

# ON THE WEIGHTED ORTHOGONAL RICCI CURVATURE

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ABSTRACT. We introduce the weighted orthogonal Ricci curvature – a two-parameter version of Ni–Zheng’s orthogonal Ricci curvature. This curvature serves as a very natural object in the study of the relationship between the Ricci curvature(s) and the holomorphic sectional curvature. In particular, in determining optimal curvature constraints for a compact Kähler manifold to be projective. In this direction, we prove a number of vanishing theorems using the weighted orthogonal Ricci curvature(s) in both the Kähler and Hermitian category.

## 1. INTRODUCTION

Let  $(M^n, \omega)$  be a compact Kähler manifold. One of the central problems in modern complex geometry is the relationship between the Ricci curvature  $\text{Ric}_\omega$  and the holomorphic sectional curvature  $\text{HSC}_\omega$ . The Wu–Yau theorem [34, 35, 33, 12, 40, 26, 31, 16, 3, 4] is one such manifestation of this relationship: If  $M$  supports a metric of negative holomorphic sectional curvature, then  $M$  supports a (possibly different) metric with negative Ricci curvature. It is known from Hitchin’s metrics on Hirzebruch surfaces [14], however, that  $\text{HSC}_\omega > 0$  does not imply  $\text{Ric}_\omega > 0$ .

In an attempt to further understand this relationship, Ni–Zheng [24] have studied the orthogonal Ricci curvature (which first appeared under the name “anti-holomorphic Ricci curvature” in [20]):

$$\text{Ric}_\omega^\perp : T^{1,0}M \rightarrow \mathbb{R}, \quad \text{Ric}_\omega^\perp(X) := \frac{1}{|X|_\omega^2} \text{Ric}_\omega(X, \bar{X}) - \text{HSC}_\omega(X).$$

It was observed in [24, 15, 25] that  $\text{Ric}_\omega^\perp$  is, in an appropriate sense, the trace of the orthogonal bisectonal curvature  $\text{HBC}_\omega^\perp$  (i.e., the restriction of the bisectonal curvature to pairs of orthogonal vectors). This justifies the terminology orthogonal Ricci curvature (c.f., Remark 2.6).

It is suspected (c.f., [24]) that the orthogonal Ricci curvature should give a sharp measure of the projectivity of compact Kähler manifolds. For instance,  $\text{Ric}_\omega > 0$  implies that the anti-canonical bundle  $K_M^{-1}$  is ample, hence  $M$  is projective. Recently, it was shown by Yang [39] that  $\text{HSC}_\omega > 0$  implies  $h^{p,0} = 0$  for all  $1 \leq p \leq n$ . Ni [23] showed that any compact Kähler manifold with  $\text{Ric}_\omega^\perp > 0$  must be projective and simply connected (also see [24]).

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With this aim of generating a sharp curvature constraint for compact Kähler manifolds to be projective, we introduce the following *weighted orthogonal Ricci curvature*:

**Definition 1.1.** Let  $(M^n, \omega)$  be a compact Kähler manifold. For  $\alpha, \beta \in \mathbb{R}$ , we define the *weighted orthogonal Ricci curvature* to be the function

$$\text{Ric}_{\alpha, \beta}^{\perp} : T^{1,0}M \rightarrow \mathbb{R}, \quad \text{Ric}_{\alpha, \beta}^{\perp} := \frac{\alpha}{|X|_{\omega}^2} \text{Ric}_{\omega}(X, \bar{X}) + \beta \text{HSC}_{\omega}(X).$$

**Remark 1.2.** Of course, when  $\alpha = 1$ ,  $\beta = -1$ , we recover the orthogonal Ricci curvature  $\text{Ric}^{\perp}$ . Chu–Lee–Tam [11] showed that, for  $\alpha > 0$  and  $\beta > 0$ , a compact Kähler manifold with  $\text{Ric}_{\alpha, \beta}^{\perp} < 0$  must be projective with ample canonical bundle, extending the aforementioned Wu–Yau theorem. Further, they also show that a compact Kähler manifold with  $\text{Ric}_{\alpha, \beta}^{\perp} > 0$  for  $\alpha > 0$ ,  $\beta > 0$ , is projective and simply connected.

By considering the weighted orthogonal Ricci curvature, in place of merely the orthogonal Ricci curvature, natural questions concerning the *extent* to which the holomorphic sectional curvature and Ricci curvature are related easily proliferate. For instance, the following question is of tremendous interest:

**Question 1.3.** Determine all  $(\alpha, \beta) \in \mathbb{R}^2$  such that a compact Kähler manifold with  $\text{Ric}_{\alpha, \beta}^{\perp} > 0$  or  $\text{Ric}_{\alpha, \beta}^{\perp} < 0$  is projective.

In this direction, we have the following theorem:

**Theorem 1.4.** Let  $(M^n, \omega)$  be a compact Kähler manifold with  $\text{Ric}_{\alpha, \beta}^{\perp} > 0$  for some  $\alpha > 0 > \beta$ . If, moreover,  $3\alpha + 2\beta > 0$ , then  $M$  is projective.

In a similar spirit, we have the following natural question:

**Question 1.5.** Determine all  $(\alpha, \beta) \in \mathbb{R}^2$  such that a compact Kähler manifold with  $\text{Ric}_{\alpha, \beta}^{\perp} > 0$  satisfies  $h^{p,0} = 0$ .

We have the following partial answer to the above question:

**Theorem 1.6.** Let  $(M^n, \omega)$  be a compact Kähler manifold with  $\text{Ric}_{\alpha, \beta}^{\perp} > 0$  for some  $\alpha > 0$  and  $\beta < 0$ . If, moreover,  $(p+1)\alpha + 2\beta > 0$ , then  $h^{p,0} = 0$  for all  $1 \leq p \leq n$ . In particular,  $M$  is projective.

We will exhibit a number of theorems of this type in this manuscript. The utility of such theorems is also seen if one has knowledge of the Hodge numbers. Indeed, we can show that certain manifolds do not support metrics with particular relations on the Ricci and

holomorphic sectional curvature. To state one such instance of this, let us first observe that we can define Hermitian extensions of the weighted orthogonal Ricci curvature:

**Definition 1.7.** Let  $(M^n, \omega)$  be a Hermitian manifold. For  $1 \leq k \leq 4$ , we define the *weighted  $k$ th orthogonal Ricci curvature* of  $\omega$  to be the function

$$\text{Ric}_{\alpha, \beta}^{(k)} : T^{1,0}M \rightarrow \mathbb{R}, \quad \text{Ric}_{\alpha, \beta}^{(k)}(X) := \frac{\alpha}{|X|_\omega^2} \text{Ric}_\omega^{(k)}(X, \bar{X}) + \beta \text{HSC}_\omega(X).$$

Since we know the Hodge numbers of the Iwasawa threefold, we can prove the following:

**Theorem 1.8.** There is no balanced metric on the Iwasawa threefold such that  $\text{Ric}_{\alpha, \beta}^{(k)} > 0$  for  $\alpha > 0 > \beta$  and  $3\alpha + 2\beta > 0$ .

## 2. SOME REMINDERS OF CURVATURE IN COMPLEX GEOMETRY

Let  $(M^n, \omega)$  be a Hermitian manifold. The Chern connection  $\nabla$  on  $T^{1,0}M$  is the unique Hermitian connection whose torsion has vanishing  $(1, 1)$ -part. Fix a point  $p \in M$ . In a local coordinate frame  $\left\{ \frac{\partial}{\partial z_i} \right\}$  of  $T_p^{1,0}M$ , the components of the Chern curvature tensor  $R$  read

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}_j}.$$

**Reminder 2.1: Chern–Ricci and scalar curvatures.** The Chern curvature is an  $\text{End}(T^{1,0}M)$ -valued  $(1, 1)$ -form. The first Chern–Ricci curvature is the contraction over the endomorphism part:

$$\text{Ric}_\omega^{(1)} = \sqrt{-1} \text{Ric}_{i\bar{j}}^{(1)} dz_i \wedge d\bar{z}_j = \sqrt{-1} g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} dz_i \wedge d\bar{z}_j,$$

and is a  $(1, 1)$ -form representing the first Chern class the anti-canonical bundle  $c_1(K_M^{-1})$ . The second Chern–Ricci curvature is a contraction over the  $(1, 1)$ -part:

$$\text{Ric}_\omega^{(2)} = \text{Ric}_{k\bar{\ell}}^{(2)} = g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}.$$

Similarly, the third and fourth Chern–Ricci curvature are defined:

$$\text{Ric}_\omega^{(3)} = \text{Ric}_{k\bar{j}}^{(3)} = g^{i\bar{\ell}} R_{i\bar{j}k\bar{\ell}}, \quad \text{Ric}_\omega^{(4)} = \text{Ric}_{i\bar{\ell}}^{(4)} = g^{k\bar{j}} R_{i\bar{j}k\bar{\ell}}.$$

The contraction

$$\text{Scal}_\omega := g^{i\bar{j}} \text{Ric}_{i\bar{j}}^{(1)} = g^{i\bar{j}} g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}$$

is the Chern scalar curvature. The contraction

$$\widehat{\text{Scal}}_\omega := g^{i\bar{\ell}} \text{Ric}_{k\bar{j}}^{(3)} = g^{i\bar{\ell}} g^{k\bar{j}} R_{i\bar{j}k\bar{\ell}}$$

will be referred to as the altered Chern scalar curvature.

**Remark 2.2.** If the Hermitian metric is Kähler, then the Ricci curvatures all coincide. Of course, this implies that the scalar curvatures coincide if the metric is Kähler, too. In fact, these statements are true in the more general setting of Kähler-like metrics. That is, a metric is said to be Kähler-like [38] if the Chern curvature tensor satisfies the symmetries of the Kähler curvature tensor. The Iwasawa threefold shows that a Kähler-like manifold is not necessarily Kähler.

For convenience, let us restate the definition of the curvatures of primary interest in this manuscript:

**Definition 2.3.** Let  $(M^n, \omega)$  be a compact Hermitian manifold. For  $1 \leq k \leq 4$ , we define the *weighted  $k$ th orthogonal Ricci curvature* of  $\omega$  to be the function

$$\text{Ric}_{\alpha, \beta}^{(k)} : T^{1,0}M \rightarrow \mathbb{R}, \quad \text{Ric}_{\alpha, \beta}^{(k)}(X) := \frac{\alpha}{|X|_\omega^2} \text{Ric}_\omega^{(k)}(X, \bar{X}) + \beta \text{HSC}_\omega(X).$$

In particular, if the metric is Kähler<sup>1</sup>, we can speak of the weighted orthogonal Ricci curvature:

**Definition 2.4.** Let  $(M^n, \omega)$  be a compact Kähler manifold. For  $\alpha, \beta \in \mathbb{R}$ , we define the *weighted orthogonal Ricci curvature* to be the function

$$\text{Ric}_{\alpha, \beta}^\perp : T^{1,0}M \rightarrow \mathbb{R}, \quad \text{Ric}_{\alpha, \beta}^\perp := \frac{\alpha}{|X|_\omega^2} \text{Ric}_\omega(X, \bar{X}) + \beta \text{HSC}_\omega(X).$$

We say that  $\text{Ric}_{\alpha, \beta}^\perp$  (or  $\text{Ric}_{\alpha, \beta}^{(k)} > 0$  for some  $1 \leq k \leq 4$ ) is positive (negative, zero, quasi-positive, etc.) if this is true for some  $(\alpha, \beta) \in \mathbb{R}^2$ .

To justify the terminology, let us recall the Quadratic Orthogonal Bisectional Curvature introduced in [36]:

**Definition 2.5.** Let  $(M^n, \omega)$  be a Hermitian manifold. The *Quadratic Orthogonal Bisectional Curvature* (from now on, QOBC) is the function

$$\text{QOBC}_\omega : \mathcal{F}_M \times \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}, \quad \text{QOBC}_\omega : (\vartheta, v) \mapsto \frac{1}{|v|_\omega^2} \sum_{\alpha, \gamma=1}^n R_{\alpha \bar{\alpha} \gamma \bar{\gamma}} (v_\alpha - v_\gamma)^2,$$

where  $R_{\alpha \bar{\alpha} \gamma \bar{\gamma}}$  denote the components of the Chern connection of  $\omega$  with respect to the unitary frame  $\vartheta$  (a section of the unitary frame bundle  $\mathcal{F}_M$ ).

This curvature first appeared implicitly in [2] and is the Weitzenböck curvature operator (in the sense of [27, 28, 29]) acting on real  $(1, 1)$ -forms. See [5, 6, 7] for alternative descriptions of the QOBC. From [17], the QOBC is strictly weaker than the orthogonal bisectional curvature

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<sup>1</sup>Or more generally, Kähler-like.

$\text{HBC}_\omega^\perp$  (the restriction of the holomorphic bisectional curvature  $\text{HBC}_\omega$  to pairs of orthogonal  $(1, 0)$ -tangent vectors). From [13], the Kähler–Ricci flow on a compact Kähler manifold with  $\text{HBC}_\omega^\perp \geq 0$  converges to a Kähler metric  $\text{HBC}_\omega \geq 0$ . Hence, Mok’s extension [21] of the solution of the Frankel conjecture [22, 30] shows that all compact Kähler manifolds with  $\text{HBC}_\omega^\perp \geq 0$  are biholomorphic to a product of Hermitian symmetric spaces (of rank  $\geq 2$ ) and projective spaces. In particular, although  $\text{HBC}_\omega^\perp$  is an algebraically weaker curvature notion than  $\text{HBC}_\omega$ , the positivity of  $\text{HBC}_\omega^\perp$  does not generate new examples.

**Remark 2.6.** Recall that the orthogonal Ricci curvature  $\text{Ric}_\omega^\perp$ , which first appeared in [20] under the name *anti-holomorphic Ricci curvature*, is defined

$$\text{Ric}_\omega^\perp : T^{1,0}M \rightarrow \mathbb{R}, \quad \text{Ric}_\omega^\perp(X) := \frac{1}{|X|_\omega^2} \text{Ric}_\omega(X, \bar{X}) - \text{HSC}_\omega(X).$$

From [24, 15, 25], we observe that, in a fixed unitary frame, we have

$$\text{QOBC}_\omega = \sum_{i,k} R_{i\bar{i}k\bar{k}} \rho_{i\bar{k}} \rho_{i\bar{k}} - \sum_{i,j} R_{i\bar{i}j\bar{j}} \rho_{i\bar{j}} \rho_{j\bar{j}} - \sum_{i \neq \ell \text{ or } k \neq j} R_{i\bar{i}k\bar{j}} \rho_{i\bar{k}} \rho_{j\bar{\ell}}.$$

Choosing  $\rho_{i\bar{j}} = 0$  for all  $i, j$ , except  $\rho_{1\bar{1}}$ , we see that

$$\text{QOBC}_\omega = (R_{1\bar{1}} - R_{1\bar{1}1\bar{1}}) |\rho_{1\bar{1}}|^2 \geq 0.$$

This implies that  $\text{Ric}^\perp \geq 0$ . Since  $\text{Ric}_{\alpha,\beta}^\perp > 0$  is defined for some  $\alpha, \beta \in \mathbb{R}$ , it is immediate that  $\text{Ric}^\perp > 0$  implies  $\text{Ric}_{\alpha,\beta}^\perp > 0$ . Hence, the quadratic orthogonal bisectional curvature dominates the weighted orthogonal Ricci curvature  $\text{Ric}_{\alpha,\beta}^\perp$ .

In light of this remark, we pose the following:

**Question 2.7.** Determine all values of  $(\alpha, \beta) \in \mathbb{R}^2$  such that  $\text{QOBC}_\omega \geq 0$  implies  $\text{Ric}_{\alpha,\beta}^\perp \geq 0$ .

Let  $\int_{\mathbb{S}^{2n-1}} := \frac{1}{\text{vol}(\mathbb{S}^{2n-1})} \int_{\mathbb{S}^{2n-1}}$ . If  $(M^n, \omega)$  is a compact Kähler manifold, then we have

$$\text{Scal}_\omega = 2n \int_{\mathbb{S}^{2n-1}} \text{Ric}_\omega, \quad \text{and} \quad \text{Scal}_\omega = n(n+1) \int_{\mathbb{S}^{2n-1}} \text{HSC}_\omega.$$

Hence,

$$\int_{\mathbb{S}^{2n-1}} \text{Ric}_{\alpha,\beta}^\perp = \frac{\alpha(n+1) + 2\beta}{2n(n+1)} \text{Scal}_\omega,$$

and subsequently,  $\text{Ric}_{\alpha,\beta}^\perp > 0$  implies  $\text{Scal}_\omega > 0$  if  $\alpha(n+1) + 2\beta > 0$ . Since  $\text{QOBC}_\omega \geq 0$  implies  $\text{Scal}_\omega \geq 0$  (see [9]), this argument gives a partial answer to Question 2.7:

**Proposition 2.8.** Let  $(M^n, \omega)$  be a compact Kähler manifold with  $\text{QOBC}_\omega \geq 0$ . If  $\text{Ric}_{\alpha,\beta}^\perp \geq 0$ , then  $\alpha(n+1) + 2\beta \geq 0$ .

## 3. VANISHING THEOREMS

In this section, we will prove a number of vanishing theorems for Hodge numbers, using the weighted orthogonal Ricci curvature(s).

An important technique that will be used throughout is Ni's technique of viscosity considerations (see [23, §3]). Let us briefly describe this technique for the purposes needed here. Let  $V$  a vector space, and let  $\eta$  be a  $k$ -covector. We say that  $\eta$  is *simple* if there exist  $v_1, \dots, v_k$  such that  $\eta = v_1 \wedge \dots \wedge v_k$ . Let  $\mathcal{S}(1) := \{\eta : \eta \text{ is simple with } \|\eta\| = 1\}$ . Here,  $\|\cdot\|$  is the norm induced by the scalar product defined on simple covectors  $\eta = v_1 \wedge \dots \wedge v_k$  and  $\omega = u_1 \wedge \dots \wedge u_k$  by the formula

$$\langle \eta, \omega \rangle := \det(\langle v_i, u_j \rangle)$$

and subsequently extended bilinearly to all  $r$ -covectors which are linear combinations of simple covectors. Define the *comass* of a  $k$ -covector  $\rho$ :

$$\|\rho\|_0 := \sup_{\eta \in \mathcal{S}(1)} |\rho(\eta)|.$$

Let  $\sigma \in \Omega^{p,0}(M)$  be a holomorphic  $(p, 0)$ -form. Write  $\|\sigma\|_0(x)$  for its comass at  $x$ . Let  $x_0$  be the point at which  $\|\sigma\|_0$  attains its maximum. The idea is to construct a simple  $(p, 0)$ -form  $\tilde{\sigma}(x)$  in a neighborhood of  $x_0$  such that the  $L^2$ -norm of  $\tilde{\sigma}$  attains its maximum at  $x_0$ . See [23, p. 277] for details. Computing  $\partial_v \partial_{\bar{v}} \log \|\tilde{\sigma}\|^2$  at  $x_0$  then yields

$$0 \geq \sum_{j=1}^p R_{v\bar{v}j\bar{j}},$$

for all  $v \in T_{x_0}^{1,0}M$ .

**Reminder 3.1.** Fix a point  $x \in M$ , and let  $\Sigma \subset T_x^{1,0}M$  be a  $k$ -dimensional subspace. Write  $\mathbb{S}^{2k-1} \subset \Sigma$  for the unit sphere inside  $\Sigma$ . Recall that the  $k$ -scalar curvature is defined

$$\text{Scal}_k(x, \Sigma) := k \int_{\mathbb{S}^{2n-1}} \text{Ric}_\omega(X, \bar{X}) d\vartheta(X)$$

Using Ni's method of viscosity considerations, we have the following:

**Theorem 3.2.** Let  $(M^n, \omega)$  be a compact Kähler manifold with  $\text{Ric}_{\alpha, \beta}^\perp > 0$  for some  $\alpha > 0$  and  $\beta < 0$ . If, moreover,  $(p+1)\alpha + 2\beta > 0$ , then  $h^{p,0} = 0$  for all  $1 \leq p \leq n$ . In particular,  $M$  is projective.

*Proof.* Assume there is a non-zero holomorphic  $(p, 0)$ -form  $\sigma \in H_{\bar{\partial}}^{p,0}(M) \simeq H^0(M, \Omega^p(M))$ . Let  $x_0 \in M$  be the point at which the comass  $\|\sigma\|_0$  attains its maximum. From Ni's viscosity

considerations, we have (in a fixed unitary frame  $e_k$  near  $x_0$ )

$$\sum_{k=1}^p R_{v\bar{v}k\bar{k}} \leq 0 \quad (3.1)$$

for any  $v \in T_{x_0}^{1,0}M$ . Let  $\Sigma$  denote the span of  $\{e_1, \dots, e_p\}$ , and write  $\text{Scal}_p(x_0, \Sigma)$  for the scalar curvature of  $R|_{\Sigma}$ . The inequality (3.1) implies that  $\text{Scal}_p(x_0, \Sigma) \leq 0$ . From  $\text{Ric}_{\alpha, \beta}^{\perp} > 0$ , we observe that

$$\begin{aligned} 0 &< \int_{X \in \Sigma, |X|=1} \alpha \text{Ric}_{\omega}(X, \bar{X}) + \beta \text{HSC}_{\omega}(X) d\vartheta(X) \\ &= \frac{\alpha}{p} \sum_{k=1}^p \text{Ric}_{k\bar{k}} + \frac{2\beta}{p(p+1)} S_p(x_0, \Sigma) \\ &= \frac{\alpha}{p} \left( S_p(x_0, \Sigma) + \sum_{\ell=p+1}^n \sum_{k=1}^p R_{\ell\bar{\ell}k\bar{k}} \right) + \frac{2\beta}{p(p+1)} S_p(x_0, \Sigma) \\ &\leq \frac{\alpha}{p} S_p(x_0, \Sigma) + \frac{2\beta}{p(p+1)} S_p(x_0, \Sigma) = \frac{(p+1)\alpha + 2\beta}{p(p+1)} S_p(x_0, \Sigma). \end{aligned}$$

If  $(p+1)\alpha + 2\beta > 0$ , then  $S_p(x_0, \Sigma) > 0$ , furnishing the desired contradiction.  $\square$

Let us raise the following natural question:

**Question 3.3.** Let  $(M^n, \omega)$  be a compact Kähler manifold. Determine all  $(\alpha, \beta) \in \mathbb{R}^2$  such that  $h^{p,0} = 0$  for all  $1 \leq p \leq n$ . In particular, determine the  $(\alpha, \beta) \in \mathbb{R}^2$  such that  $M$  is projective and simply connected.

To address the simply connectedness problem, let us exhibit the following extension of the diameter estimate in [11, Proposition 6.2]:

**Proposition 3.4.** Let  $(M^n, \omega)$  be a complete Kähler manifold with  $\text{Ric}_{\alpha, \beta}^{\perp} \geq \lambda$  for some  $\alpha > 0 > \beta$ , with  $3\alpha + 2\beta > 0$  and some  $\lambda > 0$ . Then  $M$  is compact with

$$\text{diam}(M, \omega) \leq \pi \sqrt{\frac{\alpha(2n-1) + \beta}{\lambda}}.$$

*Proof.* Analogous to the argument in the proof of the Bonnet–Meyer theorem, for  $p, q \in M$ , let  $\gamma : [0, \ell] \rightarrow M$  be a minimizing geodesic which connects  $p$  and  $q$ . Following [11], we will show that

$$\ell \leq \pi \sqrt{\frac{\alpha(2n-1) + \beta}{\lambda}}.$$

Let  $e_i$  be an orthonormal set of parallel vector fields along  $\gamma$  such that  $e_{2n-1} = J\dot{\gamma}$  and  $e_{2n} = \dot{\gamma}$ . Here,  $J$  is the underlying complex structure. For each  $1 \leq i \leq 2n-1$ , set

$$\begin{aligned} V_i(t) &:= \sin\left(\frac{\pi t}{\ell}\right) e_i(t), \\ \phi_i(t, s) &:= \exp_{\gamma(t)}(sV_i(t)), \\ L_i(s) &:= \text{length}(\phi_i(\cdot, s)). \end{aligned}$$

For each  $t$ , we have  $\phi_i(t, 0) = \gamma(t)$ , and since  $\gamma$  is minimizing,  $L_i$  has a minimum at  $s = 0$ . Hence, the second variation of arc length formula yields

$$\begin{aligned} 0 &\leq \left. \frac{d^2}{ds^2} \right|_{s=0} L_i(s) \\ &= \int_0^\ell (|\nabla V_i|_g^2 - R(V_i, \gamma', \gamma', V_i)) dt \\ &= \int_0^\ell \left( \left(\frac{\pi}{\ell}\right)^2 \cos^2\left(\frac{\pi t}{\ell}\right) - \sin^2\left(\frac{\pi t}{\ell}\right) R(e_i, \gamma', \gamma', e_i) \right) dt. \end{aligned}$$

Let  $X = \frac{1}{\sqrt{2}}(\gamma' - \sqrt{-1}J\gamma')$ . Then  $\text{Ric}_{\alpha, \beta}^\perp(X) \geq \lambda$  implies

$$\alpha \sum_{i=1}^{2n-1} R(e_i, \gamma', \gamma', e_i) + \beta R(e_{2n-1}, \gamma', \gamma', e_{2n-1}) \geq \lambda.$$

Since  $3\alpha + 2\beta > 0$  implies  $\alpha(2n-1) + \beta > 0$  for  $n \geq 2$ , we see that

$$\begin{aligned} 0 &\leq \left. \frac{d^2}{ds^2} \right|_{s=0} \left( \alpha \sum_{i=1}^{2n-1} L_i(s) + \beta L_{2n-1}(s) \right) \\ &= \int_0^\ell (\alpha(2n-1) + \beta) \left(\frac{\pi}{\ell}\right)^2 \cos^2\left(\frac{\pi t}{\ell}\right) dt - \int_0^\ell \sin^2\left(\frac{\pi t}{\ell}\right)^2 \text{Ric}_{\alpha, \beta}^\perp(X) dt \\ &\leq \int_0^\ell \left( (\alpha(2n-1) + \beta) \left(\frac{\pi}{\ell}\right)^2 \cos^2\left(\frac{\pi t}{\ell}\right) - \lambda \sin^2\left(\frac{\pi t}{\ell}\right) \right) dt \\ &= \frac{\ell}{2} \left( (\alpha(2n-1) + \beta) \left(\frac{\pi}{\ell}\right)^2 - \lambda \right). \end{aligned}$$

□

**Remark 3.5.** Note that Chu–Lee–Tam [11] assume that  $\alpha, \beta > 0$ . Of course, the argument requires only that  $\alpha(2n-1) + \beta > 0$ .

**Proposition 3.6.** Let  $(M^n, \omega)$  be a compact Kähler manifold with  $\text{Ric}_{\alpha, \beta}^\perp > 0$  for some  $\alpha > 0 > \beta$  and  $\alpha + \beta > 0$ . Then  $M$  is simply connected.



*Proof.* Let  $\mathcal{O}_M$  denote the structure sheaf of  $M$ . The Euler characteristic is given by  $\chi(\mathcal{O}_M) := \sum_{k=0}^n (-1)^k h^{k,0}$ . For any finite  $\nu$ -sheeted covering  $\widetilde{M} \rightarrow M$ , the Riemann–Roch–Hirzebruch formula states that

$$\chi(\mathcal{O}_{\widetilde{M}}) = \nu \chi(\mathcal{O}_M).$$

If there is a metric on  $M$  with  $\text{Ric}_{\alpha,\beta}^\perp > 0$  for some  $\alpha > 0 > \beta$  and  $\alpha + \beta > 0$ , then  $h^{p,0} = 0$  for all  $1 \leq p \leq n$ . From [24, Theorem 3.2], the universal cover  $\widetilde{M}$  affords a metric with  $\text{Ric}_{\alpha,\beta}^\perp > \delta > 0$ . In particular,  $\widetilde{M}$  is compact and projective with  $\chi(\mathcal{O}_{\widetilde{M}}) = 1$ . It follows that  $\nu = 1$ , and thus  $\pi_1(M) = 0$ .  $\square$

**Conjecture 3.7.** Let  $(M^n, \omega)$  be a compact Hermitian manifold.

- (i) If  $\text{Ric}_{\alpha,\beta}^{(1)} > 0$  for some  $\alpha > 0$  and  $\beta < 0$ , then  $h^{p,0} = 0$  for all  $1 \leq p \leq n$ .
- (ii) If  $\text{Ric}_{\alpha,\beta}^{(2)} > 0$  for  $\alpha > 0$  and  $\beta < 0$ , then  $h^{p,0} = 0$  for all  $1 \leq p \leq n$ .

In this direction, we have the following:

**Proposition 3.8.** Let  $(M^n, \omega)$  be a compact Hermitian manifold.

- (i) If  $\text{Ric}_{\alpha,-\alpha}^{(1)} > 0$  for some  $\alpha > 0$ , then  $h^{n-1,0} = 0$ .
- (ii) If  $\text{Ric}_{\alpha,\beta}^{(2)} > 0$  for some  $\alpha > 0$ ,  $\beta < 0$  with  $\alpha + \beta \geq 0$ , then  $h^{1,0} = 0$ .

*Proof.* We observe that Ni’s technique of viscosity considerations extends to the Hermitian category. Let  $\sigma \in H_{\bar{\partial}}^{n-1,0}(M)$  be a non-trivial holomorphic  $(n-1,0)$ -form on  $M$ . Let  $p \in M$  be the point at which the comass  $\|\sigma\|_0$  achieves its maximum. The Bochner technique, together with the maximum principle implies that

$$\sum_{k=1}^{n-1} R_{v\bar{v}k\bar{k}} \leq 0 \tag{3.2}$$

for all  $v \in T_p^{1,0}M$ . Hence, for any  $\alpha > 0$ ,

$$0 \geq \alpha \sum_{k=1}^{n-1} R_{n\bar{n}k\bar{k}} = \alpha \sum_{k=1}^n R_{n\bar{n}k\bar{k}} - \alpha R_{n\bar{n}n\bar{n}} = \text{Ric}_{\alpha,-\alpha}^{(1)},$$

contradicting  $\text{Ric}_{\alpha,-\alpha}^{(1)} > 0$ . This proves (i).

Similarly for (ii), let  $\sigma \in H_{\bar{\partial}}^{1,0}(M)$  be a non-trivial holomorphic  $(1,0)$ -form. The same argument implies (in a fixed unitary frame  $e_1, \dots, e_n$ )

$$R_{v\bar{v}1\bar{1}} \leq 0 \tag{3.3}$$

for any  $v \in T_p^{1,0}M$ , where  $p$  is the point where  $|\sigma|^2$  achieves its maximum. Choose  $v$  such that  $v = e_1$ , then

$$\alpha \sum_{k=2}^n R_{k\bar{k}1\bar{1}} \leq 0, \quad (\alpha + \beta)R_{1\bar{1}1\bar{1}} \leq 0.$$

Hence,

$$0 \geq \alpha \sum_{k=2}^n R_{k\bar{k}1\bar{1}} + (\alpha + \beta)R_{1\bar{1}1\bar{1}} = \text{Ric}_{\alpha,\beta}^{(2)},$$

contradicting  $\text{Ric}_{\alpha,\beta}^{(2)} > 0$  if  $\alpha + \beta \geq 0$ . This proves (ii).  $\square$

**Corollary 3.9.** Let  $M^3$  be a compact Kähler threefold. If  $M$  supports a Hermitian metric with  $\text{Ric}_{\alpha,-\alpha}^{(1)} > 0$  for some  $\alpha > 0$ , then  $M$  is projective.

*Proof.* From part (i) of Proposition 3.8, we have  $h^{2,0} = 0$ . It is well-known that a compact Kähler manifold satisfying  $h^{2,0} = 0$  is projective.  $\square$

Recall that the Chern scalar curvature of a Hermitian metric  $\omega$  is defined by

$$\text{Scal}_\omega := g^{i\bar{j}}g^{k\bar{\ell}}R_{i\bar{j}k\bar{\ell}}.$$

We also have the altered scalar curvature  $\widehat{\text{Scal}}_\omega := g^{i\bar{\ell}}g^{k\bar{j}}R_{i\bar{j}k\bar{\ell}}$ .

**Theorem 3.10.** Let  $(M^n, \omega)$  be a compact Hermitian manifold with  $\text{Ric}_{\alpha,\beta}^{(k)} > 0$  for some  $1 \leq k \leq 4$ , and  $\alpha > 0 > \beta$ . If  $\text{Scal}_p(\omega) = \widehat{\text{Scal}}_p(\omega)$  and  $(p+1)\alpha + 2\beta$ , then  $h^{p,0} = 0$  for all  $1 \leq p \leq n$ .

*Proof.* Assume there is a non-zero holomorphic  $(p, 0)$ -form  $\sigma \in H_{\bar{\partial}}^{p,0}(M) \simeq H^0(M, \Omega^p(M))$ . Let  $x_0 \in M$  be the point at which the comass  $\|\sigma\|_0$  attains its maximum. From Ni's viscosity considerations, we have (in a fixed unitary frame  $e_i$  near  $x_0$ )

$$\sum_{i=1}^p R_{v\bar{v}i\bar{i}} \leq 0 \tag{3.4}$$

for any  $v \in T_{x_0}^{1,0}M$ . Let  $\Sigma := \text{span}\{e_1, \dots, e_p\}$ , and write  $\text{Scal}_p(x_0, \Sigma)$  for the Chern scalar curvature of  $R|_\Sigma$ . Similarly, write  $\widehat{\text{Scal}}_p(x_0, \Sigma)$  for the altered scalar curvature of  $R|_\Sigma$ . If  $\omega$  is balanced, then  $\text{Scal}_p(x_0, \Sigma) = \widehat{\text{Scal}}_p(x_0, \Sigma)$ , and (3.4) implies  $\text{Scal}_p(x_0, \Sigma) \leq 0$ . Assume

$\text{Ric}_{\alpha,\beta}^{(k)} > 0$  for some  $1 \leq k \leq 3$ , then

$$\begin{aligned}
0 &< \int_{\mathbb{S}^{2p-1}} \alpha \text{Ric}_{\omega}^{(k)}(X, \bar{X}) + \beta \text{HSC}_{\omega}(X) d\vartheta \\
&= \frac{\alpha}{p} \sum_{i=1}^p \text{Ric}_{i\bar{i}}^{(k)} + \frac{2\beta}{p(p+1)} (\text{Scal}_p(x_0, \Sigma) + \widehat{\text{Scal}}_p(x_0, \Sigma)) \\
&= \frac{\alpha}{p} \text{Scal}_p(x_0, \Sigma) + \frac{2\beta}{p(p+1)} (\text{Scal}_p(x_0, \Sigma) + \widehat{\text{Scal}}_p(x_0, \Sigma)) \\
&= \frac{(\alpha(p+1) + \beta) \text{Scal}_p(x_0, \Sigma) + \beta \widehat{\text{Scal}}_p(x_0, \Sigma)}{2p(p+1)} \\
&= \frac{\alpha(p+1) + 2\beta}{2p(p+1)} \text{Scal}_p(x_0, \Sigma),
\end{aligned}$$

where the last equality makes use of the balanced condition. If  $\alpha(p+1) + 2\beta > 0$ , we have the desired contradiction.  $\square$

From [18], we have the pointwise equality

$$\text{Scal}_{\omega} = \widehat{\text{Scal}}_{\omega} + \langle \bar{\partial} \bar{\partial}^* \omega, \omega \rangle.$$

In particular, if  $\omega$  is balanced, then  $\text{Scal}_{\omega} = \widehat{\text{Scal}}_{\omega}$ :

**Corollary 3.11.** Let  $(M^n, \omega)$  be a compact Hermitian manifold with a balanced metric of  $\text{Ric}_{\alpha,\beta}^{(k)} > 0$  for some  $1 \leq k \leq 4$ , and  $\alpha > 0 > \beta$ . If  $\alpha(n+1) + 2\beta > 0$ , then  $h^{n,0} = 0$ .

Of course, when  $n = 2$ , i.e., on compact complex surfaces, the balanced condition is equivalent to the Kähler condition.

**Remark 3.12.** Petersen–Wink [29] have established estimates on the Hodge numbers of Kähler manifolds in terms of the eigenvalues of the Kähler curvature operator. In particular, they show that if  $\mathfrak{R} : \mathfrak{u}(n) \rightarrow \mathfrak{u}(n)$  denotes the Kähler curvature operator with eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{n^2}$ , then  $h^{2,0} = 0$  if

$$\lambda_1 + \dots + \lambda_{n-1} > 0.$$

That is,  $M$  is projective if the metric is  $(n-1)$ -positive. Let us remark that is quite strong: for  $n = 3$ , this reduces to the 2-positivity of the Kähler curvature operator, which is known to be equivalent to the positivity of the orthogonal bisectional curvature  $\text{HBC}_{\omega}^{\perp}$ . In particular, such metrics are all biholomorphically isometric to  $(\mathbb{P}^3, \omega_{\text{FS}})$ .

If  $(M, \omega)$  is a compact Kähler surface, the orthogonal Ricci curvature  $\text{Ric}_{\omega}^{\perp}$  is equivalent to the orthogonal bisectional curvature  $\text{HBC}_{\omega}^{\perp}$ . From the work of Gu–Zhang [13], any compact simply connected Kähler surface with  $\text{Ric}_{\omega}^{\perp} > 0$ , therefore, deforms under the Kähler–Ricci

flow to a metric with  $\text{HBC}_\omega > 0$ . It follows from the solution of the Frankel conjecture that  $M$  is biholomorphic to  $\mathbb{P}^2$ .

**Proposition 3.13.** Let  $(M^n, \omega)$  be a compact Kähler manifold with  $\text{Ric}_{\alpha, \beta}^\perp < 0$  for some  $\alpha > 0$  and  $\beta < 0$ . If  $\alpha + \beta \geq 0$ , then  $M$  does admit any non-trivial holomorphic vector fields.

*Proof.* Let  $X$  be a non-trivial holomorphic vector field. The Bochner formula gives

$$\left\langle \sqrt{-1} \partial \bar{\partial} |X|^2, \frac{1}{\sqrt{-1}} v \wedge \bar{w} \right\rangle = \langle \nabla_v X, \overline{\nabla_w X} \rangle - R_{v\bar{w}X\bar{X}}.$$

At the point  $p \in M$  where  $|X|^2$  attains its non-zero maximum, we have

$$R_{v\bar{w}X\bar{X}} \geq 0, \quad (3.5)$$

for all  $v \in T_p^{1,0}M$ . Let  $\{e_1, \dots, e_n\}$  be a local unitary frame near  $p$ , such that  $X|_p = e_1|_p$ . From (3.5), we have

$$\alpha \left( \sum_{k=2}^n R_{k\bar{k}1\bar{1}} \right) \geq 0, \quad (\alpha + \beta) \text{HSC}_\omega(X) \geq 0. \quad (3.6)$$

Since  $\text{Ric}_{\alpha, \beta}^\perp < 0$ , however,

$$\alpha \left( \sum_{k=2}^n R_{k\bar{k}1\bar{1}} \right) + (\alpha + \beta) \text{HSC}_\omega(X) < 0,$$

violating (3.6). □

Immediate from the argument in the Kähler category, is the following:

**Proposition 3.14.** Let  $(M^n, \omega)$  be a compact Hermitian manifold. Suppose  $\text{Ric}_{\alpha, \beta}^{(2)} < 0$  for  $\alpha > 0$ ,  $\beta < 0$ , with  $\alpha + \beta \geq 0$ . Then  $M$  does not admit any non-trivial holomorphic vector fields.

Let us close this section by extending an old result of Cheung [10]:

**Theorem 3.15.** Let  $(M^2, \omega)$  be a Kähler–Einstein surface with  $\text{Ric}_\omega = \lambda\omega$ . The metric  $\omega$  has negative holomorphic sectional curvature if and only if

$$\text{Ric}_{2, -1}^\perp < 0 \quad \text{and} \quad |R_{1\bar{2}1\bar{2}}|^2 < |\text{Ric}_{2, -1}^\perp|^2.$$

*Proof.* Fix a point  $p \in M$ , and assume the holomorphic sectional curvature achieves a minimum in the direction  $e_1$ . Taking partial derivatives of  $\sum_{i, j, k, \ell} R_{i\bar{j}k\bar{\ell}} v_i \bar{v}_j v_k \bar{v}_\ell$ , we see that  $R_{1\bar{1}1\bar{2}} = R_{1\bar{1}2\bar{1}} = 0$  at  $p$ . Since the metric is Kähler–Einstein, we further deduce that the following components of the curvature vanish:

$$R_{1\bar{1}1\bar{2}} = R_{1\bar{1}2\bar{1}} = R_{1\bar{2}1\bar{1}} = R_{1\bar{2}2\bar{2}} = R_{2\bar{1}1\bar{1}} = R_{2\bar{1}2\bar{2}} = R_{2\bar{2}1\bar{1}} = R_{2\bar{2}2\bar{1}} = 0.$$

The holomorphic sectional curvature at  $p$ , in the unit direction  $(v_1, v_2)$  is given by

$$\begin{aligned} \sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} v_i \bar{v}_j v_k \bar{v}_\ell &= R_{1\bar{1}1\bar{1}}(v_1 \bar{v}_1)^2 + 4R_{1\bar{1}2\bar{2}}v_1 \bar{v}_1 v_2 \bar{v}_2 + R_{2\bar{1}2\bar{1}}(v_2 \bar{v}_1)^2 \\ &\quad + R_{1\bar{2}1\bar{2}}(v_1 \bar{v}_2)^2 + R_{2\bar{2}2\bar{2}}(v_2 \bar{v}_2)^2 \\ &= R_{1\bar{1}1\bar{1}} + 2(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}})|v_1 \bar{v}_2|^2 + 2\operatorname{Re}(R_{1\bar{2}1\bar{2}}(v_1 \bar{v}_2)^2). \end{aligned}$$

Write  $R_{1\bar{2}1\bar{2}} = |R_{1\bar{2}1\bar{2}}|e^{i\vartheta_1}$  and consider the direction  $v_1 = \frac{1}{\sqrt{2}}$ ,  $v_2 = \frac{\sqrt{-1}}{\sqrt{2}}e^{i\vartheta_1/2}$ . The holomorphic sectional curvature in this direction is therefore

$$\begin{aligned} R_{1\bar{1}1\bar{1}} + \frac{1}{2}(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}}) + 2\operatorname{Re}\left(-\frac{1}{4}|R_{1\bar{2}1\bar{2}}|e^{i\vartheta_1}e^{-i\vartheta_1}\right) \\ = R_{1\bar{1}1\bar{1}} + \frac{1}{2}(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} - |R_{1\bar{2}1\bar{2}}|). \end{aligned}$$

Since  $e_1$  minimizes the holomorphic sectional curvature, we have

$$2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} \geq |R_{1\bar{2}1\bar{2}}|,$$

which implies  $2R_{1\bar{1}2\bar{2}} \geq R_{1\bar{1}1\bar{1}}$ , i.e.,  $2\lambda \geq 3R_{1\bar{1}1\bar{1}}$ .

Extending the above calculation,

$$\begin{aligned} \sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} v_i \bar{v}_j v_k \bar{v}_\ell &\leq R_{1\bar{1}1\bar{1}} + 2(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}})|v_1 \bar{v}_2|^2 + 2|R_{1\bar{2}1\bar{2}}||v_1 \bar{v}_2|^2 \\ &= R_{1\bar{1}1\bar{1}} + 2(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} + |R_{1\bar{2}1\bar{2}}|)|v_1 \bar{v}_2|^2 \\ &\leq R_{1\bar{1}1\bar{1}} + \frac{1}{2}(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} + |R_{1\bar{2}1\bar{2}}|)(|v_1|^2 + |v_1|^2) \\ &= R_{1\bar{1}1\bar{1}} + \frac{1}{2}(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} + |R_{1\bar{2}1\bar{2}}|). \end{aligned}$$

A variation argument similar to the one above shows that the upper bound is achieved when  $v_1 = \frac{1}{\sqrt{2}}$  and  $v_2 = \frac{1}{\sqrt{2}}e^{i\vartheta_1/2}$ . Hence, the holomorphic sectional curvature is maximized at  $p$  with value

$$R_{1\bar{1}1\bar{1}} + \frac{1}{2}(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} + |R_{1\bar{2}1\bar{2}}|).$$

Since this equality is negative if and only if

$$\lambda - \frac{1}{2}R_{1\bar{1}1\bar{1}} + \frac{1}{2}|R_{1\bar{2}1\bar{2}}| < 0$$

if and only if

$$2\lambda - R_{1\bar{1}1\bar{1}} < 0 \quad \text{and} \quad |R_{1\bar{2}1\bar{2}}|^2 < |2\lambda - R_{1\bar{1}1\bar{1}}|^2,$$

this completes the proof.  $\square$

## 4. EXAMPLES

**Example 4.1. The Iwasawa Threefold.** Let  $X = G/\Gamma$  denote the Iwasawa threefold given by the quotient of

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : (z_1, z_2, z_3) \in \mathbb{C}^3 \right\}$$

by the discrete group

$$\Gamma := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_1, z_2, z_3 \in \mathbb{Z} + \sqrt{-1}\mathbb{Z} \right\}.$$

It is well-known that  $X$  is non-Kähler, but supports a balanced metric. Indeed, the projection map  $f : X \rightarrow \mathbb{Z}[\sqrt{-1}]$  given by

$$f : \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto z_1$$

is a surjective holomorphic map with Kähler fibers. The map

$$\sigma : z_1 \mapsto \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$$

defines a holomorphic section of  $f$ . By [19, Theorem 5.5],  $X$  admits a balanced metric. The Hodge numbers of  $X$  are detailed in [1, p. 49]. In particular,

$$h^{1,0} = 3, \quad h^{2,0} = 3, \quad h^{3,0} = 1.$$

We, therefore, have the following:

**Corollary 4.2.** Let  $X$  be the Iwasawa threefold. There is no balanced Hermitian metric on  $X$  with  $\text{Ric}_{\alpha,\beta}^{(k)} > 0$  for  $\alpha > 0 > \beta$  and  $3\alpha + 2\beta > 0$ .

**Example 4.3.  $U(n)$ -invariant Kähler metrics on  $\mathbb{C}^n$ .** In the standard coordinates on  $\mathbb{C}^{n \geq 3}$ , a  $U(n)$ -invariant Kähler metric is given by

$$g_{i\bar{j}} = f(r)\delta_{ij} + f'(r)\bar{z}_i z_j,$$

where  $r = \sum_{k=1}^n |z_k|^2$ , and the function  $f$  is smooth on  $[0, \infty)$ . Set  $h = (rf)'$ .

We first recall the following lemma of Wu-Zheng [37]:

**Lemma 4.4.** The  $U(n)$ -invariant metric  $g$  defined above is a complete Kähler metric if and only if  $f > 0$ ,  $h > 0$ , and

$$\int_0^\infty \sqrt{\frac{h}{r}} dr = \infty.$$

For  $h > 0$ , the function  $\xi = -rh'/h$  is smooth on  $[0, \infty)$ , with  $\xi(0) = 0$ . The components of the curvature tensor of a  $U(n)$ -invariant Kähler metric in the unitary frame

$$e_1 := \frac{1}{\sqrt{h}} \partial_{z_1}, \quad e_k := \frac{1}{\sqrt{f}} \partial_{z_k}, \quad k \geq 2,$$

at the point  $p = (z_1, 0, \dots, 0)$  are given by

$$\begin{aligned} A &:= R_{1\bar{1}1\bar{1}} = \xi'/h \\ B &:= R_{1\bar{1}i\bar{i}} = \frac{1}{(rf)^2} \left[ rh - (1 - \xi) \int_0^r h ds \right], \quad i \geq 2, \\ C &:= R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = \frac{2}{r^2 f^2} \left( \int_0^r h ds - rh \right), \quad 2 \leq i \neq j. \end{aligned}$$

All other components (not given by symmetries of the above) vanish. By the unitary-invariance, it suffices to calculate the curvature at the point  $p$ .

In the above notation (c.f., [15, p. 6]), we have

$$\begin{aligned} R_{1\bar{1}} &= A + (n-1)B, \\ R_{1\bar{1}1\bar{1}} &= R_{1\bar{1}} - (n-1)B \\ R_{i\bar{i}} &= B + C + \frac{(n-2)}{2}C = B + \frac{n}{2}C, \quad i \geq 2, \\ R_{i\bar{i}i\bar{i}} &= R_{i\bar{i}} - B - \frac{(n-2)}{2}C, \quad i \geq 2. \end{aligned}$$

**Proposition 4.5.** Let  $\omega$  be the complete  $U(n)$ -invariant metric above. In the above notation,

$$\text{Ric}_{\alpha, \beta}^\perp(e_1) = \alpha(n-1)B + \beta A,$$

and for each  $i \geq 2$ ,

$$\text{Ric}_{\alpha, \beta}^\perp(e_i) = \left( \frac{\alpha n}{2} + \beta \right) C + \alpha B.$$

**Example 4.6. Hopf Manifolds.** Let  $X = \mathbb{S}^{2n-1} \times \mathbb{S}^1$  be the standard Hopf manifold of dimension  $n \geq 2$ . On  $X$  there is a natural metric (inherited from the cyclic group action of  $z \mapsto \frac{1}{2}z$  on  $\mathbb{C}^n - \{0\}$ ):

$$\omega_0 = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j = \sqrt{-1} \frac{4\delta_{ij}}{|z|^2} dz_i \wedge d\bar{z}_j.$$

From [18, §6.1], the components of the Chern curvature tensor read

$$R_{i\bar{j}k\bar{\ell}} = g^{p\bar{q}} \frac{\partial g_{p\bar{\ell}}}{\partial z_i} \frac{\partial g_{k\bar{q}}}{\partial \bar{z}_j} - \frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_j} = \frac{4\delta_{k\ell}(\delta_{ij}|z|^2 - z_j\bar{z}_i)}{|z|^6}.$$

Let  $v \in T^{1,0}X$  be a  $(1, 0)$ -tangent vector of unit length. Then

$$\begin{aligned} \text{HSC}_\omega(v) &= \sum_{i,j,k,\ell=1}^n \frac{4\delta_{k\ell}(\delta_{ij}|z|^2 - z_j\bar{z}_i)}{|z|^6} v_i \bar{v}_j v_k \bar{v}_\ell \\ &= \frac{4}{|z|^6} \sum_{k=1}^n |v_k|^2 \sum_{i \neq j=1}^n (|v_i|^2 |z|^2 - z_j \bar{z}_i v_i \bar{v}_j). \end{aligned}$$

Moreover, we have that

$$\text{Ric}_\omega^{(1)}(v) = n\sqrt{-1}\partial\bar{\partial} \log |z|^2$$

and

$$\text{Ric}_\omega^{(2)}(v) = \frac{n-1}{4} \left( \frac{4}{|z|^2} \sum_{i=1}^n |v_i|^2 \right) = \frac{n-1}{|z|^2}.$$

Hence,

$$\text{Ric}_{\alpha,\beta}^{(2)} = \frac{\alpha(n-1)}{|z|^2} + \frac{4\beta}{|z|^6} \sum_{k=1}^n \sum_{i \neq j=1}^n (|v_i|^2 |z|^2 - z_j \bar{z}_i v_i \bar{v}_j)$$

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## REFERENCES

- [1] Angella, D., Cohomological aspects of non-Kähler manifolds, Lecture Notes in Mathematics, 2095, Springer, Cham, 2014.
- [2] R. L. Bishop and S. I. Goldberg, On the second cohomology group of a Kaehler manifold of positive curvature, Proc. Amer. Math. Soc. 16 (1965), 119–122. MR0172221
- [3] Broder, K., The Schwarz lemma in Kähler and non-Kähler geometry, arXiv:2109.06331
- [4] Broder, K., The Schwarz lemma: An Odyssey, arXiv:2110.04989
- [5] Broder, K., On the non-negativity of the Dirichlet energy of a weighted graph. (submitted)
- [6] Broder, K., An eigenvalue characterization of the dual EDM cone, to appear in the Bull. of the Aust. Math. Soc.
- [7] Broder, K., Remarks on the Quadratic Orthogonal Bisectional Curvature, (in preparation).
- [8] A. Chau and L.-F. Tam, Kähler C-spaces and quadratic bisectional curvature, J. Differential Geom. 94 (2013), no. 3, 409–468. MR3080488
- [9] Chau, A., Tam, L.-F., On quadratic orthogonal bisectional curvature, J. Diff. Geom. **92** (2012), no. 2, 187–200.



- [10] Cheung, C.-K., Negative holomorphic sectional curvature and hyperbolic manifold, Ph.D. thesis (University of California, Berkeley, 1988).
- [11] Chu, J. C., Lee, M. C., Tam L.-F., Kähler manifolds with negative  $k$ -Ricci Curvature, arXiv: 2009.06297
- [12] Diverio, S., Trapani, S., Quasi-negative holomorphic sectional curvature and positivity of the canonical bundle, *J. Differential Geom.* **111** (2019), no. 2, 303–314.
- [13] Gu, H., Zhang, Z., An extension of Mok’s theorem on the generalized Frankel conjecture. *Sci. China Math.* **53**, 1–12 (2010).
- [14] Hitchin, N., On the curvature of rational surfaces, *Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973)*, pp. 65–80. Amer. Math. Soc., Providence, R. I., 1975
- [15] Huang, S. C., Tam, L.-F.,  $U(n)$ -invariant Kähler metrics with nonnegative quadratic bisectional curvature. *Asian J. Math.* **19** (1), 1–16 (2015)
- [16] Lee, M. C., Streets, J., Complex manifolds with negative curvature operator, *Int. Math. Res. Not.*, accepted. arXiv: 1903.12645.
- [17] Li, Q., Wu, D., and Zheng, F., An example of compact Kähler manifold with non-negative quadratic bisectional curvature, *Proc. Amer. Math. Soc.* 141 (2013), no. 6, 2117–2126. MR3034437
- [18] Liu, K., Yang, X., Ricci curvatures on Hermitian manifolds. *Trans. Amer. Math. Soc.* 369 (2017), no. 7, 5157–5196. MR3632564
- [19] Michelson, M. L., On the existence of special metrics in complex geometry, *Acta Math.* **149** (1982), no. 3–4, 261–295
- [20] Miquel, V., Palmer, V., Mean curvature comparison for tubular hypersurfaces in Kähler manifolds and some applications, *Compos. Math.* **86** (3), 317–335 (1993)
- [21] Mok, N., The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature, *J. Diff. Geom.* **27** (1988) 179–214.
- [22] Mori, S., Projective manifolds with ample tangent bundles, *Ann. of Math. (2)* **110** (1979), no. 3, 593–606.
- [23] Ni, L., The fundamental group, rational connectedness, and the positivity of Kähler manifolds, *J. Reine Angew. Math.* 774 (2021), 267–299, DOI 10.1515/crelle-2020-0040
- [24] Ni, L., Zheng, F., Comparison and vanishing theorems for Kähler manifolds, *Calc. Var.* **57** (151), (2018)
- [25] Niu, Y.-Y., A note on nonnegative quadratic orthogonal bisectional curvature. *Proc. Am. Math. Soc.* **142** (11), 1856–1870 (2014)
- [26] Nomura, R., Kähler manifolds with negative holomorphic sectional curvature, Kähler-Ricci flow approach, *Int. Math. Res. Not. IMRN*, 2018, 21: 6611–6616.
- [27] Petersen, P., *Riemannian Geometry*, third ed., Graduate Texts in Mathematics, vol. 171, Springer, 2016.
- [28] Petersen, P., Wink, M., New Curvature Conditions for the Bochner Technique, *Invent. math.* 224, 33-54 (2021)
- [29] Petersen, P., Wink, M., *Vanishing and estimation results for Hodge numbers*, *J. Reine Angew. Math.* (2021)
- [30] Siu, Y.-T., Yau, S.-T., Compact Kähler manifolds of positive bisectional curvature, *Invent. Math.* **59** (1980), no. 2, 189–204.
- [31] Tang, K., On real bisectional curvature and Kähler-Ricci flow, *Proc. Amer. Math. Soc.*, 2019, 147(2): 793-798.
- [32] Tang, K., Holomorphic sectional curvature and Kähler-like metrics, preprint 2020, to appear in *Sci. China - Math (Chinese series)*.
- [33] Tosatti, V., Yang, X.-K., An extension of a theorem of Wu–Yau, *J. Differential Geom.* 107(3): 573–579

- [34] Wu, D., Yau, S.-T., Negative holomorphic sectional curvature and positive canonical bundle, *Invent. Math.* **204** (2016), no. 2, 595–604
- [35] Wu, D., Yau, S.-T., A remark on our paper “Negative holomorphic sectional curvature and positive canonical bundle”, *Comm. Anal. Geom.* **24** (2016), no. 4, 901–912.
- [36] D. Wu, S.-T. Yau, and F. Zheng, A degenerate Monge-Ampère equation and the boundary classes of Kähler cones, *Math. Res. Lett.* 16 (2009), no. 2, 365–374. MR2496750
- [37] Wu, H., Zheng, F., Examples of Positively Curved Complete Kähler Manifolds, *Geometry and analysis*. No. 1, pp. 517–542, *Adv. Lect. Math.*, 17, Int. Press, Somerville, MA, 2011.
- [38] Yang, B., Zheng, F., On curvature tensors of Hermitian manifolds, *Communications in Analysis and Geometry*, vol. **26** (2018), no. 5, pp. 1195–1222
- [39] Yang, X., RC-positivity, rational connectedness, and Yau’s conjecture. *Camb. J. Math.* 6 (2018), no. 2, 183–212.
- [40] Yang, X., Zheng, F., On the real bisectional curvature for Hermitian manifolds, *Trans. Amer. Math. Soc.* **371** (2019), no. 4, 2703–2718
- [41] Yau, S.-T., On the Ricci curvature of compact Kähler manifolds and the complex Monge-Ampère equation, *Comm. Pure Appl. Math.* **31** (1978), 339–411

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