and diverges for $k < 0$.
Thus combining Subcase (a) and Subcase (b), Definition 2 follows.

Definition 2 ⇒ Definition 1.

Since f is of weak type τ_f, by Definition 2, for arbitrary positive ε the integral $\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f+\varepsilon+1}} dr \; (r_0 > 0)$ converges.
Then by Lemma 2,

$$\liminf_{r\to\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f+\varepsilon}} = 0.$$

So we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f+\varepsilon}} &< \varepsilon \\ i.e.,\ \exp\{T(r,f)\} &< \varepsilon.\left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f+\varepsilon} \\ i.e.,\ T(r,f) &< \log\varepsilon + (\tau_f+\varepsilon)\, r^{\lambda_f} \\ i.e.,\ \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} &\leq \tau_f + \varepsilon. \end{aligned}$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} \leq \tau_f. \tag{5}$$

On the otherhand the divergence of the integral $\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f-\varepsilon+1}} dr \; (r_0 > 0)$ implies that

$$\liminf_{r\to\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f-\varepsilon}} = \infty,$$

i.e., for all sufficiently large values of r,

$$\begin{aligned} \exp\{T(r,f)\} &> \left[\exp\left(r^{\lambda_f}\right)\right]^{\tau_f-\varepsilon} \\ i.e.,\ T(r,f) &> (\tau_f-\varepsilon)\, r^{\lambda_f}. \end{aligned}$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} \geq \tau_f. \tag{6}$$

So from (5) and (6) we obtain that

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} = \tau_f.$$

This proves the theorem. ■

In the following theorem we obtain a relationship between τ_f and σ_f.

Theorem 2 *Let f be a meromorphic function such that λ_f and ρ_f are both finite. Also let f be of regular growth i.e., $\lambda_f = \rho_f$. Then $\tau_f = \sigma_f$.*

Proof. Since f is of regular growth, we get that

$$\begin{aligned} \sigma_f &= \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} \\ &\geq \liminf_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} = \tau_f. \end{aligned} \tag{7}$$

On the otherhand Definition $7\,(cf.[2])$ implies that the integral $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\rho_f}\right)\right]^{k+1}} dr\ (r_0 > 0)$ converges for $k > \sigma_f$ and diverges for $k < \sigma_f$.
From Definition 2 it follows that the integral $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr\ (r_0 > 0)$ is convergent for $k > \tau_f$ and diverges for $k < \tau_f$.

Also all the quantities in the expression

$$\left[\frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} - \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\rho_f}\right)\right]^{k+1}}\right]$$

are of non negative type. So

$$\int\limits_{r_0}^{\infty} \left[\frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} - \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\rho_f}\right)\right]^{k+1}}\right] dr\ (r_0 > 0) \geq 0.$$

$$i.e., \int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\lambda_f}\right)\right]^{k+1}} dr \geq \int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\left(r^{\rho_f}\right)\right]^{k+1}} dr \text{ for } r_0 > 0.$$

$$i.e., \ \tau_f \geq \sigma_f. \tag{8}$$

Hence from (7) and (8) we obtain that

$$\tau_f = \sigma_f.$$

Thus the theorem is established. ∎

Theorem 3 *If f be a meromorphic function of regular growth i.e., $\lambda_f = \rho_f$, then the following quantities*

$$(i)\, \sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, \ \ (ii)\, \tau_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}},$$

$$(iii) \liminf_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} \ \textit{and} \ (iv) \limsup_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}}$$

are all equivalent.

Proof.
$(i) \Rightarrow (ii)$.
In view of Theorem 2, as f is of regular growth

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} = \tau_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}}.$$

$(ii) \Rightarrow (iii)$.
Since f is of regular growth i.e., $\lambda_f = \rho_f$ we get that

$$\tau_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}.$$

$(iii) \Rightarrow (iv)$.
In view of Theorem 2 and the condition $\lambda_f = \rho_f$ it follows that

$$\begin{aligned} \liminf_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} &= \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} = \tau_f = \sigma_f \\ &= \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}}. \end{aligned}$$

$(iv) \Rightarrow (i)$.
As f is of regular growth i.e., $\lambda_f = \rho_f$, we obtain that

$$\limsup_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}} = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} = \sigma_f.$$

Thus the theorem follows. ■

Theorem 4 *Let f be a meromorphic function and g be an entire function satisfying* $(i)\, 0 < \lambda_g \le \rho_g < \infty$, $(ii)\, \tau_g > 0$, $(iii)\, \sigma_g < \infty$, $(iv)\, 0 < \lambda_{f\circ g} < \infty$ *and* $(v)\, 0 < \tau_{f\circ g} < \infty$. *Then*

(a). $\liminf_{r\to\infty} \frac{T(r,f\circ g)}{T(r,g)} \ge \frac{\tau_{f\circ g}}{\sigma_g}$ *if* $\lambda_{f\circ g} = \rho_g$ *and*

(b). $\liminf_{r\to\infty} \frac{T(r,f\circ g)}{T(r,g)} \le \frac{\tau_{f\circ g}}{\tau_g}$ *if* $\lambda_{f\circ g} = \lambda_g$.

Proof. From the definitions of $\tau_{f\circ g}, \tau_g$ and σ_g we get for arbitrary positive ε and for all sufficiently large values of r,

$$T(r, f\circ g) \ge (\tau_{f\circ g} - \varepsilon)\, r^{\lambda_{f\circ g}} \tag{9}$$

$$T(r, g) \ge (\tau_g - \varepsilon)\, r^{\lambda_g} \tag{10}$$

$$T(r,g) \leq (\sigma_g + \varepsilon) r^{\rho_g}. \tag{11}$$

Also for a sequence of values of r tending to infinity,

$$T(r, f \circ g) \leq (\tau_{f \circ g} + \varepsilon) r^{\lambda_{f \circ g}}. \tag{12}$$

Hence from (9) and (11) we obtain for all sufficiently large values of r that

$$\frac{T(r, f \circ g)}{T(r,g)} \geq \frac{(\tau_{f \circ g} - \varepsilon) r^{\lambda_{f \circ g}}}{(\sigma_g + \varepsilon) r^{\rho_g}}.$$

Since $\lambda_{f \circ g} = \rho_g$ and $\varepsilon\,(> 0)$ is arbitrary it follows from above that

$$\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r,g)} \geq \frac{\tau_{f \circ g}}{\sigma_g}.$$

This proves Theorem 4 (a).
Also for a sequence of values of r tending to infinity we get from (10) and (12) that

$$\frac{T(r, f \circ g)}{T(r,g)} \leq \frac{(\tau_{f \circ g} + \varepsilon) r^{\lambda_{f \circ g}}}{(\tau_g - \varepsilon) r^{\lambda_g}}.$$

Since $\varepsilon\,(> 0)$ is arbitrary and $\lambda_{f \circ g} = \lambda_g$, we obtain from above that

$$\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r,g)} \leq \frac{\tau_{f \circ g}}{\tau_g}.$$

This proves Theorem 4 (b). ■

Corollary 1 *If in addition g be of regular growth i.e., $\lambda_g = \rho_g$ then*

$$\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r,g)} = \frac{\tau_{f \circ g}}{\sigma_g} = \frac{\tau_{f \circ g}}{\tau_g}.$$

Proof. Since g be of regular growth *i.e.*, $\lambda_g = \rho_g$, Corollary 1 follows from Theorem 2 and Theorem 4. ■

Theorem 5 *Let f be a meromorphic function and g be an entire function with $(i)\, 0 < \lambda_g = \rho_{f \circ g} < \infty$, $(ii)\, \sigma_{f \circ g} < \infty$ and $(iii)\, \tau_g > 0$. Then*

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r,g)} \leq \frac{\sigma_{f \circ g}}{\tau_g}.$$

Proof. From the definition of σ_{fog} we get for all sufficiently large values of r that

$$T(r, f \circ g) \leq (\sigma_{fog} + \varepsilon) r^{\rho_{f \circ g}}. \tag{13}$$

Now from (10) and (13) it follows for all sufficiently large values of r that

$$\frac{T(r, f \circ g)}{T(r, g)} \leq \frac{(\sigma_{fog} + \varepsilon) r^{\rho_{f \circ g}}}{(\tau_g - \varepsilon) r^{\lambda_g}}.$$

Since $\varepsilon\,(> 0)$ is arbitrary and $\lambda_g = \rho_{f \circ g}$, we obtain from above that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq \frac{\sigma_{fog}}{\tau_g}.$$

Thus the theorem is established. ■

Theorem 6 *If f be meromorphic and g be entire such that $(i)\, 0 < \rho_g < \infty$, $(ii)\, \sigma_g < \infty$, $(iii)\, \rho_g = \lambda_f$, $(iv)\, 0 < \lambda_f \leq \rho_f < \infty$ and $(iv)\, \tau_f > 0$, then*

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \rho_f . \frac{\sigma_g}{\tau_f}.$$

Proof. In view of Lemma 1 and $T(r, g) \leq \log^+ M(r, g)$, we get for all sufficiently large values of r that

$$\begin{aligned} \log T(r, f \circ g) &\leq o(1) + \log T(M(r, g), f) \\ i.e., \ \log T(r, f \circ g) &\leq o(1) + (\rho_f + \varepsilon) \log M(r, g). \end{aligned} \tag{14}$$

Now from the definition of σ_g we obtain for all sufficiently large values of r that

$$\log M(r, g) \leq (\sigma_g + \varepsilon) r^{\rho_g}. \tag{15}$$

Also from the definition of τ_f we have for all sufficiently large values of r,

$$T(r, f) \geq (\tau_f - \varepsilon) r^{\lambda_f}. \tag{16}$$

So from (14), (15) and (16) it follows for all sufficiently large values of r that

$$\frac{\log T(r, f \circ g)}{T(r, f)} \leq \frac{o(1) + (\rho_f + \varepsilon)(\sigma_g + \varepsilon) r^{\rho_g}}{(\tau_f - \varepsilon) r^{\lambda_f}}.$$

Since $\varepsilon\,(> 0)$ is arbitrary and $\rho_g = \lambda_f$ we get from above that

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} \leq \rho_f . \frac{\sigma_g}{\tau_f}.$$

This proves the theorem. ■

Theorem 7 *Let f be meromorphic and g be entire satisfying* $(i)\, 0 < \lambda_g < \infty$, $(ii)\, \rho_f < \infty$, $(iii)\, \sigma_g < \infty$ *and* $(iv)\, \tau_g > 0$. *Also let g be of regular growth i.e.,$\lambda_g = \rho_g$. Then*

$$\limsup_{r\to\infty} \frac{\log T\,(r, f\circ g)}{T\,(r,g)} \leq \rho_f.$$

Proof. In view of (10), (14) and (15) we obtain for all sufficiently large values of r that

$$\frac{\log T\,(r, f\circ g)}{T\,(r,g)} \leq \frac{o\,(1) + (\rho_f + \varepsilon)\,(\sigma_g + \varepsilon)\, r^{\rho_g}}{(\tau_g - \varepsilon)\, r^{\lambda_g}}.$$

Since $\varepsilon\,(>0)$ is arbitrary and $\lambda_g = \rho_g$,Theorem 7 follows from Theorem 2. ■

References

[1] Bergweiler, W. : *On the Nevanlinna Characteristic of a composite function,* Complex Variables 10 (1988) , pp. 225 – 236.

[2] Datta, S. K. : *A note on the order and type of a meromorphic function,* J. Indian Acad. Math., Vol. 27, No. 2 (2005), pp. 317 – 327.

[3] Hayman, W.K. : *Meromorphic functions,* The Clarendon Press, Oxford, 1964.

[4] Valiron, G. : *Lectures on the general theory of integral functions,* Chelsea Publishing Company (1949).

Received: July, 2008