ON THE WEDDERBURN PRINCIPAL THEOREM(1)

BY

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Introduction. Let A be a strictly power-associative algebra with radical N and such that the difference algebra $A-N$ is separable. Then we say that A has a Wedderburn decomposition if A has a subalgebra $S \cong A-N$ with $A=S+N$ (vector space direct sum).

It is known that associative [1], alternative [15] and Jordan [3], [13] algebras have Wedderburn decompositions. In addition, if A is commutative and $A-N$ separable with no nodal subalgebras such that either the simple summands of $A-N$ have degree $\geqq 3$ or A is stable, then A has a Wedderburn decomposition [5].

For our purposes, an algebra is a finite dimensional vector space on which a multiplication is defined which satisfies both distributive laws. We define $x^1=x$ and $x^{k+1}=x^k x$. An algebra A is called power-associative if $x^\alpha x^\beta = x^{\alpha+\beta}$ for all positive integers α and β and every x in A. An algebra is called strictly power-associative if A_K is power-associative for every scalar extension K of the base field. As a consequence of [11], if char $\neq 2, 3, 5$ then a power-associative algebra is strictly power-associative.

In this paper, the radical N of A is the maximal nil ideal and a nonnil algebra with zero radical is said to be semisimple. An algebra is separable if it is semisimple over every scalar extension of the base field. We will call an algebra simple if it is semisimple and contains no proper ideals. The associator $(x, y, z)=(xy)z-x(yz)$ and the commutator $(x, y)=xy-yx$. An algebra is nodal if each element can be written as $\alpha \cdot 1+z$ with z nilpotent where the set of nilpotent elements is not a subalgebra. The center of A is the set of all elements that commute and associate with all of A. It is known [16, p. 16] that the center of a simple algebra is a field and is a finite extension of the base field.

For char $\neq 2$, define $x \cdot y=(xy+yx)/2$ and define A^+ as the vector space A with multiplication defined by $x \cdot y$. When char $\neq 2$, it has been proved [2] that if A is power-associative and if e is an idempotent ($e^2=e\neq 0$) then

(1) $A=A_e(1)+A_e(1/2)+A_e(0)$ where $A_e(t)=\{x : x\cdot e=tx\}$. In addition, we have [2]:

(2) $A_e(1)A_e(0)=A_e(0)A_e(1)=0$,

(3) $xe=ex=0$ for x in $A_e(0)$,

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(4) $xe=ex=x$ for x in $A_e(1)$,

(5) $A_e(1/2)\cdot A_e(1/2)\subseteq A_e(1)+A_e(0)$,

(6) $A_e(1)\cdot A_e(1/2)\subseteq A_e(1/2)+A_e(0)$,

(7) $A_e(0)\cdot A_e(1/2)\subseteq A_e(1/2)+A_e(1)$.

An idempotent e is principal if $A_e(0)$ contains only nilpotent elements. An idempotent e is primitive if e is the only idempotent in $A_e(1)$ and e is absolutely primitive if e is primitive in A_K for K any extension of the base field. At times it will simplify notation to write $A(e, t)$ for $A_e(t)$.

We will define K to be a splitting field of a nonnil strictly power-associative algebra A if every primitive idempotent e of A_K-N_K is absolutely primitive and if every element in $(A_K-N_K)_e(1)$ for e primitive can be written as $ke+y$ with y nilpotent and k in K. If K is the algebraic closure of the base field F of A, it clearly is a splitting field of A but may not be a finite extension of F.

We will give two fairly general approaches to proving the Wedderburn Principal Theorem. Our first approach will be to reduce the question to the case where $A-N$ is simple and A has a unity element. This approach will apply to any decomposable class of algebras (a term defined in the next section). When A has a unity element and $A-N$ is simple, we will give conditions on A that force A to have a Wedderburn decomposition.

In our second approach, we give a set of conditions on an algebra A that force A to have a Wedderburn decomposition.

Finally, we apply the material in §§2 and 3 to derive the Wedderburn Principal Theorem for certain classes of algebras [2], [4], [6], [7], [8], [9], [10], [12], [14].

2. **First approach.** We will first prove the following lemma. Our proof is, for the most part, an adaptation to the noncommutative case of the proof given by Hemminger in the commutative case [5].

LEMMA 2.1. *Let* $[u_1],\ldots,[u_t]$ *be pairwise orthogonal idempotents in* $A-M$ (M *a nil ideal of* A *and* $[x]$ *the image of* x *in* $A\to A-M$) *for* A *a strictly power-associative algebra over a field of* char $\neq 2, 3$ *and let* $u=u_1+\cdots+u_t$. *Then, there exists an idempotent* e *in* A *and pairwise orthogonal idempotents* $e_1,\ldots,e_t$ *in* $A_e(1)$ *such that*

(8) $[e]=[u]$ *and*

(9) $[e_i]=[u_i]$ $(i=1,\ldots,t)$. *Furthermore, if* A *has a unity element* 1 *and* $[1]=[u]$, *then* $e=1$.

Proof. We will prove the first part by induction on t. When $t=1$, $u=u_1$. In addition, $[u^k]=[u]^k=[u]$ so u is not nilpotent. Hence, the associative algebra of all polynomials in u, denoted by $F[u]$, is not nilpotent and thus contains an idempotent $e=f(u)$ for $f(x)$ a polynomial. Then $[e]=[f(u)]=a[u]$ where $a=f(1)$ is in F. Thus $a[u]=[e]=[e]^2=a^2[u]^2=a^2[u]$. Since e is not in M, $a[u]\neq 0$ and $a=1$.

Let $w=u_1+u_2$ for $t\geqq 2$. Then $u=w+u_3+\cdots+u_t$ for pairwise orthogonal idempotents $[w], [u_3],\ldots,[u_t]$. By the inductive hypothesis, there exists an idempotent e and pairwise orthogonal idempotents $f, e_3,\ldots,e_t$ in $A_e(1)$ such that

$e=f+e_3+\cdots+e_t$, $[e]=[u]$, $[f]=[w]$, and $[e_i]=[u_i]$ for $i=3,\ldots,t$. Now $[f][u_1]=[w][w-u_2]=[w]-[u_2]=[u_1]$ and $[u_1][f]=[u_1]$. If $[f][x]=[x]=[x][f]$ for x in A we can write $x=x_1+x_{1/2}+x_0$ for x_a in $A_f(a)$ and have $[x_1]+[x_{1/2}]+[x_0]=[x]=(1/2)([f][x]+[x][f])=[x_1]+(1/2)[x_{1/2}]$ so $[x]=[x_1]$. Therefore, there is an element x_1 in $A_f(1)$ with $[x_1]=[u_1]$. Also, x_1 is not nilpotent for $[x_1^k]=[u_1^k]=[u_1]^k=[u_1]$ so, as in the case $t=1$, we have an idempotent e_1 in $F[x_1]\subseteq A_f(1)$ (powers of x_1 are in $A_f(1)$ for $A_f(1)\cdot A_f(1)\subseteq A_f(1)$) such that $[e_1]=[x_1]=[u_1]$. Now $e_2=f-e_1$ is an idempotent in $A_f(1)$, $e_1e_2=e_1(f-e_1)=0=e_2e_1$ and $[e_2]=[f-e_1]=[f]-[e_1]=[w]-[u_1]=[u_2]$. Also $A_f(1)A_f(0)=A_f(0)A_f(1)=0$ so $e_ie_j=e_je_i=0$ for $i=1,2; j=3,\ldots,t$. If x were in $A_f(1)\cap A_{e-f}(a)$ for $a=1/2$, 1, then $x\cdot e=x+ax=(1+a)x$. Let $x=x_1+x_{1/2}+x_0$ for x_b in $A_e(b)$. We obtain $x\cdot e=x_1+(1/2)x_{1/2}$. Therefore $(1+a)x_1=x_1$, $2(1+a)x_{1/2}=x_{1/2}$, and $(1+a)x_0=0$. Since char $\neq 2, 3$ and $a=1/2$ or 1, $x=0$. Hence, $A_f(1)\subseteq A_e(1)$ and $e_1,\ldots,e_t$ are in $A_e(1)$. We have proved the first part of the lemma. If $[1]=[e]$, then $1-e$ is in M and is nilpotent. Hence it is zero and $e=1$.

In reducing the theorem to the case where $A-N$ is simple, we will need the following definitions.

Definition 2.1. A class of algebras will be called a Wedderburn class if each algebra A in the class has a Wedderburn decomposition.

Definition 2.2. A class P of algebras will be called a decomposable class if for each A in P:

(a) A is strictly power-associative over a field of char $\neq 2, 3$.

(b) $A-N$ is in P.

(c) If B is a subalgebra of A whose image in $A\to A-N$ is a nonnil ideal in $A-N$, then B is in P.

(d) A semisimple implies $A=A_1\oplus\cdots\oplus A_t$ where each A_i is simple with a unity element.

(e) $A_e(t)A_e(t)\subseteq A_e(t)$, $(t=0, 1)$ if e is an idempotent in A.

Definition 2.3. A member A of a class P is in the center $C(P)$ if A has a unity element and $A-N$ is simple.

We remark that each specific class of algebras shown to be a Wedderburn class [1], [3], [5], [13], [15] has been a decomposable class.

Theorem 2.1. *A decomposable class P is a Wedderburn class if and only if $C(P)$ is a Wedderburn class.*

Proof. In one direction, the implication is obvious. Suppose that $C(P)$ is a Wedderburn class. For A in P, $A-N=B_1\oplus\cdots\oplus B_t$ where each B_i is simple. From Lemma 2.1 and the fact that each B_i has a unity element $[u_i]$, there exists pairwise orthogonal idempotents $e_i,\ldots,e_t$ in A such that $[e_i]$ is the unity element of B_i $(i=1,\ldots,t)$. Denote $A(e_i, 1)$ by A_i. Clearly, $A_i\subseteq A(e_j, 0)$ $(j\neq i)$ for if $x\cdot e_i=x$ and $x\cdot e_j=tx$, $t=1/2$ or 1; then $x\cdot(e_i+e_j)=(3/2)x$ or $2x$ contrary to fact. Hence, from (2) and condition (e), $A_iA_j\subseteq\delta_{ij}A_i$ (Kronecker delta) for $i,j=1,\ldots,t$. Also,

$e=e_t+\cdots+e_t$ is principal and $A_e(1/2)+A_e(0)\subseteq N$. It is clear that the radical N_i of A_i contains $A_i \cap N$. In addition, $A(e_i, 1/2)\subseteq N$ for $A=A_1+\cdots+A_t+M$ where $M\subseteq N$. Now, $(N_i+N)A_i\subseteq N_i+N$ and $A_i(N_i+N)\subseteq N_i+N$. Since $A(e_i, 1/2) \subseteq N$, $A(e_i, 1/2)N_i \cup N_iA(e_i, 1/2)\subseteq N$. Also, $(N_i+N)(A(e_i, 0))=NA(e_i, 0)\subseteq N$ and $(A(e_i, 0))(N_i+N)\subseteq N$. Therefore N_i+N is an ideal in A. For x in N_i and y in N, $(x+y)^k=x^k+y_1$ where y_1 is in N. If $x^n=0$, $(x+y)^n=y_1$ in N and therefore $x+y$ is nilpotent. Hence, $N_i+N\subseteq N$ and $N_i\subseteq A_i \cap N$. Therefore $A_i-N_i\cong B_i$. But, then, A_i is in $C(P)$ so there is a subalgebra S_i of A_i with $A_i=S_i+N_i$ and $S_i\cong B_i$. If $S= S_1 \oplus\cdots\oplus S_t$, $A=S+N$ and $S\cong A-N$.

In the following discussion, $H(e)$ is the ideal generated by $A_e(1/2)$.

THEOREM 2.2. *Let A be an algebra in the center of a decomposable class over a splitting field F of* char $\neq 2, 3$. *If we further assume that $A-N$ is not nodal then the following are equivalent*:

(10) *A has a Wedderburn decomposition.*

(11) *There exists a primitive idempotent e for which either $H(e)$ has a Wedderburn decomposition or $H(e)\subseteq N$.*

(12) *Either $A-N\cong F$ or there exists a nonnil ideal $H(e)$ which has a Wedderburn decomposition.*

(13) *There exists a nonnil ideal of A which has a Wedderburn decomposition.*

(14) *There exists a nonnil subalgebra B with a Wedderburn decomposition such that, in the mapping $T\colon A \to A-N$, the image $T(B)$ is an ideal.*

Proof. Suppose (10) and let $A=S+N$ with $S\cong A-N$. Let e be a primitive idempotent of S and assume that $H(e)\nsubseteq N$. Now $H([e])$, the ideal of $A-N$ generate by $(A-N)([e], 1/2)$ is the image of $H(e)$ in the mapping $A \to A-N$. Consequently $H([e])\neq 0$ so, since $A-N$ is simple, $H([e])=A-N$. In the mapping $A \to A-N$, S is mapped onto $A-N$. Clearly then $S_e(t)$ is mapped onto $(A-N)([e], t)$ for $t=1$, $1/2$, 0. (Since e is in S, $S=S_e(1)+S_e(1/2)+S_e(0)$.) Hence, $S_e(1/2)\neq 0$ and $H(e) \cap S\neq 0$. Now, $H(e) \cap S$ is an ideal in the simple algebra S so $S\subseteq H(e)$. If M is the radical of $H(e)$ then $H(e)=S+M$ is a Wedderburn decomposition for $H(e)$ and (11) is established.

To prove that (11) $\Rightarrow$ (12), let us suppose that no $H(e)$ has a Wedderburn decomposition. Then by (11) there exists a primitive idempotent e with $H(e)\subseteq N$. In the mapping $A \to A-N$, $A_e(t)$ is mapped onto $(A-N)([e], t)$ for $t=1$, $1/2$, 0. Hence $(A-N)([e], 1/2)=0$ and by (2) and (e) of Definition 2.2,

$$A-N = (A-N)([e], 1) \oplus (A-N)([e], 0).$$

But, $A-N$ is simple and $[e]$ is in $(A-N)([e], 1)$ so $(A-N)([e], 0)=0$. If $[e]= [u] \oplus [v]$ for idempotents u and v then Lemma 2.1 guarantees the existence of idempotents f and g in $A_e(1)$ with $e=f\oplus g$ contrary to the primitiveness of e. Thus, $[e]$ is primitive. Now, F is a splitting field, so any element in $A-N= (A-N)([e], 1)$ has the form $\alpha[e]+z$ with α in F and z nil. Since $A-N$ is not nodal then M, the set of all nilpotent elements of $A-N$, is a subalgebra of $A-N$. Clearly, M is an ideal of $A-N$ so $M=0$. We then have $A-N=\{\alpha\cdot[e] : \alpha \text{ in } F\}\cong F$ and (12) is proved. If $A-N\cong F$ then $A-N=\{\alpha\cdot[e] : \alpha \text{ in } F\}$ and by Lemma 2.1, e can be chosen as an idempotent of A. Under this condition, $A=F\cdot e+N$ is a Wedderburn decomposition for A which may be regarded as an ideal of A.

The implication (12) $\Rightarrow$ (13) is now easy to see.

To prove (13) $\Rightarrow$ (14), we merely observe that the ideal whose existence is guaranteed by (13) satisfies the conditions of (14).

Finally, let B be a nonnil subalgebra satisfying (14). Since $A-N$ is simple and $T(B)$ is an ideal then $T(B)=0$ or $T(B)=A-N$. But B is not nil so $T(B)\neq 0$. Therefore, $T(B)=A-N$. If $B=S+M$ with $S\cong A-N$ is a Wedderburn decomposition for B, then we claim that $A=S+N$ is a Wedderburn decomposition for A. For $S\cong A-N$, $S\cap N=0$ and $\dim A=\dim S+\dim N$ ($\dim A$=dimension of A as a vector space). Thus, (14) $\Rightarrow$ (10) and the theorem is proved.

We remark here that the condition that $A-N$ be not nodal cannot be removed. Let E_1 be the algebra spanned by 1, x, x^2 and y with $xy=-yx=1$ and $x^2x=xx^2=y^2=x^2y=yx^2=x^2x^2=0$. Let $u=\alpha\cdot 1+v$ where $v=\beta x+\gamma y+\delta x^2$. Then $v^2=\beta^2x^2+\beta\gamma-\gamma\beta=\beta^2x^2$ and $v^2v=vv^2=0$. By induction, we see that $u^n=\alpha^n\cdot 1+n\alpha^{n-1}v+((n^2-n)/2)\alpha^{n-2}v^2$. From this, it is easy to show that E_1 is power-associative. In addition, $N=\{\alpha x^2\}$ is the radical of E_1 and E_1-N is simple and nodal. (Also, E_1 is nearly antiflexible (see [14]).) If S is a proper subalgebra of E_1 then $S=\{\alpha\cdot 1\}$ which is not isomorphic to E_1-N or S has a nonzero radical. Let $u=\alpha\cdot 1+v$ with $v=\beta x+\gamma y+\delta x^2$ be in S. Suppose we have $\alpha=0$ for some $u\neq 0$ in S. Then $u^2=v^2=\beta^2x^2$ so $N\subseteq S$ or $\beta=0$. But, if $\beta=0$, then $\{\varepsilon u\}=\{\varepsilon\gamma y+\varepsilon\delta x^2\}$ is a nil ideal in S. Now suppose we have $\alpha\neq 0$ for every $u\neq 0$ in S. Since $u^2-\alpha u=\alpha v+v^2=\alpha\beta x+\alpha\gamma y+(\alpha\delta+\beta^2)x^2$ is in S then $u^2=\alpha u$ for the coefficient of 1 in $u^2-\alpha u$ is 0. But this implies (u/α) is an idempotent. However, 1 is primitive so we have $u/\alpha=1$ or $u=\alpha\cdot 1$. Since this holds for every $u\neq 0$ in S, $S=\{\alpha\cdot 1\}$. Hence, E_1 does not possess a Wedderburn decomposition. However, E_1 trivially satisfies (11) for 1 is its only idempotent and $H(1)=0\subseteq N$.

The following lemma generalizes Lemmas 2.1 and 2.2 of [5].

Lemma 2.2. *Let Q be a class of algebras such that*:

(15) *If A is in Q, then $A-N$ is simple.*

(16) *If A is in Q, and if $M\subseteq N$ is an ideal of A then $A-M$ is in Q.*

(17) *If B is a subalgebra of Q whose image in $A\to A-N$ is a nonnil ideal of $A-N$ then B is in Q.*

If every algebra B in Q for which $\dim B<\dim C$ *has a Wedderburn decomposition and if C contains an ideal M other than* 0, *N and C then C has a Wedderburn decomposition.*

Proof. The result is obvious if $\dim C=1$. We will assume then that $\dim C>1$ and will first prove the assertion for the case when M is nil. By the homomorphism theorems, $C-N\cong(C-M)-(N-M)$ and $N-M$ is the radical of $C-M$. Now $C-M$ is in Q by (16) and $\dim(C-M)<\dim C$ so there exists $C_0\subseteq C-M$ with $C_0\cong(C-M)-(N-M)\cong C-N$. Also, by the homomorphism theorems, there exists a C_1 in C with $C_1\neq 0$, C such that $M\subseteq C_1$ and $C_0\cong C_1-M$. Now C_1 is nonnil and C_1-M is simple so $(C_1-M)\cap(N-M)=0$. Therefore, if $T(C_1)$ is the image of C_1 in the mapping $C\to C-N$, then

$$T(C_1)\cong C_1-M\cong C_0\cong C-N.$$

Thus $T(C_1)=C-N$ and by (17), C_1 is in Q. Since also $\dim C_1<\dim C$, there is a subalgebra S in $C_1\subseteq C$ with $C_1=S+M$ and $S\cong C_1-M$. Now, S is simple so $S\cap N=0$. Also, $S\cong C-N$. We then have $\dim C=\dim S+\dim N$ so $C=S+N$ is a Wedderburn decomposition.

Now suppose M is not nil so that $M\nsubseteq N$. If $N\subset M$ with $N\neq M$, then $N-M$ is an ideal in $C-N$ which implies $M-N=C-N$. By (17), M is in Q and there is a subalgebra S of $M\subseteq C$ with $S\cong M-N=C-N$. Since S is simple, $S\cap N=0$ and, by a dimension argument, $C=S+N$. If $M\cap N\neq 0$ and $N\nsubseteq M$ then $M\cap N$ is a nil ideal unequal to 0, N and C and C has a Wedderburn decomposition by the first part of this proof.

We now assume that $M\cap N=0$. From this we see that M is not nil. Consequently, the image $T(M)$ of M in $C\to C-N$ is a nonnil ideal of $C-N$ so M is in Q. Hence, there exists a subalgebra S of $M\subseteq C$ with $S\cong T(M)$. From the simplicity of $C-N$, we have $T(M)=C-N$ which implies S is simple. Thus, $S\cap N=0$ and a dimension argument implies $C=S+N$ is a Wedderburn decomposition for C.

3. **Second approach.** The main theorem of this section is

THEOREM 3.1. *Let A be a nonnil strictly power-associative algebra over a splitting field with* char $\neq 2, 3$ *and assume the following*:

(18) *$A-N$ contains no simple nodal subalgebras for N the radical of A.*

(19) *For every idempotent e, the following are equivalent*:

(a) *$H(e)$ is nil.*

(b) *$H(e)\cap A_e(1)$ is nil.*

(c) $\mathrm{H}(e)\cap A_e(0)$ *is nil.*

(20) *For every idempotent e and for $t\neq 1/2$, $A_e(t)A_e(t)\subseteq A_e(t)$.*

(21) *There exists a set $e_1,\ldots,e_n$ of pairwise orthogonal primitive idempotents with $e=\sum_{i=1}^n e_i$ principal such that $L=\sum_{i=1}^n H(e_i)$ has a Wedderburn decomposition if L is not nil.*

With these assumptions, A has a Wedderburn decomposition.

We will prove this theorem in a series of lemmas. First let us note that the algebra E_1 constructed in the last section satisfies (19), (20) and (21) so (18) cannot be eliminated in the hypotheses. It is not known if (19) or (21) can be removed. To see that (18), (19) and (21) are not sufficient, let E_2 be the algebra spanned by e,

x, x^2, y with $xy=-yx=e$, $e^2=e$ and all other products zero. It is easily seen that E_2 satisfies (18), (19) and (21) but does not have a Wedderburn decomposition.

LEMMA 3.1. *If A is a semisimple power-associative algebra over a field of* char $\neq 2$ *satisfying* (19) *and* (20), *then A is a direct sum of simple algebras each of which has an identity element.*

Proof. The result is obvious if dim $A=1$. Suppose dim $A=n$ and the lemma is true for all algebras with dimension less than n satisfying the hypotheses.

Let J be an ideal of A of smallest nonzero dimension. Since A is semisimple, J is not nil so it must have an idempotent. Let e be any idempotent in J. If x is in $A_e(1)$, then $x=xe \in AJ \subseteq J$ so $A_e(1) \subseteq J$. If x is in $A_e(1/2)$ then $x=xe+ex \in AJ+JA \subseteq J$ so $A_e(1/2) \subseteq J$. Therefore $H(e) \subseteq J$. But J is an ideal of A of smallest nonzero dimension so $H(e)=J$ or $H(e)=0$. Now if e is an idempotent principal in J then $J_e(0)$ is nil. However, $H(e) \subseteq J$ so $H(e) \cap A_e(0) \subseteq J_e(0)$. Therefore, by (19), $H(e)$ is nil. But A is semisimple so $H(e)=0$. Therefore, by (20) and (2), $A=A_e(1) \oplus A_e(0)$. Since $A_e(1) \subseteq J$, $e \in A_e(1)$ and J is an ideal of smallest nonzero dimension, $J=A_e(1)$.

Let K be an ideal of J. Since $K \subseteq A_e(1)$, $KA_e(0)=A_e(0)K=0$ so $KA \subseteq KJ \subseteq K$ and $AK \subseteq JK \subseteq K$. Therefore, $K=0$ or J and we have shown that J is simple and possesses an identity element. If $A_e(0) \neq 0$, then $A_e(0)$ is a semisimple power-associative algebra over a field of char $\neq 2$ and satisfies (20). If K is an ideal of $A_e(0)$, $KA_e(1)=A_e(1)K=0$ so $KA+AK \subseteq K$ and K is an ideal of A. From this it is easy to verify that (19) holds in $A_e(0)$. Hence, by the inductive hypothesis, $A_e(0)$ is a direct sum of simple algebras each of which has an identity element. Consequently, A is a direct sum of simple algebras each of which has an identity element.

In the light of Lemma 3.1, one would expect to find that the class of all algebras satisfying the hypotheses of Theorem 3.1 is a decomposable class. The answer is not fully known but the following two lemmas will show how close we are to the answer.

LEMMA 3.2. *The class C_1 of all nonnil strictly power-associative algebras over splitting fields with* char $\neq 2, 3$ *that satisfy* (18), (19) *and* (20) *is a decomposable class.*

Proof. Conditions (a) and (e) are directly assumed and condition (d) was proved in Lemma 3.1. To verify (b), we need only show that if A is in C_1 then $A-N$ satisfies (19) and (20). Let $[u]$ be any idempotent of $A-N$ with u in A. Lemma 2.1 implies the existence of an idempotent e in A with $[e]=[u]$. Since (19) and (20) hold for e in A, it is ceasily shown that they hold for $[e]$ in $A-N$.

Condition (c) is all that we now need to verify. Let B be a subalgebra of A whose image in $T: A \to A-N$ is a nonnil ideal in $A-N$ where A is in C_1. Write $A'=A-N$ and $B'=T(B)$. Since B' is not nil, there exists an idempotent e principal in B'. If x is in $A'_e(1)$, then $x=xe \in A'B' \subseteq B'$ so $A'_e(1) \subseteq B'$. Also, if x is in $A'_e(1/2)$, then $x=xe+ex \in A'B'+B'A' \subseteq B'$ so $A'_e(1/2) \subseteq B'$. Consequently, $H(e) \subseteq B'$. Now,

$B'_e(0)$ is nil for e is principal. Also, $H(e) \cap A'_e(0) \subseteq B' \cap A'_e(0) \subseteq B'_e(0)$ so $H(e) \cap A'_e(0)$ is nil. Since $A' = A - N$ is in C_1, (19) holds in A' so $H(e)$ is nil. But, A' is semisimple so $H(e) = 0$. Now, by (2) and (20),

$$\begin{aligned} B'_e(0)(A') &= B'_e(0)(A'_e(1) + A'_e(0)) \\ &= B'_e(0)A'_e(0) \\ &\subseteq B' \cap A'_e(0) = B'_e(0). \end{aligned}$$

Also, $A'B'_e(0) \subseteq B'_e(0)$ so $B'_e(0)$ is a nil ideal of A'. Thus, $B'_e(0) = 0$ and $A' = B' \oplus A'_e(0)$. As a result of this, any ideal of B' is also an ideal of A'. Hence, B' is semisimple and satisfies (18), (19) and (20). Since B satisfies (18), (20) and is strictly power-associative, we need only verify that B satisfies (19). Let e be any idempotent of B. Clearly, in (19), (a) implies both (b) and (c). To avoid confusion denote by $K(e)$ the ideal in B generated by $B_e(1/2)$. Suppose $K(e) \cap B_e(1)$ is nil. It is easily seen that $T(K(e) \cap B_e(1)) = H([e]) \cap B'([e], 1)$ since B' is a direct summand of A'. Therefore $H([e]) \cap B'([e], 1)$ is nil and, since (19) holds in B', $H([e])$ is nil. By the semisimplicity of B_1, $H([e]) = 0$ so $K(e)$ is nil. Similarly in (19), (c) implies (a) so B is in C_1 and C_1 is a decomposable class.

LEMMA 3.3. *The class C of all algebras that satisfy the hypotheses of Theorem* 3.1 *satisfies conditions* (a), (b), (d) *and* (e) *of Definition* 2.2.

Proof. As a consequence of Lemma 3.2, conditions (a), (d) and (e) are satisfied. Since $A - N$ trivially is in C, condition (b) is satisfied.

We now assume that A satisfies the hypotheses of Theorem 3.1 and let $e, e_1, \ldots, e_n$ be the idempotents guaranteed by (21). Throughout the argument T is the natural mapping $A \to A - N$ and $[x] = T(x)$. We begin by renumbering the e_i so that e_i is in $L + N$ (not necessarily a vector space direct sum) if and only if $i > m$. If $H(e_i)$ is not nil then $H(e_i) \cap A(e_i, 1)$ contains an element x with x not nil. The subalgebra generated by x contains an idempotent e'. Since $H(e_i) \cap A(e_i, 1)$ is a subalgebra, $e' \in H(e_i) \cap A(e_i, 1) \subseteq A(e_i, 1)$. But e_i is primitive so $e_i = e' \in H(e_i) \subseteq L$ and $i > m$. Consequently, $H(e_i) \subseteq N$ whenever $i \leq m$. It is possible that $m = 0$ or $m = n$. We will dispense with these cases first.

LEMMA 3.4. *If $m = 0$ or $m = n$ then A has a Wedderburn decomposition.*

Proof. Assume $m = 0$. Now e is principal in A so $[e]$ is principal in $A - N$. By Lemma 3.1, $[e]$ is the identity element of $A - N$. Also, $T(L)$ is an ideal of $A - N$ with $[e]$ in $T(L)$. Therefore $T(L) = A - N$. By (21), $L = S + M$ where M is the radical of L and $S \cong L - M$. We know that $N \cap L \subseteq M$. Since M is a nil ideal of L, $T(M)$ is a nil ideal of $T(L)$ so $T(M) = 0$. Therefore $M \subseteq N$ so $N \cap L = M$ and $S \cong T(L) = A - N$. By a dimension argument, $A = S + N$ is a Wedderburn decomposition for A.

Now suppose $m = n$ so $H(e_i) \subseteq N$ for $i = 1, \ldots, n$. Then (2) and (20) imply that, for each e_i,

$$A - N = (A - N)([e_i], 1) \oplus (A - N)([e_i], 0). \tag{22}$$

Since F is a splitting field and $(A-N)([e_i], 1)$ is not nodal, we have $(A-N)([e_i], 1) = \{\alpha \cdot [e_i]\}$ for $i=1, \ldots, n$. Write $S_i = \{\alpha e_i\}$, $i=1, \ldots, n$. By induction, for each $p \leqq n$,

$$A-N = T(S_1) \oplus \cdots \oplus T(S_p) \oplus (A-N)([e_1]+\cdots+[e_p], 0). \tag{23}$$

For, (23) is true when $p=1$ by (22). Now, assume (23) when $p=k$. It is easily seen that $S_{k+1} \subseteq A(e_1+\cdots+e_k, 0)$ so $T(S_{k+1}) \subseteq (A-N)([e_1]+\cdots+[e_k], 0)$. This together with (22) for $i=k+1$ yields (23) with $p=k+1$. Also, since $[e]$ is an identity for $A-N$, $(A-N)([e], 0)=0$ so

$$A-N = T(S_1) \oplus \cdots \oplus T(S_n).$$

From the fact that $e_i e_j = \delta_{ij} e_i$ (Kronecker delta), we have $S_i S_j = \delta_{ij} S_i$. We then write $S = S_1 \oplus \cdots \oplus S_n$ and note that $S \subseteq A$. Also, $S \cong A-N$ so $A = S+N$ is a Wedderburn decomposition of A.

From now on, assume $0<m<n$. Let $g = \sum_{i=1}^m e_i$ and $h = \sum_{i=m+1}^n e_i$.

Lemma 3.5. *The algebra*

$$A-N = (A-N)([h], 1) \oplus (A-N)([e_1], 1) \oplus \cdots \oplus (A-N)([e_m], 1).$$

Proof. Since $e=g+h$ and $[e]$ is an identity for $A-N$ then $[g]+[h]$ is an identity for $A-N$. If $[x]$ is in $(A-N)([g], 1/2)$ then $[x][h]+[h][x] = [x][e]+[e][x]-[x][g]-[g][x] = [x]$ so $[x]$ is in $(A-N)([h], 1/2)$. Also, $(A-N)([h], 1/2) \subseteq (A-N)([g], 1/2)$ so $(A-N)([g], 1/2) = (A-N)([h], 1/2)$. However, $(A-N)([e_i], 1/2)=0$ for $i=1, \ldots, m$ so $(A-N)([g], 1/2)=0$.

If we consider the decomposition of an algebra B relative to two orthogonal idempotents a and b we have B the vector space direct sum of nine possible subspaces of the form $B_a(s) \cap B_b(t)$ where $s, t = 0, 1/2, 1$. Since $ab=ba=0$, $a+b$ is an idempotent and B has the decomposition

$$B = B_{a+b}(1)+B_{a+b}(1/2)+B_{a+b}(0).$$

In addition if, for an idempotent e', $x \cdot e' = \alpha x$ with α in F then $\alpha = 1$, $1/2$ or 0. Now $B_a(s) \cap B_b(t) \subseteq B_{a+b}(s+t)$ so this intersection is zero unless $s+t=0$, $1/2$ or 1. Consequently, the following relations hold:

$$B_{a+b}(1) = B_a(1) \cap B_b(0)+B_a(1/2) \cap B_b(1/2)+B_a(0) \cap B_b(1). \tag{24}$$

$$B_{a+b}(1/2) = B_a(1/2) \cap B_b(0)+B_a(0) \cap B_b(1/2). \tag{25}$$

$$B_{a+b}(0) = B_a(0) \cap B_b(0). \tag{26}$$

Since $H([e_i])=0$ for $i=1, \ldots, m$, (22) holds when $i \leqq m$. From (24), (2) and (20), if $B_a(1/2)=0$ or $B_b(1/2)=0$, then

$$B_{a+b}(1) = B_a(1) \cap B_b(0) \oplus B_a(0) \cap B_b(1). \tag{27}$$

Also, $B_a(1) \cap B_b(t) \subseteq B_{a+b}(1+t)$ so $B_a(1) \cap B_b(t)=0$ unless $t=0$. Therefore $B_a(1) \subseteq B_b(0)$ and $B_b(1) \subseteq B_a(0)$ so (27) becomes

$$B_{a+b}(1) = B_a(1) \oplus B_b(1). \tag{28}$$

Inductively, we have

$$(29)\qquad (A-N)([g],1) = (A-N)([e_1],1)\oplus\cdots\oplus(A-N)([e_m],1).$$

Also,

$$(30)\qquad A-N = (A-N)([g]+[h],1) = (A-N)([h],1)\oplus(A-N)([g],1).$$

These last two results give the conclusion of the lemma.

LEMMA 3.6. *If $L=S+M$ is a Wedderburn decomposition for L with M the radical of L then* $\dim S=\dim(A-N)-m$ *and* $S\cong(A-N)([h],1)$.

Proof. Since $h=\sum_{i=m+1}^{n} e_i$, then h is in L. If x is in $A_h(1)$ then $x=xh\in Al\subseteq L$ so $A_h(1)\subseteq L$. Therefore $(A-N)([h],1)\subseteq T(L)$. If $L_h(0)$ is not nil, there is an idempotent e' in $L_h(0)$. Now $[e']$ is in $(A-N)([h],0)$ so $[e']=\sum_{i=1}^{m}\alpha_i[e_i]$ with α_i in F, $i=1,\dots,m$. From $[e']^2=[e']\neq 0$ we find that for some $j\leqq m$, $\alpha_j\neq 0$ and $\alpha_j^2=\alpha_j$. Hence $\alpha_j=1$. But $[e'][e_j]=[e_j]$. Since $T(L)$ is an ideal of $A-N$ and $[e']$ is in $T(L)$, then $[e_j]$ is in $T(L)$ which contradicts the fact that e_j is not in $L+N$ (not necessarily supplementary). Therefore h is principal in L so $[h]$ is principal in $T(L)$. Consequently, $(T(L))([h],0)$ is nil. Lemma 3.5 then implies that $(T(L))([h],0)=0$ so $T(L)=(A-N)([h],1)$. But then $T(M)$ is an ideal of a direct summand of $A-N$ so $T(M)$ is an ideal of $A-N$. Since M is nil, $T(M)$ is nil so $T(M)=0$. Therefore, $S\cong(A-N)([h],1)$. Now, F is a splitting field and $(A-N)([e_i],1)$ is not nodal so $(A-N)([e_i],1)=\{\alpha[e_i]\}$ and has dimension 1. Thus, from Lemma 3.5, $\dim S=\dim(A-N)-m$.

LEMMA 3.7. *There exists a set $\{f_i\}_{i=0}^{m}$ of pairwise orthogonal idempotents such that f_0 is the identity of S, $[f_0]=[h]$, and $[f_i]=[e_i]$ for $i=1,\dots,m$.*

Proof. Since $S\cong(A-N)([h],1)$, it contains an identity element f_0. Clearly, $[f_0]=[h]$. Now, each $[e_i]$ is in $T(A(f_0,0))$, $i=1,\dots,m$. The kernel of the homomorphism $A(f_0,0)\to T(A(f_0,0))$ is a subset of N so, by the homomorphism theorems, there is a nil ideal M of $A(f_0,0)$ with $A(f_0,0)-M\cong T(A(f_0,0))$. Thus, by Lemma 2.1, there exist pairwise orthogonal idempotents $f_1,\dots,f_m$ in $A(f_0,0)$ with $[f_i]=[e_i]$, $i=1,\dots,m$.

We are now able to prove the theorem. Let $B=S+F\cdot f_1+\cdots+F\cdot f_m$. Since f_0 is the identity of S and the set $\{f_i\}_{i=0}^{m}$ is a set of pairwise orthogonal idempotents then $B=S\oplus F\cdot f_1\oplus\cdots\oplus F\cdot f_m$. Also, $B\cong A-N$ so a dimension argument guarantees the fact that $A=B+N$ and we are done.

4. **Applications.** For a general application, we derive the following theorem. The set $A_{ij}(e)=\{x : ex=ix \text{ and } xe=jx\}$.

THEOREM 4.1. *Let B be a nonnil strictly power-associative algebra over a splitting field F with char $\neq 2, 3$ and assume the following*:

(31) *$B-N$ contains no simple nodal subalgebras where N is the radical of B.*

(32) *For any idempotent e in B*, $B=B_{11}(e)+B_{10}(e)+B_{01}(e)+B_{00}(e)$.

(33) *The product* $B_{ij}(e)B_{km}(e)\subseteq\delta_{jk}B_{im}(e)$ *with the exception that, for* $i\neq j$, $(B_{ij}(e))^2\subseteq B_{ji}(e)$ *with* $x_{ij}2=0$.

(34) *The set* $B_{10}(e)B_{01}(e)+B_{10}(e)+B_{01}(e)+B_{01}(e)B_{10}(e)$ *is an alternative ideal.*

With these assumptions, B has a Wedderburn decomposition.

Proof. Clearly $B_{11}(e)=B_e(1)$, $B_{00}(e)=B_e(0)$ and $B_{10}(e)+B_{01}(e)=B_e(1/2)$. Hence $H(e)=B_{10}(e)B_{01}(e)+B_{10}(e)+B_{01}(e)+B_{01}(e)B_{10}(e)$ where $H(e)$ is the ideal of B generated by $B_e(1/2)$. From the fact that the Wedderburn Principal Theorem holds for alternative algebras [15], we have condition (21) holding in B of Theorem 3.1. Also, (18) and (20) both hold in B.

In (19), the implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) always hold. Let $H(e)\cap B_e(1)=B_{10}(e)B_{01}(e)$ be nil. If $H(e)$ is not nil it has a radical N_1 and $D=H(e)-N_1$ is semi-simple. Now, $H(e)$ is alternative so D is alternative and possesses an identity element v. Define $D_{ij}=H_{ij}(e)-(N_1\cap H_{ij}(e))$ and write $v=v_{11}+v_{10}+v_{01}+v_{00}$ with v_{ij} in D_{ij}. By the definition of $H(e)$, $H(e)$ is nil if and only if $A_{10}(e)+A_{01}(e)\subseteq N_1$. Hence, $H(e)$ is nil if and only if $D_{10}+D_{01}=0$. The D_{ij} multiply as the $B_{ij}(e)$ do so, from $v^2=v$, we obtain $v_{11}=v_{11}^2+v_{10}v_{01}$, $v_{10}=v_{11}v_{10}+v_{10}v_{00}$, $v_{01}=v_{01}v_{11}+v_{00}v_{01}$ and $v_{00}=v_{00}^2+v_{01}v_{10}$. But v is the identity element of D so $v_{10}=v_{10}v=v_{10}v_{00}+v_{10}v_{01}$. Therefore $v_{10}=v_{10}v_{00}$ and $v_{10}v_{01}=0$. Inductively, $v_{11}=v_{11}^2=\cdots=v_{11}^k$. Since $H_{11}(e)$ is nil, $v_{11}=v_{11}^n=0$. Also, $v_{10}=vv_{10}=v_{11}v_{10}+v_{01}v_{10}$ so $v_{10}=v_{11}v_{10}$ and $v_{01}v_{10}=0$. This implies $v_{10}=v_{11}v_{10}=0$. Now $v_{01}=v_{01}v=v_{01}v_{11}+v_{01}v_{10}=v_{01}v_{11}=0$. Consequently, $v=v_{00}$. Let $x_{10}+x_{01}$ be an element of $D_{10}+D_{01}$. Because $v=v_{00}$ is an identity, $x_{10}=v_{00}x_{10}=0$ and $x_{01}=x_{01}v_{00}=0$ so $D_{10}+D_{01}=0$. Therefore, $H(e)$ is nil. Similarly, if $H(e)\cap A_e(0)$ is nil then $H(e)$ is nil so (19) holds in B. Hence, B has a Wedderburn decomposition.

THEOREM 4.2. *Let A satisfy* (20). *Then the algebra A contains a nodal subalgebra if and only if* $A-N$ *contains a nodal subalgebra.*

Proof. If B is a nodal subalgebra of A then $T(B)$ is a nodal subalgebra of $A-N$. Suppose now that B is a nodal subalgebra of $A-N$. By the homomorphism theorems, there is a subalgebra C of A with $N\subseteq C$ and $T(C)=B$. We know that B has a primitive idempotent $[e]$ and, by Lemma 2.1, e can be chosen as an idempotent in C. We claim that $C_e(1)$ is a nodal subalgebra of A. For $C_e(1)=C\cap A_e(1)$ and the intersection of two subalgebras of A is a subalgebra of A so $C_e(1)$ is a subalgebra of A. If $e=f+g$ with $fg=gf=0$, then $[e]=[f]+[g]$ with $[f][g]=[g][f]=0$. Hence, e is primitive. If x is in $C_e(1)$ then $[x]=\alpha[e]+[z]=[\alpha e+z]$ where α is in F (the base field) and $[z]$ is nil. Also, $[x-\alpha e]=[z]$ so $x-\alpha e=z+n$ with n in N. Now, for some k, $[(z+n)^k]=[z+n]^k=[z]^k=0$ so $(z+n)^k$ is in N. This implies $z+n$ is nilpotent. In addition $z+n=x-\alpha e$ is in $C_e(1)$. We need only show that M, the set of nilpotent elements of $C_e(1)$ is not a subalgebra. First, recall that $C=C_e(1)+C_e(1/2)+C_e(0)$. If x is in $C_e(t)$, then $[x][e]+[e][x]=[xe+ex]=[2tx]=2t[x]$ so $[x]$ is in $B([e],t)$. Since $B=B([e],1)+B([e],1/2)+B([e],0)$, we have $T(C_e(t))=$

$B([e], t)$. But $B=B([e], 1)=T(C_e(1))$. As a consequence, $T(M)$ is the set of nilpotent elements of B. Since B is nodal, $T(M)$ is not a subalgebra. Hence, M is not a subalgebra of $C_e(1)$.

THEOREM 4.3. *Let B be a (γ, δ) algebra with $\delta\neq 0, 1$ over a splitting field F of* char $\neq 2, 3, 5$. *If B contains no nodal subalgebras, then B has a Wedderburn decomposition.*

Proof. A (γ, δ) algebra is an algebra which satisfies the following identities:

$$(z, x, y)+\gamma(x, z, y)+\delta(y, z, x) = 0. \tag{35}$$

$$(x, y, z)-\gamma(x, z, y)+(1-\delta)(y, z, x) = 0. \tag{36}$$

It is also assumed that $\gamma^2-\delta^2+\delta=1$. We will show that B satisfies the hypotheses of Theorem 4.1. From Theorem 4.2, condition (31) is satisfied. Now, the results of [9, pp. 250, 251] state that B satisfies (32) and (33). In addition, B is shown to be power-associative [9, Theorem 2]. Furthermore $B_{11}(e)+B_{10}(e)+B_{01}(e)+B_{01}(e)B_{10}e$ and $B_{00}(e)+B_{10}(e)+B_{01}(e)+B_{10}(e)B_{01}(e)$ are ideals of B [9, p. 254]. Thus, $H(e)=B_{10}(e)B_{01}(e)+B_{10}(e)+B_{01}(e)+B_{01}(e)B_{10}(e)$ is an ideal of B. From Lemma 2 in [9], $H(e)$ is associative so (34) is satisfied and we are done.

THEOREM 4.4. *Let B be a strictly power-associative algebra over a splitting field F of* char $\neq 2, 3$. *Assume that B satisfies*

$$\alpha(y, x, x)-(\alpha+1)(x, y, x)+(x, x, y) = 0 \tag{37}$$

for $\alpha\neq 0, 1, -1/2, -2$ and that B contains no nodal subalgebras. If $\alpha\neq -1$ or if, for each idempotent e in B, $B_e(t)B_e(t)\subseteq B_e(t)$ for $t\neq 1/2$ then B has a Wedderburn decomposition.

Proof. By Theorem 4.2, (31) is satisfied. Clearly, B satisfies (37) with the same value of α. Theorem 2 of [8] implies (32), Theorem 3 of [8] implies (33) and Lemma 4 of [8] implies (34) so B has a Wedderburn decomposition.

We now turn our attention to a class of associator dependent algebras studied in [6] and defined by

$$(x, y, z)+\alpha(y, z, x)+\alpha^2(z, x, y) = 0 \tag{38}$$

where $\alpha^3=1$, $\alpha\neq 1$. A special subclass of this is the class of algebras satisfying

$$(x, y, z) = \alpha(y, z, x) \tag{39}$$

with $\alpha^3=1$, $\alpha\neq 1$. While an algebra satisfying (38) is not necessarily power-associative, we will show that an algebra satisfying (39) is power-associative.

LEMMA 4.1. *If* char *is prime to* 30 *and* $1-\alpha$ *then a ring satisfying* (39) *is power-associative.*

Proof. We first prove that, if any associator involving any three of w, x, y or z is zero then $(wx, y, z)+(zw, x, y)=0$. In any ring we have

$$(ab, c, d)-(a, bc, d)+(a, b, cd) = a(b, c, d)+(a, b, c)\,d. \tag{40}$$

Hence, we have:

$$(wx, y, z)-(w, xy, z)+(w, x, yz) = 0. \tag{41}$$

$$(xy, z, w)-(x, yz, w)+(x, y, zw) = 0. \tag{42}$$

Using (42) and (39) we derive

$$(w, xy, z)-(w, x, yz)+(zw, x, y) = 0. \tag{43}$$

Now (41) and (43) imply

$$(wx, y, z)+(zw, x, y) = 0. \tag{44}$$

We will now prove Lemma 4.1 by induction. The identity (39) implies $xx^2=x^3$. Let $n\geqq 4$ and assume $x^a x^b=x^{a+b}$ for $a+b<n$. If we let $z=y=x$ and $w=x^{n-3}$ then (44) implies $2(x^{n-2}, x, x)=0$ so $(x^{n-2}, x, x)=0$. From (39), we then derive $(x, x^{n-2}, x)=0$ and $(x, x, x^{n-2})=0$. For $n=4$, these three identities imply $x^{4-a}x^a=x^4$ for any a. Now let $n\geqq 5$. Now, Lemma 2 of [2] gives

$$x^{n-a}x^a = x^n+((a-1)/2)(x^{n-1}, x).$$

However, $(x^{n-1}, x)=x^n-xx^{n-1}=(x, x^{n-2}, x)=0$. Hence, the ring is power-associative.

THEOREM 4.5. *Let B be a nonnil power-associative algebra over a splitting field F of char $\neq 2, 3$. If B satisfies* (38) *and contains no nodal subalgebras, then B has a Wedderburn decomposition.*

Proof. Since B also satisfies (38), the results of [6] imply the hypotheses of Theorem 4.1 and we are done.

5. **Algebras in** [12]. In this section and the next, we will study certain classes of algebras whose Wedderburn decomposition cannot be so easily derived from Theorem 4.1. In [12], Kosier studied algebras satisfying

$$(x^2, y, z) = 2x\cdot(x, y, z) \tag{45}$$

$$(z, y, x^2) = 2x\cdot(z, y, x) \tag{46}$$

$$(x, x, x) = 0. \tag{47}$$

Such algebras are power-associative [12, Theorem 1]. If e is an idempotent, define

$$L(e) = \{x : x\in A_e(1/2) \text{ and } ax,\ xa\in A_e(1/2) \text{ for all } a \text{ in } A\}.$$

It is known [12, Theorem 4] that $L(e)$ is an ideal of A and for any x in $L(e)$, $x^2=0$. Suppose char $\neq 2, 3$. If K is a splitting field of A and $L(e)=0$ for each idempotent e

of $B=A_K$ then Theorems 5 and 6 of [12] imply (32) and (33). In addition, Theorem 7 and the proof of Theorem 8 of [12] also implies (34). We thus have this result.

THEOREM 5.1. *Let A be an algebra satisfying* (45), (46) *and* (47) *over a splitting field F of* char $\neq 2, 3$. *If A contains neither nodal subalgebras nor ideals L with x in L implying $x^2=0$ then A has a Wedderburn decomposition.*

We would like to remove the condition that A has no ideals L with x in L implying $x^2=0$. The condition cannot be removed as the following example will show. Let A be the five dimensional algebra over a field F of char $\neq 2, 3$ spanned by e, x, y, f and z whose multiplication relative to this basis is given by

$$\begin{aligned}(48)\qquad &(a_1, a_2, a_3, a_4, a_5)(b_1, b_2, b_3, b_4, b_5)\\ &= (a_1b_1+a_2b_3, a_1b_2+a_2b_4, a_3b_1+a_4b_3, a_3b_2+a_4b_4,\\ &\qquad a_1(b_5-b_2)+a_2(b_1-b_4)+a_4b_2+a_5b_4).\end{aligned}$$

Now, if $N=\{\alpha z\}$ with α in F then $A-N\cong M_2$ (2×2 matrices) with $[e]=e_{11}$, $[x]=e_{12}$, $[y]=e_{21}$ and $[f]=e_{22}$. Also, $e^2=e$, $f^2=f$, $A_e(1)=\{\alpha e\}$ and $A_e(0)=\{\alpha f\}$ with α in F. Now $ex=x-z$ and $xe=z$ so x is in $A_e(1/2)$; $ey=0$ and $ye=y$ so y is in $A_e(1/2)$; $ez=z$ and $ze=0$ so z is in $A_e(1/2)$. Therefore $A_e(1/2)$ is the vector space spanned by x, y and z. Now, one finds by checking that $z^2=0$ and AN, $NA\subseteq N$ so, since $A-N$ is simple, N is the radical of A. In addition $e+f$ is an identity element for A.

Linearizing (45), (46) and (47) gives

$$(49)\qquad (u\cdot s, v, w) = u\cdot(s, v, w)+s\cdot(u, v, w),$$

$$(50)\qquad (w, v, u\cdot s) = u\cdot(w, v, s)+s\cdot(w, v, u),$$

$$(51)\qquad (u, v, w)+(v, w, u)+(w, u, v)+(w, v, u)+(v, u, w)+(u, w, v) = 0.$$

Since F is of char $\neq 2, 3$, (49) $\Rightarrow$ (45), (50) $\Rightarrow$ (46) and (51) $\Rightarrow$ (47). Here is a list of all nonzero associators involving e, x, y, f and z.

$$(52)\qquad z = (e, e, x) = (x, f, e) = (x, e, f) = (y, x, x) = (x, y, z) = (f, f, x)$$

$$(53)\qquad -z = (x, e, e) = (e, f, x) = (f, e, x) = (x, x, y) = (z, y, x) = (x, f, f).$$

By simply checking, we see that (49), (50) and (51) are all satisfied. If $S\subseteq A$ with $A=S+N$ and $S\cong A-N$ then S must be spanned by e', x', y' and f' with $[e']=[e]$, $[x']=[x]$, $[y']=[y]$ and $[f']=[f]$. Hence, in particular, $x'=x+\alpha z$ and $e'=e+\beta z$. Now, from (48), $x'e'=z$ which is not in S. Therefore A does not possess a Wedderburn decomposition.

An algebra A with identity element 1 is of degree t if, for K a splitting field of A, $1=e_1+\cdots+e_t$ where $\{e_i\}_{i=1}^t$ is a collection of pairwise orthogonal primitive idempotents in A_K. The algebra of the preceding paragraph is of degree 2.

THEOREM 5.2. *Let A be an algebra satisfying* (45), (46) *and* (47) *over a splitting field F of* char $\neq 2, 3$. *If A contains no nodal subalgebras and if $A-N$ contains no simple ideals of degree* 2, *then A has a Wedderburn decomposition.*

Proof. Let P be the class of all algebras satisfying the hypotheses of Theorem 5.2. We claim that P is a decomposable class. For, P clearly satisfies (a), (b) and (c). Furthermore, Theorem 3 of [12] implies (e) and Theorem 10 of [12] implies (d). By Theorem 2.1, it suffices to show that $C(P)$ is a Wedderburn class. We will prove this by induction on dim A. If dim $A=1$, the result is obvious. Let dim $A=n$ and assume that B has a Wedderburn decomposition if dim $B<n$. If A has degree 1 then $A-N\cong F$ so by Theorem 2.2, A has a Wedderburn decomposition. Suppose then that degree $A>1$. If, for some primitive idempotent e, $L(e)=0$, then by the proof of Theorem 8 of [12], $H(e)$ is alternative. Hence, by Theorem 2.2, A has a Wedderburn decomposition. If for some idempotent $L(e)\neq 0$, N, then Lemma 2.2 implies A has a Wedderburn decomposition for $L(e)\subseteq N\neq A$. Therefore, we need only prove that A has a Wedderburn decomposition if $L(e)=N$ for every primitive idempotent. Let $t=$degree A. Since $t\neq 2$, $t>1$ we have $t\geqq 3$ so $1=e_1+\cdots+e_t$, with $\{e_i\}_{i=1}^t$ a set of pairwise orthogonal primitive idempotents. Also, $L_i=N$, $i=1,\ldots,t$. However, $L_i\subseteq A(e_i, 1/2)$, $i=1,\ldots,t$. If x is in $A(e_1, 1/2)\cap A(e_2, 1/2)\cap A(e_3, 1/2)$, then $xe+ex=3x$ where $e=e_1+e_2+e_3$. But, e is an idempotent so this is impossible unless $x=0$. Therefore, $N=L_1\cap L_2\cap L_3=0$. Hence, A has a Wedderburn decomposition $A=A+0$.

6. **Nearly** (1, 1) **algebras.** We will call A a nearly (1, 1) algebra if it is strictly power-associative and satisfies

$$(x, y, x) = (x, x, y). \tag{54}$$

These algebras were studied in [8]. The (1, 1) algebras are power-associative [10] and satisfy

$$(x, y, z) = (x, z, y). \tag{55}$$

A (nearly) $(-1, 0)$ algebra is one which is anti-isomorphic to a (nearly) (1, 1) algebra. The (1, 1) and $(-1, 0)$ are special cases of (γ, δ) algebras and the nearly (1, 1) algebras satisfy (37) with $\alpha=0$. We will assume throughout that F, the base field, is of char $\neq 2, 3$. For any idempotent e in A, $A=A_{11}(e)+A_{10}(e)+A_{01}(e)+A_{00}(e)$ [8, Theorem 2]. Furthermore the subspaces satisfy the relation

$$A_{ij}(e)A_{km}(e) \subseteq \delta_{jk}A_{im}(e)$$

with the following exceptions: for $i\neq j$, $A_{ij}(e)A_{ij}(e)\subseteq A_{ji}(e)$; $x_{ij}^2=0$; $A_{ii}(e)A_{ij}(e)\subseteq A_{ij}(e)+A_{jj}(e)$; $A_{ij}(e)A_{ii}(e)\subseteq A_{jj}(e)$ [8, Theorem 3]. In addition, if x_{km} is in $A_{km}(e)$, then for $i\neq j$ [8, Theorem 3],

$$(x_{ii}y_{ij}-y_{ij}x_{ii}) \in A_{ij}(e). \tag{56}$$

Defining $G_i(e)=A_{ji}(e)A_{ij}(e)$ for $j\neq i$, it is proved that $G(e)=G_1(e)+G_0(e)$ is an ideal of A with $G(e)G(e)=0$ [8, Lemma 3]. We will also need the following relations:

$$A_{ij}(e)G_i(e) \subseteq G_j(e) \quad i\neq j \tag{57}$$

$$G_i(e)A_{ij}(e) \subseteq A_{ij}(e)+G_j(e) \quad i\neq j \tag{58}$$

$$A_{ji}(e)G_i(e) = 0 \quad i\neq j \tag{59}$$

$$G_i(e)A_{ji}(e) = 0 \quad i\neq j. \tag{60}$$

The first two are a consequence of the definition and the last two follow from [8, Lemma 2].

LEMMA 6.1. *Let A be a nearly* (1, 1) *algebra with identity element over a field F of* char $\neq 2, 3$ *and $N=G(e)$ for each idempotent $e\neq 1$ in A. If $A-N$ contains a total matric algebra M_t whose identity is the identity of $A-N$ then there exists a subalgebra S of A with $S\cong M_t$.*

Proof. Now, M_t is spanned by $\{[u_{ij}]\}_{i,j=1}^t$ with

$$[u_{ij}][u_{km}] = \delta_{jk}[u_{im}]. \tag{61}$$

By the homomorphism theorems there is a subalgebra B of A with $T(B)=M_t$ and $N\subseteq B$. If B has a subalgebra $S\cong M_t$, then A has the same subalgebra $S\cong M_t$. Hence, it suffices to consider the case $A-N=M_t$. In this case each $[u_{ii}]$ is primitive. By Lemma 2.1, there exist pairwise orthogonal idempotents $e_1,\dots,e_t$ with $1=e_1+\cdots+e_t$ and $[e_i]=[u_{ii}]$. Furthermore, each e_i is primitive. If $t=1$, $S=F\cdot 1\cong M_1$ and we are done.

Assume $t>1$. We make the following definitions:

$$\begin{aligned} A_{ii} &= A_{11}(e_i) \quad i=1,\dots,t \\ A_{ij} &= A_{10}(e_i)\cap A_{01}(e_j) \quad i\neq j;\, i,j=1,\dots,t \\ G_i &= G_1(e_i). \end{aligned}$$

An induction on (24), (25) and (26) will imply (see [14]) $A=\sum_{i,j=1}^t A_{ij}$.

Also,

$$N = G(e_j) = \sum_{i=1}^t G_i \quad j=1,\dots,t.$$

For, $G(e_j)=G_1(e_j)+G_0(e_j)$ and $G_0(e_j)=A_{10}(e_j)A_{11}(e_j)=A_{01}(1-e_j)A_{00}(1-e_j)$. If $i\neq j$, then $G_i=G_1(e_i)=A_{01}(e_i)A_{00}(e_i)\subseteq G_0(e_j)$. It is clear that $A_{01}(1-e_j)\subseteq\sum_{i\neq j}A_{01}(e_j)$ and (see (26)) $A_{00}(1-e_j)=\bigcap_{i\neq j}A_{00}(e_j)$. Therefore, $G_0(e_j)\subseteq\sum_{i\neq j}G_i$. Hence,

$$G(e_j)\subseteq \sum_{i=1}^t G_i \subseteq G(e_j).$$

If $i\neq j$, then $A_{ij}\cap N\subseteq A_{10}(e_i)\cap(A_{11}(e_i)+A_{00}(e_i))=0$. Clearly, $T(A_{ij}(e_k))=(A-N)_{ij}([e_k])=(A-N)_{ij}([u_{kk}])$ for $i,j=0,1$; $k=1,\dots,t$. Thus $T(A_{ij})=F\cdot[u_{ij}]$;

$i, j=1, \ldots, t$. Consequently, when $i \neq j$, $\dim A_{ij}=1$. Therefore, for each $i \neq j$ there is a unique element e_{ij} in A_{ij} with $[e_{ij}]=[u_{ij}]$. We also define $e_{ii}=e_i$, $i=1, \ldots, t$. Using the relations on the subspaces $A_{ij}(e_k)$ for $i, j=0, 1$; $k=1, \ldots, t$ we have

$$A_{ij}A_{km} \subseteq \delta_{jk}A_{im} \quad i, j, k, m = 1, \ldots, t \tag{62}$$

with these exceptions:

$$A_{ij}A_{ii} \subseteq G_j \qquad i \neq j \tag{63}$$

$$A_{ii}A_{ij} \subseteq A_{ij}+G_j \qquad i \neq j. \tag{64}$$

In addition, (57), (58), (59) and (60) imply

$$A_{ij}G_i \subseteq G_j \qquad i \neq j \tag{65}$$

$$G_iA_{ij} \subseteq A_{ij}+G_j \qquad i \neq j \tag{66}$$

$$A_{ji}G_i = G_iA_{ji} = 0 \qquad i \neq j. \tag{67}$$

We claim that $e_{ij}e_{km}=\delta_{jk}e_{im}$ if $i \neq m$ or $j \neq k$. For, since e_{ij} is in $A_{10}(e_{ii})$ $i \neq j$ we have $e_{ii}e_{ij}=e_{ij}$ and $e_{ij}e_{ii}=0$. If $i \neq m$, then this and (62) imply $e_{ij}e_{km}=\alpha_{jk}e_{im}$. But $[e_{ij}][e_{km}]=[u_{ij}][u_{km}]=\delta_{jk}[u_{im}]=\delta_{jk}[e_{im}]$ so $\alpha_{jk}=\delta_{jk}$; $j, k=1, \ldots, t$. If $j \neq k$ then (62) and $e_{ij}e_{ii}=0$ imply in a similar way that $e_{ij}e_{km}=0$. Now $A_{ii}=F \cdot e_{ii}+G_i$ so $A_{ij}A_{ji} \subseteq A_{ii}$ implies

$$e_{ij}e_{ji} = e_{ii}+g_i(j) \tag{68}$$

where $g_i(j)$ is in G_i. Of course, $g_i(i)=0$ and we wish to prove each $g_i(j)=0$. We first prove

$$g_i(j)e_{ik} = e_{ik}g_i(j) = 0 \quad \text{for } i \neq k. \tag{69}$$

First, from (56), $g_i(j)e_{ik}-e_{ik}g_i(j)=\alpha e_{ik}$ so $[g_i(j)][e_{ik}]-[e_{ik}][g_i(j)]=\alpha[e_{ik}]$. But $g_i(j)$ is in N so $\alpha=0$ and we have

$$g_i(j)e_{ik} = e_{ik}g_i(j) \qquad i \neq k. \tag{70}$$

Now, substitute e_{ij} for x and e_{ji} for y in (54) to obtain $(e_{ij}, e_{ji}, e_{ij})=(e_{ij}, e_{ij}, e_{ji})$. For $i \neq j$ we have $(e_{ij}, e_{ji}, e_{ij})=(e_{ii}+g_i(j))e_{ij}-e_{ij}(e_{jj}+g_j(i))=e_{ij}+g_i(j)e_{ij}-e_{ij}-0=g_i(j)e_{ij}$ and $(e_{ij}, e_{ij}, e_{ji})=-e_{ij}g_i(j)$. This with (70) implies $e_{ij}g_i(j)=0$. If $t=2$ we have established (69). Linearizing (54) yields

$$(x, y, z)+(z, y, x) = (x, z, y)+(z, x, y). \tag{71}$$

Hence,

$$(e_{ij}, e_{ji}, e_{ik})+(e_{ik}, e_{ji}, e_{ij}) = (e_{ij}, e_{ik}, e_{ji})+(e_{ik}, e_{ij}, e_{ji}). \tag{72}$$

If i, j and k are distinct the relations on the subspaces imply that $(e_{ij}, e_{ji}, e_{ik})=g_i(j)e_{ik}$, $(e_{ik}, e_{ij}, e_{ji})=-e_{ik}g_i(j)$ and the other two associators of (72) are zero. Thus, from (70) we have (69). Now, (71) gives

$$(g_i(k), e_{ij}, e_{ji})+(e_{ji}, e_{ji}, g_i(k))=(g_i(k), e_{ji}, e_{ij})+(e_{ji}, g_i(k), e_{ij}). \tag{73}$$

Recall the fact that $G^2=0$. If $i\neq j$, $(g_i(k), e_{ij}, e_{ji})=-g_i(k)$ with the other three associators zero so $g_i(k)=0$. Therefore, if S is the space spanned by $\{e_{ij}\}_{i,j=1}^t$ then S is a subalgebra of A. Hence, S is a total matric algebra and [1], $S\cong M_t$.

It is known that $A-N$ is a direct sum of simple algebras, each of which is nodal or alternative. If $A-N$ is a Cayley-Dickson algebra it is still not clear whether A has a Wedderburn decomposition or not. We content ourselves with the following three theorems.

THEOREM 6.1. *Let A be a nearly* (1, 1) *(nearly* $(-1, 0)$*) algebra over a splitting field of* char $\neq 2, 3$ *such that $A-N$ is associative. Then A has a Wedderburn decomposition.*

Proof. Clearly the class P of nearly (1, 1) algebras satisfying the hypotheses of the theorem is a decomposable class so we need only show $C(P)$ is a Wedderburn class. If A is in $C(P)$ and dim $A=1$, then A has a Wedderburn decomposition. Suppose dim $A=n$ and every algebra B in $C(P)$ with dim $B<n$ has a Wedderburn decomposition. If the identity element 1 of A is primitive, then $A=F\cdot 1+N$ is a Wedderburn decomposition for A. If 1 is not primitive and for some $e\neq 1$, $G(e)=0$, then the results in [8, pp. 478–481] imply that $H(e)$ is alternative. If $H(e)$ is nil, then $A-N=(A-N)_{11}([e])\oplus(A-N)_{00}([e])$ which is impossible for $A-N$ is simple and $[1-e]$ is in $(A-N)_{00}([e])$. Therefore, $H(e)$ is not nil so [15] and Theorem 2.2 imply A has a Wedderburn decomposition. If 1 is not primitive and for some $e\neq 1$, $G(e)\neq 0$, N, then by Lemma 2.2 (with $Q=C(P)$), A has a Wedderburn decomposition. The only other case is when 1 is not primitive and $G(e)=N$ for all $e\neq 1$. Here, Lemma 6.1 yields a Wedderburn decomposition for A. Hence, P_1 is a Wedderburn class. If A is nearly $(-1, 0)$ then A is anti-isomorphic to a nearly (1, 1) algebra B. Hence A has a Wedderburn decomposition.

As a consequence of the results in [7], if A is (1, 1) (or $(-1, 0)$), then $A-N$ is associative and we have this result.

THEOREM 6.2. *If A is a* (1, 1) *or* $(-1, 0)$ *algebra over a splitting field of char* $\neq 2, 3$ *then A has a Wedderburn decomposition.*

THEOREM 6.3. *Let A be a nearly* (1, 1) *(nearly* $(-1, 0)$*) algebra over a splitting field F of char* $\neq 2, 3$. *If A contains neither nodal subalgebras nor ideals G with* $G^2=0$ *then A has a Wedderburn decomposition.*

Proof. Since each $G(e)=0$, the results in [8] imply (32), (33) and (34) so A has a Wedderburn decomposition.

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