

Research Article

Jinxia Cen, Chao Min, Van Thien Nguyen, and Guo-ji Tang*

On the well-posedness of differential quasi-variational-hemivariational inequalities

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Abstract: The goal of this paper is to discuss the well-posedness and the generalized well-posedness of a new kind of differential quasi-variational-hemivariational inequality (DQHVI) in Hilbert spaces. Employing these concepts, we explore the essential relation between metric characterizations and the well-posedness of DQHVI. Moreover, the compactness of the set of solutions for DQHVI is delivered, when problem DQHVI is well-posed in the generalized sense.

Keywords: differential quasi-variational-hemivariational inequality, well-posedness, approximating sequence, α -monotonicity

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1 Introduction

The differential variational inequalities (DVI) have been introduced and systematically studied by Pang-Stewart [1] on a finite-dimensional space. As the powerful mathematical tools, recently, DVIs have been applied to the study of various problems involving both dynamics and constraints in the form of inequalities, which arise in many applied problems in our real life, for instance, mechanical impact problems, electrical circuits with ideal diodes, the Coulomb friction problems for contacting bodies, economical dynamics, dynamic traffic networks, and so on. Based on this motivation, Chen and Wang [2] in 2014 used the idea of DVIs to investigate a dynamic Nash equilibrium problem of multiple players with shared constraints and dynamic decision processes; Wang et al. [3] proved an existence theorem for Carathéodory weak solutions of a differential quasi-variational inequality in finite dimensional Euclidean spaces and established a convergence result on the Euler time-dependent procedure for solving the initial-value differential set-valued variational inequalities; and Migórski et al. [4] used the surjectivity of set-valued pseudomonotone operators combined with a fixed point principle to prove the unique solvability of a history-dependent DVI, and then they used the abstract frameworks to study a history-dependent frictional viscoelastic contact problem with a generalized Signorini contact condition. For more details on these topics, the reader is welcome to refer to [5–18] and references therein.

* **Corresponding author: Guo-ji Tang**, Department of Mathematics and Physics, Guangxi University for Nationalities, Nanning 530006, Guangxi Province, P. R. China, e-mail: guojvtang@126.com

Jinxia Cen: Department of Mathematics and Physics, Guangxi University for Nationalities, Nanning 530006, Guangxi Province, P. R. China; Department of Mathematics, Southwest Petroleum University, Chengdu 610500, P. R. China, e-mail: jinxiacen@163.com

Chao Min: Department of Mathematics, Southwest Petroleum University, Chengdu, Sichuan 610500, P. R. China, e-mail: minchaosc@126.com

Van Thien Nguyen: Department of Mathematics, FPT University, Education Zone, Hoa Lac High Tech Park, Km29 Thang Long Highway, Thach That Ward, Hanoi, Vietnam, e-mail: Thiennv15@fe.edu.vn

More recently, Liu et al. [19] initially introduced the notion of differential hemivariational inequalities (DHVIs), which is a generalization of variational inequalities. After that, Migórski and Zeng [20] proposed a temporally semi-discrete algorithm based on the backward Euler difference scheme together with a feedback iterative method to explore a DHVI, which is formulated by a parabolic hemivariational inequality and a nonlinear evolution equation in the framework of an evolution triple of spaces; Li and Liu [21] considered a sensitivity analysis of optimal control problems for a class of systems governed by DHVIs in Banach spaces; by applying the theory of measure of noncompactness, a fixed point theorem of a condensing multivalued map and theory of fractional calculus, Jiang et al. [22] explored an impressive existence result of the mild solutions of a global attractor for the semiflow governed by a fractional DHVI in Banach spaces. For other results on DHVIs, the reader may refer to [23–25] and references therein.

The current paper represents a continuation of [10]. In fact, the paper [10] was devoted to discuss the well-posedness and the generalized well-posedness of a differential mixed quasi-variational inequality and to provide criteria of well-posedness and well-posedness in the generalized sense of the inequality. However, the abstract frameworks of the paper [10] cannot be applied for solving the problems or phenomena described by nonconvex superpotential functions, which are locally Lipschitz. To overcome this flaw, in this paper, we are interested in studying a new kind of differential quasi-variational-hemivariational inequality (DQHVI).

Let X be a Hilbert space whose norm and scalar product are $\|\cdot\|_X$ and $\langle \cdot, \cdot \rangle_X$, respectively. In what follows, the norm convergence is denoted by “ \rightarrow ” and the weak convergence by “ \rightharpoonup ”. Let $0 < T < \infty$ and $I := [0, T]$. Note that the space $L^2(I; X)$ is endowed with the scalar product

$$\langle u_1, u_2 \rangle_{L^2(I; X)} := \int_0^T \langle u_1(t), u_2(t) \rangle_X dt \quad \text{for all } u_1, u_2 \in L^2(I; X),$$

to be a Hilbert space. Set $W(X) := \{x \in L^2(I; X) \mid \dot{x} \in L^2(I; X)\}$, where \dot{x} stands for the generalized derivative of x , namely,

$$\int_0^T \dot{x}(t)\phi(t)dt = - \int_0^T x(t)\dot{\phi}(t)dt \quad \text{for all } \phi \in C_0^\infty(I; X).$$

Indeed, it is not difficult to prove that $W(X)$ is a Hilbert space with the scalar product

$$\langle x_1, x_2 \rangle_{W(X)} := \langle x_1, x_2 \rangle_{L^2(I; X)} + \langle \dot{x}_1, \dot{x}_2 \rangle_{L^2(I; X)} \quad \text{for all } x_1, x_2 \in W(X),$$

and it is densely and continuously embedded in $C(I; X)$.

Given the Hilbert spaces X and V , let K be a fixed nonempty, closed, and convex subset of V . Let $\psi: I \times X \times V \rightrightarrows X$ be a set-valued mapping. In what follows, we denote the set-valued Nemytskii operator $\Phi: L^2(I; X) \times L^2(I; X) \rightrightarrows L^2(I; X)$ of ψ by

$$\Phi(x, u)(t) := \psi(t, x(t), u(t)) \quad \text{for all } t \in I.$$

Set $\bar{K} := \{u \in L^2(I; V): u(t) \in K, \text{ for a.e. } t \in I\}$. We also consider a set-valued mapping $S: \bar{K} \rightrightarrows \bar{K}$, which is specialized in Section 3. Let $G: L^2(I; X) \times L^2(I; V) \rightarrow L^2(I; V)$ be such that

$$G(x, u)(t) := g(t, x(t), u(t)) \quad \text{for all } t \in I,$$

where $g: I \times X \times V \rightarrow V$ is a given function. Assume that $\Gamma: X \times X \rightarrow X$, $J: L^2(I; V) \rightarrow \mathbb{R}$ is a locally Lipschitz function, and $\phi: L^2(I; V) \rightarrow \mathbb{R}$ is a convex functional, the current paper is devoted to study the following DQHVI

$$\begin{cases} \dot{x} \in \Phi(x, u), \\ u \in \text{SOL}(S(\cdot), G(x, \cdot), J, \phi), \\ \Gamma(x(0), x(T)) = 0, \end{cases} \tag{1.1}$$

where $\text{SOL}(S(\cdot), G(x, \cdot), J, \phi)$ stands for the set of solutions to the quasi-variational-hemivariational inequality: find $u \in S(u)$ such that

$$\langle G(x, u), v - u \rangle_{L^2(I; X)} + J^0(u; v - u) + \phi(v) - \phi(u) \geq 0 \quad \text{for all } v \in S(u),$$

where $J^0(u; v)$ denotes the generalized directional derivative of J at u in the direction v . Indeed, when $J = 0$, then our problem (1.1) reduces the one considered by Liu et al. [10].

The rest of the paper is organized as follows. In Section 2, we survey preliminary material needed in the sequel and introduce the concepts of well-posedness and of well-posedness in the generalized sense for problem (1.1). Section 3 is concerned with the study of the relation between metric characterizations and well-posedness of DQHVI.

2 Notation and preliminary results

In this section, we briefly review basic notation and some results which are needed in the sequel. For more details, we refer to monographs [26–28].

Let V be a Banach space. Throughout the paper, the Clarke generalized directional derivative of a locally Lipschitz function $h: V \rightarrow \mathbb{R}$ at $u \in V$ in the direction $v \in V$ is defined by

$$h^0(u; v) = \limsup_{\lambda \rightarrow 0^+, w \rightarrow u} \frac{h(w + \lambda v) - h(w)}{\lambda}.$$

However, the generalized Clarke subdifferential of h at $u \in V$ is defined by

$$\partial h(u) := \{u^* \in V^* \mid h^0(u; v) \geq \langle u^*, v \rangle_V \quad \text{for all } v \in V\}.$$

The next proposition provides basic properties of the generalized directional derivative and the generalized gradient, see, for example, [26, 29].

Proposition 2.1. *Let V be a Banach space. If $h: U \rightarrow \mathbb{R}$ is a locally Lipschitz function on a subset U of V , then*

- (i) *for every $u \in U$, the set $\partial h(u)$ is a nonempty, convex, and weakly* compact subset of V^* . More precisely, $\partial h(u)$ is bounded by the Lipschitz constant $K_u > 0$ of h near u ;*
- (ii) *the graph of ∂h is closed in $V \times (w^* - V^*)$ topology, namely, if $\{u_k\} \subset U$ and $\{\zeta_k\} \subset V^*$ are sequences such that $\zeta_k \in \partial h(u_k)$ and $u_k \rightarrow u$ in V , $\zeta_k \rightarrow \zeta$ weakly* in V^* , then we have $\zeta \in \partial h(u)$ (where $w^* - V^*$ denotes the space V^* equipped with weak* topology);*
- (iii) *the set-valued operator $U \ni u \mapsto \partial h(u) \subseteq V^*$ is upper semicontinuous from U into $w^* - V^*$;*
- (iv) *for each $v \in V$, there exists $z_v \in \partial h(u)$ such that*

$$h^0(u; v) = \max \{ \langle z, v \rangle_V, |z \in \partial h(u) \} = \langle z_v, v \rangle_V;$$

- (v) *the function $U \ni v \mapsto h^0(u; v) \in \mathbb{R}$ is finite, positively homogeneous, and subadditive on U , and satisfies $|h^0(u; v)| \leq K_u \|v\|_V$;*
- (vi) *$h^0(u; v)$ as a function of (u, v) is upper semicontinuous, and as a function of v alone is Lipschitz of rank K_u on U ;*
- (vii) *$h^0(u; -v) = (-h)^0(u; v)$ for all $u, v \in U$.*

Definition 2.2. Let V be a Banach space and $A, B \subset V$ be nonempty.

- (i) The measure of noncompactness for A is defined by

$$\mu(A) := \inf \left\{ \varepsilon > 0 \mid A = \bigcup_{i=1}^n A_i, \text{diam}(A_i) < \varepsilon, \quad i = 1, \dots, n \right\},$$

where $\text{diam}(A_i)$ denotes the diameter of the set A_i ;

(ii) The Hausdorff distance between subsets A and B is defined by

$$\mathcal{H}(A, B) := \max \{e(A, B), e(B, A)\},$$

where $e(A, B) := \sup_{a \in A} d(a, B)$ with $d(a, B) := \inf_{b \in B} \|a - b\|_V$.

Let $\{A_n\}$ be a sequence of nonempty subsets of V . We say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \rightarrow 0$, as $n \rightarrow \infty$. Moreover, we review the definitions of Painlevé-Kuratowski limits.

Definition 2.3. The Painlevé-Kuratowski strong limit inferior and (sequential) weak limit superior of a sequence $\{A_n\} \subset V$ are defined by

$$\begin{aligned} s\text{-lim inf } A_n &:= \{x \in V \mid \exists x_n \in A_n, n \in \mathbb{N}, \text{ with } x_n \rightarrow x \text{ in } V\}, \\ w\text{-lim sup } A_n &:= \{x \in V \mid \exists n_k \uparrow + \infty, \exists x_{n_k} \in A_{n_k}, k \in \mathbb{N}, \text{ with } x_{n_k} \rightharpoonup x \text{ in } V\}. \end{aligned}$$

Definition 2.4. Let V and Y be Banach spaces and $A: V \rightrightarrows Y$ be a set-valued mapping. A is called

- (i) (s,w) -closed, if $w\text{-lim sup } A(x_n) \subseteq A(x)$ as $x_n \rightarrow x$ in V ;
- (ii) (s,s) -closed, if $s\text{-lim sup } A(x_n) \subseteq A(x)$ as $x_n \rightarrow x$ in V ;
- (iii) (s,s) -lower-semicontinuous, if $A(x) \subseteq s\text{-lim inf } A(x_n)$ as $x_n \rightarrow x$ in V ;
- (iv) (s,w) -subcontinuous, if for every sequence $\{x_n\}$ strongly converging in V , every sequence $\{y_n\} \subset Y$ with $y_n \in A(x_n)$ has a weakly convergent subsequence in Y .

Obviously, from [30, Theorem 1.1.4], we can see that a set-valued mapping being (s,w) -upper semi-continuous and with closed values is sequentially (s,w) -closed as well.

In what follows, for each $\varepsilon > 0$, we set

$$\begin{aligned} \Omega(\varepsilon) &= \{(x, u) \in W(X) \times L^2(I; V) \mid d_{L^2(I; X)}(\dot{x}, \Phi(x, u)) \leq \varepsilon, d_{L^2(I; V)}(u, S(u)) \leq \varepsilon, \\ &\quad \langle G(x, u), v - u \rangle_{L^2(I; V)} + J^0(u; v - u) + \phi(v) + \phi(u) \geq -\varepsilon \|v - u\|_{L^2(I; V)} \\ &\quad \text{for all } v \in S(u), \text{ and } \|\Gamma(x(0), x(T))\|_X \leq \varepsilon\}. \end{aligned} \tag{2.1}$$

We end the section by introducing the approximating sequences, the well-posedness and the generalized well-posedness of DQHVI.

Definition 2.5. A sequence $\{(x_n, u_n)\}$ in $W(X) \times L^2(I; V)$ is called an approximating sequence for DQHVI, if there exists a sequence $\varepsilon_n \rightarrow 0^+$ as $n \rightarrow \infty$ such that for all $v \in S(u_n)$

$$\begin{cases} d_{L^2(I; X)}(\dot{x}_n, \Phi(x_n, u_n)) \leq \varepsilon_n, \\ d_{L^2(I; V)}(u_n, S(u_n)) \leq \varepsilon_n, \\ \langle G(x_n, u_n), v - u_n \rangle_{L^2(I; V)} + J^0(u_n; v - u_n) + \phi(v) + \phi(u_n) \geq -\varepsilon_n \|v - u_n\|_{L^2(I; V)}, \\ \|\Gamma(x_n(0), x_n(T))\|_X \leq \varepsilon_n. \end{cases} \tag{2.2}$$

Definition 2.6. Problem DQHVI is said to be strongly well-posed, if it has a unique solution (x_0, u_0) and every approximating sequence $\{(x_n, u_n)\}$ strongly converges to (x_0, u_0) .

Definition 2.7. Problem DQHVI is said to be strongly well-posed in the generalized sense, if the solution set Σ of DQHVI is nonempty and every approximating sequence $\{(x_n, u_n)\}$ has a subsequence which strongly converges to some point of Σ .

3 Main results

In this section, we deliver the main results concerning the essential relation between metric characterizations and the well-posedness of DQHVI, and the criteria of well-posedness and well-posedness in the generalized sense of DQHVI. Also, we prove that the set of solutions to DQHVI is compact, when DQHVI is well-posed in the generalized sense.

To do so, we impose the following assumptions.

- (A₁) $\Phi: W(X) \times L^2(I; V) \rightrightarrows L^2(I; X)$ is (s,w) -closed and sequentially (s,w) -subcontinuous;
- (A₂) $S: \bar{K} \rightrightarrows \bar{K}$ is (s,w) -closed, (s,s) -lower semicontinuous, and (s,w) -subcontinuous with convex values;
- (A₃) $J: L^2(I; V) \rightarrow \mathbb{R}$ is a locally Lipschitz function;
- (A₄) $\phi: \bar{K} \rightarrow \mathbb{R}$ is a convex and continuous function;
- (A₅) $G: L^2(I; X) \times L^2(I; V) \rightarrow L^2(I; V)$ is upper semicontinuous, that is, $u_n \rightarrow u_0$, $x_n \rightarrow x_0$, and $v_n \rightarrow v_0$ imply

$$\limsup_{n \rightarrow \infty} \langle G(x_n, u_n), v_n \rangle_{L^2(I; V)} \leq \langle G(x_0, u_0), v_0 \rangle_{L^2(I; V)};$$

- (A₆) $\Gamma: X \times X \rightarrow X$ is a continuous function.

Remark 3.1. Actually, assumption (A₂) indicates that S has closed and convex values in \bar{K} .

Besides, we recall the concept of relaxed α -monotonicity for a single-valued mapping, see [10].

Definition 3.2. Given $\alpha: L^2(I; V) \rightarrow \mathbb{R}$, the mapping $G: \bar{K} \rightarrow L^2(I; V)$ is said to be relaxed α -monotone if

$$\langle G(v) - G(u), v - u \rangle_{L^2(I; V)} \geq \alpha(v - u) \quad \text{for all } v, u \in \bar{K}.$$

Remark 3.3. However, it is obvious that

- (i) if $\alpha(u) = 0$, then G is monotone;
- (ii) if $\alpha(u) = m_G \|u\|^2$ with $m_G > 0$, then G is strongly monotone;
- (iii) if $\alpha(u) = -m_G \|u\|^2$ with $m_G > 0$, then G is relaxed monotone.

Furthermore, we suppose that the following conditions are satisfied.

- (B₁) $\Phi: W(X) \times L^2(I; V) \rightrightarrows L^2(I; X)$ is (s,s) -closed;
- (B₂) $\phi: \bar{K} \rightarrow \mathbb{R}$ is a convex and lower semicontinuous function;
- (B₃) For every $x \in W(X)$, $G(x, \cdot): L^2(I; V) \rightarrow L^2(I; V)$ is upper semicontinuous, relaxed α -monotone with $\alpha: L^2(I; V) \rightarrow \mathbb{R}$ such that $\limsup_{n \rightarrow \infty} \alpha(u_n) \geq \alpha(u)$, where $u_n \rightarrow u$, and $\lim_{t \downarrow 0} \frac{\alpha(tv)}{t} = 0$ for all $v \in L^2(I; V)$, and for every $u \in \bar{K}$, if $x_n \rightarrow x_0$ and $v_n \rightarrow v_0$, one has

$$\limsup_{n \rightarrow \infty} \langle G(x_n, u), v_n \rangle_{L^2(I; V)} \leq \langle G(x_0, u), v_0 \rangle_{L^2(I; V)}.$$

Remark 3.4. From the above two groups of conditions, we draw some conclusions:

- (i) condition (B₁) is weaker than condition (A₁). Indeed, we make use of it in Theorems 3.6 and 3.7;
- (ii) condition (B₂) is weaker than condition (A₃). We make use of it in Theorem 3.7;
- (iii) condition (B₄) is weaker than condition (A₅).

3.1 Characterizations of well-posedness for DQHVI

In this section, we are interesting to establish the metric characterizations of well-posedness for problem (1.1).

Theorem 3.5. *Let $S: \bar{K} \rightrightarrows \bar{K}$ and $\Phi: W(X) \times L^2(I; V) \rightrightarrows L^2(I; X)$ be set-valued maps. Then, DQHVI is strongly well-posed, if and only if the solution set Σ of DQHVI is nonempty and*

$$\lim_{\varepsilon \rightarrow 0} \text{diam}(\Omega(\varepsilon)) = 0, \quad (3.1)$$

where $\Omega(\varepsilon)$ is given in (2.1).

Proof. Suppose that DQHVI is strongly well-posed. Then, DQHVI has a unique solution (x_0, u_0) in $W(X) \times L^2(I; V)$, this means $\Sigma = \{(x_0, u_0)\}$. We now prove that (3.1) holds. Arguing by contradiction, we assume that $\text{diam}(\Omega(\varepsilon))$ does not tend to 0 as $\varepsilon \rightarrow 0$. Therefore, we are able to find a constant $\beta > 0$ and a positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0^+$ such that

$$\|(x_n^{(1)}, u_n^{(1)}) - (x_n^{(2)}, u_n^{(2)})\|_{W(X) \times L^2(I; V)} > \beta \quad \text{for all } n \in \mathbb{N} \quad (3.2)$$

with $(x_n^{(1)}, u_n^{(1)}), (x_n^{(2)}, u_n^{(2)}) \in \Omega(\varepsilon_n)$. Owing to the strong well-posedness of DQHVI, it finds

$$\lim_{n \rightarrow \infty} (x_n^{(1)}, u_n^{(1)}) = \lim_{n \rightarrow \infty} (x_n^{(2)}, u_n^{(2)}) = (x_0, u_0) \text{ in } W(X) \times L^2(I; V). \quad (3.3)$$

Combining (3.2) and (3.3) deduces

$$\begin{aligned} 0 < \beta < \|(x_n^{(1)}, u_n^{(1)}) - (x_n^{(2)}, u_n^{(2)})\|_{W(X) \times L^2(I; V)} \\ &\leq \|(x_n^{(1)}, u_n^{(1)}) - (x_0, u_0)\|_{W(X) \times L^2(I; V)} + \|(x_n^{(2)}, u_n^{(2)}) - (x_0, u_0)\|_{W(X) \times L^2(I; V)} \rightarrow 0. \end{aligned}$$

This generates a contradiction, hence, (3.1) is valid.

Conversely, suppose that $\Sigma \neq \emptyset$ and (3.1) is available. Then, we conclude that Σ is a singleton point set, i.e., $\Sigma = \{(x_0, u_0)\}$. Let $\{(x_n, u_n)\} \subseteq W(X) \times L^2(I; V)$ be an approximating sequence of problem DQHVI. Hence, there exists a sequence $\varepsilon_n \rightarrow 0^+$ as $n \rightarrow \infty$ such that for all $v \in S(u_n)$

$$\begin{cases} d_{L^2(I; X)}(\dot{x}_n, \Phi(x_n, u_n)) \leq \varepsilon_n, \\ d_{L^2(I; V)}(u_n, S(u_n)) \leq \varepsilon_n, \\ \langle G(x_n, u_n), v - u_n \rangle_{L^2(I; V)} + J^0(u_n; v - u_n) + \phi(v) + \phi(u_n) \geq -\varepsilon_n \|v - u_n\|_{L^2(I; V)}, \\ \|\Gamma(x_n(0), x_n(T))\|_X \leq \varepsilon_n. \end{cases}$$

This reveals $(x_n, u_n) \in \Omega(\varepsilon_n)$ for all $n \in \mathbb{N}$. Using the fact $(x_0, u_0) \in \Omega(\varepsilon_n)$ for all $n \in \mathbb{N}$ and (3.1) gives

$$\lim_{n \rightarrow \infty} \|(x_n, u_n) - (x_0, u_0)\|_{W(X) \times L^2(I; V)} \leq \lim_{n \rightarrow \infty} \text{diam}(\Omega(\varepsilon_n)) = 0.$$

This means that $\{(x_n, u_n)\}$ strongly converges to (x_0, u_0) in $W(X) \times L^2(I; V)$, as $n \rightarrow \infty$. Therefore, DQHVI is strongly well-posed. \square

Observing the proof of Theorem 3.5, we could see that the assumption $\Sigma \neq \emptyset$ plays a significant role. However, by invoking other conditions, we can remove this condition.

Theorem 3.6. *Assume that $(B_1), (A_2) - (A_6)$ are satisfied. Then DQHVI is strongly well-posed, if and only if it holds*

$$\Omega(\varepsilon) \neq \emptyset \quad \text{for all } \varepsilon > 0 \text{ and } \lim_{\varepsilon \rightarrow 0} \text{diam}(\Omega(\varepsilon)) = 0. \quad (3.4)$$

Proof. Indeed, the necessity part is a direct consequence of Theorem 3.5. Therefore, it is enough to verify the sufficiency.

Suppose (3.4) holds. For any positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$, we could find a sequence $\{(x_n, u_n)\}$ such that $(x_n, u_n) \in \Omega(\varepsilon_n)$ for all $n \in \mathbb{N}$. Note that $\Omega(\varepsilon) \subseteq \Omega(\delta)$ for $\varepsilon \leq \delta$, it follows from (3.4) that $\{(x_n, u_n)\}$ is a Cauchy sequence. Therefore, $\{(x_n, u_n)\}$ converges strongly to some $(x_0, u_0) \in W(X) \times L^2(I; V)$, as $n \rightarrow \infty$. Furthermore, we show the following claims.

Claim 1. $u_0 \in S(u_0)$ and $\Gamma(x_0(0), x_0(T)) = 0$.

Recall that $x_n \rightarrow x_0$ in $W(X)$ and the embedding from $W(X)$ to $C(I; X)$ is continuous, it yields $x_n(0) \rightarrow x_0(0)$ and $x_n(T) \rightarrow x_0(T)$ in X , as $n \rightarrow \infty$. The latter combined with the continuity of Γ (see (A_6)) and the inequality $\|\Gamma(x_n(0), x_n(T))\|_X \leq \varepsilon_n$ indicates

$$\Gamma(x_0(0), x_0(T)) = \lim_{n \rightarrow \infty} \Gamma(x_n(0), x_n(T)) = 0.$$

On the other side, we assert that

$$d_{L^2(I;V)}(u_0, S(u_0)) \leq \liminf_{n \rightarrow \infty} d_{L^2(I;V)}(u_n, S(u_n)). \quad (3.5)$$

If the above inequality is not true, then we could find a constant $\gamma > 0$ such that

$$\liminf_{n \rightarrow \infty} d_{L^2(I;V)}(u_n, S(u_n)) < \gamma < d_{L^2(I;V)}(u_0, S(u_0)). \quad (3.6)$$

Hence, along subsequence $\{u_{n_k}\}$ and $\{p_{n_k}\}$ with $p_{n_k} \in S(u_{n_k})$, it holds

$$\|u_{n_k} - p_{n_k}\|_{L^2(I;V)} < \gamma \quad \text{for all } k \in \mathbb{N}.$$

However, the reflexivity of $L^2(I; V)$ and the boundedness of p_{n_k} allow us to assume that $p_{n_k} \rightharpoonup p_0$ in $L^2(I; V)$. But, the (s, w) -closedness of S (see condition (A_2)) guarantees $p_0 \in S(u_0)$. Following the above analysis and (3.6), it leads to the contradiction

$$\gamma < d(u_0, S(u_0)) \leq \|u_0 - p_0\|_{L^2(I;V)} \leq \liminf_{k \rightarrow \infty} \|u_{n_k} - p_{n_k}\|_{L^2(I;V)} \leq \gamma.$$

So, (3.5) holds. Using (3.5) and the fact $d_{L^2(I;V)}(u_n, S(u_n)) \leq \varepsilon_n$, we verify Claim 1.

Claim 2. $u_0 \in \text{SOL}(S(\cdot), G(x, \cdot), J, \phi)$.

Let $v \in S(u_0)$ be fixed. Invoking the (s, s) -lower semicontinuity of S (see (A_2)), there exists a sequence $\{v_n\} \subset L^2(I; V)$ with $v_n \in S(u_n)$ such that $v_n \rightarrow v$ in $L^2(I; V)$. Taking into account (A_3) , (A_4) , and (A_5) , it yields

$$\begin{aligned} & \langle G(x_0, u_0), v - u_0 \rangle_{L^2(I;V)} + J^0(u_0; v - u_0) + \phi(v) - \phi(u_0) \\ & \geq \limsup_{n \rightarrow \infty} \left[\langle G(x_n, u_n), v_n - u_n \rangle_{L^2(I;V)} + J^0(u_n; v_n - u_n) + \phi(v_n) - \phi(u_n) \right] \\ & \geq \limsup_{n \rightarrow \infty} [-\varepsilon_n \|v_n - u_n\|_{L^2(I;V)}] = 0, \end{aligned}$$

for all $v \in S(u_0)$. This implies $u_0 \in \text{SOL}(S(\cdot), G(x, \cdot), J, \phi)$.

Claim 3. $\dot{x}_0 \in \Phi(x_0, u_0)$.

By virtue of $d_{L^2(I;X)}(\dot{x}_n, \Phi(x_n, u_n)) \leq \varepsilon_n$, there exists $p_n \in \Phi(x_n, u_n)$ satisfying

$$\|\dot{x}_n - p_n\|_{L^2(I;X)} \leq 2\varepsilon_n.$$

Employing the hypothesis (B_1) and the convergence $x_n \rightarrow x_0$ in $W(X)$, it gives $\dot{x}_n \rightarrow \dot{x}_0$ in $L^2(I; X)$ and $\dot{x}_0 \in \Phi(x_0, u_0)$.

Claim 4. Σ contains the only one element (x_0, u_0) .

Let $(x_0, u_0), (x_1, u_1) \in \Sigma$. Hence, $(x_0, u_0), (x_1, u_1) \in \Omega(\varepsilon)$ for all $\varepsilon > 0$. Therefore, (3.4) gives

$$\|(x_0, u_0) - (x_1, u_1)\|_{W(X) \times L^2(I;V)} \leq \text{diam}(\Omega(\varepsilon)) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Consequently, Σ is a singleton set. □

Additionally, if the set-valued mapping is specialized by $S(u) \equiv \bar{K}$, then we have the following results.

Theorem 3.7. *Assume that conditions (B_1) , (B_2) , (A_3) , (A_5) , and (A_6) hold or conditions (B_1) – (B_3) , (A_3) , and (A_6) hold. Then DQHVI is strongly well-posed, if and only if*

$$\Omega(\varepsilon) \neq \emptyset, \text{ for all } \varepsilon > 0, \text{ and } \lim_{\varepsilon \rightarrow 0} \text{diam}(\Omega(\varepsilon)) = 0.$$

Proof. The necessity part could be obtained directly by employing the same arguments with the proof of Theorem 3.6.

We now prove the sufficiency part. Indeed, when assumptions (B_1) , (B_2) , (A_3) , (A_5) , and (A_6) are satisfied, obviously, the desired conclusion is a direct consequence of Theorem 3.6.

In contrast, we assume that conditions (B_1) – (B_3) , (A_3) , and (A_6) hold. Let $\{(x_n, u_n)\}$ be an approximating sequence to DQHVI. Note that $\text{diam}(\Omega(\varepsilon))$ tends to 0, as $\varepsilon \rightarrow 0$, so, $\{(x_n, u_n)\}$ is a Cauchy sequence. We may suppose that $\{(x_n, u_n)\}$ converges strongly to some point $(x_0, u_0) \in W(X) \times \bar{K} \subseteq W(X) \times L^2(I; V)$. The latter together with the continuity of Γ deduces

$$\Gamma(x_0(0), x_0(T)) = 0.$$

Since $\{(x_n, u_n)\}$ is an approximating sequence of problem DQHVI,

$$\langle G(x_n, u_n), v - u_n \rangle_{L^2(I; V)} + J^0(u_n; v - u_n) + \phi(v) - \phi(u_n) \geq -\varepsilon_n \|v - u_n\|_{L^2(I; V)}$$

for all $v \in S(u_n)$. Let $v \in S(u_0)$ be fixed. Invoking the (s, s) -lower semicontinuity of S (see (A_2)), there exists a sequence $\{v_n\} \subset L^2(I; V)$ with $v_n \in S(u_n)$ such that $v_n \rightarrow v$ in $L^2(I; V)$. Using conditions (B_2) , (B_3) , (A_3) , (A_6) and relaxed α -monotonicity of G , we obtain

$$\begin{aligned} & \langle G(x_0, v), v - u_0 \rangle_{L^2(I; V)} + J^0(u_0; v - u_0) + \phi(v) - \phi(u_0) \\ & \geq \limsup_{n \rightarrow \infty} [\langle G(x_n, v_n), v_n - u_n \rangle_{L^2(I; V)} + J^0(u_n; v_n - u_n) + \phi(v_n) - \phi(u_n)] \\ & \geq \limsup_{n \rightarrow \infty} [\langle G(x_n, u_n), v_n - u_n \rangle_{L^2(I; V)} + J^0(u_n; v_n - u_n) + \phi(v_n) - \phi(u_n) + \alpha(v_n - u_n)] \\ & \geq \limsup_{n \rightarrow \infty} [\alpha(v_n - u_n) - \varepsilon_n \|v_n - u_n\|_{L^2(I; V)}] \geq \alpha(v - u_0) \end{aligned}$$

for all $v \in S(u_0)$. Because $S(u_0) \subset L^2(I; V)$ is convex. For each $v \in S(u_0)$ and $\lambda \in [0, 1]$, we insert $v_\lambda := \lambda v + (1 - \lambda)u_0 \in S(u_0)$ into the above inequality to yield

$$\langle G(x_0, v_\lambda), v_\lambda - u_0 \rangle_{L^2(I; V)} + J^0(u_0; v_\lambda - u_0) + \phi(v_\lambda) - \phi(u_0) \geq \alpha(v_\lambda - u_0).$$

However, the positive homogeneity of $v \mapsto J^0(u; v)$ and convexity of ϕ guarantee

$$\langle G(x_0, v_\lambda), v - u_0 \rangle_{L^2(I; V)} + J^0(u_0; v - u_0) + \phi(v) - \phi(u_0) \geq \frac{\alpha(\lambda(v - u_0))}{\lambda}$$

for all $v \in S(u_0)$. Letting $\lambda \rightarrow 0$, it finds

$$\langle G(x_0, u_0), v - u_0 \rangle_{L^2(I; V)} + J^0(u_0; v - u_0) + \phi(v) - \phi(u_0) \geq 0$$

for all $v \in S(u_0)$, where we have used the upper hemicontinuity of $v \mapsto G(x, v)$ and hypothesis (B_3) . Recall that $d_{L^2(I; X)}(\dot{x}_n, \Phi(x_n, u_n)) \leq \varepsilon_n$, we are able to find $p_n \in \Phi(x_n, u_n)$ such that

$$\|\dot{x}_n - p_n\|_{L^2(I; V)} \leq 2\varepsilon_n.$$

Taking into account that Φ is (s, s) -closed as known from (B_1) and that $\dot{x}_n \rightarrow \dot{x}_0$, we conclude $\dot{p}_n \rightarrow \dot{x}_0 \in \Phi(x_0, u_0)$, namely, (x_0, u_0) is a solution of DQHVI.

It remains to illustrate the uniqueness of solution to DQHVI. Let (x^*, u^*) , (x_0, u_0) be two solutions to DQHVI, so, (x^*, u^*) , $(x_0, u_0) \in \Omega(\varepsilon)$, for all $\varepsilon > 0$. But, the estimate

$$\|(x_0, u_0) - (x^*, u^*)\|_{W(X) \times L^2(I; V)} \leq \text{diam}(\Omega(\varepsilon)) \rightarrow 0, \text{ as } \varepsilon \rightarrow 0,$$

points out $(x^*, u^*) = (x_0, u_0)$. □

3.2 Characterizations of well-posedness in the generalized sense of DQHVI

In this section, we explore the metric characterizations of well-posedness in the generalized sense for DQHVI. Besides, the compactness of solution set of DQHVI is proved, when DQHVI is well-posed in the generalized sense. First, we deliver the following important result that for each $\varepsilon > 0$, the set $\Omega(\varepsilon)$ introduced in (2.1) is closed.

Lemma 3.8. *Assume that conditions (A_1) – (A_6) hold. Then, for every $\varepsilon > 0$, $\Omega(\varepsilon)$ is closed.*

Proof. Let $\{(x_n, u_n)\} \subset \Omega(\varepsilon)$ be a sequence such that $(x_n, u_n) \rightarrow (x_0, u_0)$ in $W(X) \times L^2(I; V)$, as $n \rightarrow \infty$. Hence, it holds for all $v \in S(u_n)$

$$\begin{cases} d_{L^2(I; X)}(\dot{x}_n, \Phi(x_n, u_n)) \leq \varepsilon, \\ d_{L^2(I; V)}(u_n, S(u_n)) \leq \varepsilon, \\ \langle G(x_n, u_n), v - u_n \rangle_{L^2(I; V)} + J^0(u_n; v - u_n) + \phi(v) + \phi(u_n) \geq -\varepsilon \|v - u_n\|_{L^2(I; V)}, \\ \|\Gamma(x_n(0), x_n(T))\|_X \leq \varepsilon. \end{cases}$$

The continuity of the embedding from $W(X)$ to $C(I; X)$ indicates $x_n \rightarrow x_0$ in $C(X)$, so $x_n(0) \rightarrow x_0(0)$ and $x_n(T) \rightarrow x_0(T)$ in X . Therefore, from the continuity of Γ , it has

$$\|\Gamma(x_0(0), x_0(T))\|_X \leq \varepsilon. \tag{3.7}$$

We now assert

$$d_{L^2(I; V)}(u_0, S(u_0)) \leq \liminf_{n \rightarrow \infty} d_{L^2(I; V)}(u_n, S(u_n)). \tag{3.8}$$

If (3.8) were not true, there would exist $\gamma > 0$ such that

$$\liminf_{n \rightarrow \infty} d_{L^2(I; V)}(u_n, S(u_n)) < \gamma < d_{L^2(I; V)}(u_0, S(u_0)). \tag{3.9}$$

Without loss of generality, we could find subsequences $\{u_{n_k}\}$ of $\{u_n\}$ and $p_{n_k} \in S(u_{n_k})$ satisfying

$$\|u_{n_k} - p_{n_k}\|_{L^2(I; V)} < \gamma \quad \text{for all } k \in \mathbb{N}.$$

In addition, we may suppose that $p_{n_k} \rightarrow p_0$ in $L^2(I; V)$ and $p_0 \in S(u_0)$ due to hypothesis (A_2) . Using (3.9), we reach the contradiction

$$\gamma < d_{L^2(I; V)}(u_0, S(u_0)) \leq \|u_0 - p_0\|_{L^2(I; V)} \leq \liminf_{k \rightarrow \infty} \|u_{n_k} - p_{n_k}\|_{L^2(I; V)} \leq \gamma.$$

Consequently, we can conclude via (3.8) that

$$d_{L^2(I; V)}(u_0, S(u_0)) \leq \varepsilon. \tag{3.10}$$

Similarly, by (A_1) , it yields

$$d_{L^2(I; V)}(\dot{x}_0, \Phi(x_0, u_0)) \leq \varepsilon. \tag{3.11}$$

Finally, for any $v \in S(u_0)$, by the (s, s) -lower semicontinuity of S in (A_2) , there exists a sequence $\{v_n\}$ with $v_n \in S(u_n)$ such that $v_n \rightarrow v$. It follows from (A_3) – (A_5) that

$$\begin{aligned} & \langle G(x_0, u_0), v - u_0 \rangle_{L^2(I; V)} + J^0(u_0; v - u_0) + \phi(v) + \phi(u_0) \\ & \geq \limsup_{n \rightarrow \infty} [\langle G(x_n, u_n), v_n - u_n \rangle_{L^2(I; V)} + J^0(u_n; v_n - u_n) + \phi(v_n) + \phi(u_n)] \geq -\varepsilon \|v - u_0\|_{L^2(I; V)}. \end{aligned} \tag{3.12}$$

Since $v \in S(u_0)$ is arbitrary, we can apply (3.7) and (3.10)–(3.12) to conclude $(x_0, u_0) \in \Omega(\varepsilon)$. □

Theorem 3.9. *Problem DQHVI is strongly well-posed in the generalized sense if and only if, the solution set Σ of DQHVI is nonempty, compact, and*

$$\lim_{\varepsilon \rightarrow 0^+} e(\Omega(\varepsilon), \Sigma) = 0. \quad (3.13)$$

Proof. Suppose that DQHVI is strongly well-posed in the generalized sense. Then, Σ is nonempty and $\Sigma \subset \Omega(\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. We now claim that Σ is compact. For all $\varepsilon > 0$, let $\{(x_n, u_n)\} \subset \Sigma \subset \Omega(\varepsilon)$ be an arbitrary sequence and $\{\varepsilon_n\}$ be such that $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$. So, $\{(x_n, u_n)\}$ with $(x_n, u_n) \in \Omega(\varepsilon_n)$ is also an approximating sequence for DQHVI. However, by virtue of Definition 2.7, we infer that $\{(x_n, u_n)\}$ has a subsequence converging to some point of Σ , which shows that Σ is compact.

Next, we demonstrate (3.13). Arguing by contradiction, we assume that for every sequence $\varepsilon_n \rightarrow 0^+$, there exist $\beta > 0$ and $(x_n^*, u_n^*) \subset \Omega(\varepsilon_n)$ such that

$$d_{W(X) \times L^2(I; V)}((x_n^*, u_n^*), \Sigma) > \beta \quad \text{for all } n \in \mathbb{N}.$$

Note that $\{(x_n^*, u_n^*)\}$ is an approximating sequence for DQHVI, from the generalized well-posedness, we are able to find a subsequence $\{(x_{n_k}^*, u_{n_k}^*)\}$ of $\{(x_n^*, u_n^*)\}$ strongly converging to some point of Σ . This means

$$0 < \beta < d_{W(X) \times L^2(I; V)}((x_{n_k}^*, u_{n_k}^*), \Sigma) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which generates a contradiction.

Conversely, assume that (3.13) holds. Let $\{(x_n, u_n)\} \subset W(X) \times L^2(I; V)$ be an approximating sequence of DQHVI, i.e., there exists $\varepsilon_n \rightarrow 0^+$ as $n \rightarrow \infty$ such that $(x_n, u_n) \in \Omega(\varepsilon_n)$ for all $n \in \mathbb{N}$. However, (3.13) ensures us to take a sequence $\{(x_n^*, u_n^*)\}$ in Σ such that

$$\|(x_n, u_n) - (x_n^*, u_n^*)\|_{W(X) \times L^2(I; V)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Invoking the compactness of Σ , we are able to find a subsequence $\{(x_{n_k}^*, u_{n_k}^*)\}$ of $\{(x_n^*, u_n^*)\}$ strongly converging to some point $(x_0, u_0) \in \Sigma$, which leads to

$$\begin{aligned} \|(x_{n_k}, u_{n_k}) - (x_0, u_0)\|_{W(X) \times L^2(I; V)} &\leq \|(x_{n_k}, u_{n_k}) - (x_{n_k}^*, u_{n_k}^*)\|_{W(X) \times L^2(I; V)} + \|(x_{n_k}^*, u_{n_k}^*) - (x_0, u_0)\|_{W(X) \times L^2(I; V)} \\ &\rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Therefore, DQHVI is strongly well-posed in the generalized sense. \square

In fact, we can observe that the compactness of Σ plays a central role for the proof of Theorem 3.9. However, under certain circumstances we can develop a different approach.

Theorem 3.10. *Assume that $(A_1) - (A_6)$ hold. If, in addition, $S: \bar{K} \rightrightarrows \bar{K}$ is a closed, convex set-valued mapping, then DQHVI is strongly well-posed in the generalized sense, if and only if*

$$\Omega(\varepsilon) \neq \emptyset, \quad \text{for all } \varepsilon > 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \mu(\Omega(\varepsilon)) = 0. \quad (3.14)$$

Proof. Suppose that DQHVI is well-posed in the generalized sense. Then, for every $\varepsilon > 0$ it is true $\Sigma \subset \Omega(\varepsilon) \neq \emptyset$. It follows from Theorem 3.9 that

$$\mathcal{H}(\Omega(\varepsilon), \Sigma) = \max \{e(\Omega(\varepsilon), \Sigma), e(\Sigma, \Omega(\varepsilon))\} = e(\Omega(\varepsilon), \Sigma) \quad (3.15)$$

for all $\varepsilon > 0$, and

$$\mu(\Sigma) = 0. \quad (3.16)$$

Combining (3.15) and (3.16) implies

$$\mu(\Omega(\varepsilon)) \leq 2\mathcal{H}(\Omega(\varepsilon), \Sigma) + \mu(\Sigma) = 2e(\Omega(\varepsilon), \Sigma). \quad (3.17)$$

The above inequality and (3.13) indicate $\lim_{\varepsilon \rightarrow 0} \mu(\Omega(\varepsilon)) = 0$.

Conversely, assume that (3.14) holds. However, Lemma 3.8 points out that for every $\varepsilon > 0$ the set $\Omega(\varepsilon)$ is closed. Set $\Omega = \bigcap_{\varepsilon>0} \Omega(\varepsilon)$. From the generalized Cantor theorem (see cf. [31]), we conclude $\lim_{\varepsilon \rightarrow 0} \mathcal{H}(\Omega(\varepsilon), \Sigma) = 0$ and $\Omega \neq \emptyset$. Next, we prove $\Omega = \Sigma$. Obviously, $\Sigma \subseteq \Omega$, therefore, it reminds us to illustrate that $\Omega \subseteq \Sigma$. For any $(x_0, u_0) \in \Omega$ and $\varepsilon > 0$ fixed, it has $d_{W(X) \times L^2(I;V)}((x_0, u_0), \Omega(\varepsilon)) = 0$. Besides, for each $n \in \mathbb{N}$, there exists $(x_n, u_n) \in \Omega(\varepsilon_n)$ such that

$$\|(x_0, u_0) - (x_n, u_n)\|_{W(X) \times L^2(I;V)} \leq \varepsilon_n$$

with $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$. This infers $x_n \rightarrow x_0$ in $W(X)$ and $u_n \rightarrow u_0$ in $L^2(I; V)$, as $n \rightarrow \infty$. As before we have done, it is not difficult to show that $x_n(0) \rightarrow x_0(0)$ and $x_n(T) \rightarrow x_0(T)$ in X , as $n \rightarrow \infty$. This couples with the continuity of Γ to get

$$\Gamma(x_0(0), x_0(T)) = \lim_{n \rightarrow \infty} \Gamma(x_n(0), x_n(T)) = 0. \tag{3.18}$$

By using the same arguments as in Theorem 3.6 and Remark 3.4, it concludes

$$d_{L^2(I;V)}(u_0, S(u_0)) \leq \liminf_{n \rightarrow \infty} d_{L^2(I;V)}(u_n, S(u_n)) \leq \lim_{n \rightarrow \infty} \varepsilon_n = 0,$$

and

$$d_{L^2(I;X)}(\dot{x}_0, \Phi(x_0, u_0)) \leq \liminf_{n \rightarrow \infty} d_{L^2(I;X)}(\dot{x}_n, \Phi(x_n, u_n)) \leq \lim_{n \rightarrow \infty} \varepsilon_n = 0.$$

The last two inequalities indicate

$$u_0 \in S(u_0) \text{ and } \dot{x}_0 \in \Phi(x_0, u_0). \tag{3.19}$$

Furthermore, we verify that u_0 is a solution of $\text{SOL}(S(\cdot), G(x_0, \cdot), J, \phi)$. Let $v \in S(u_0)$ be fixed. Recall that S is (s, s) -lower semicontinuous, there exists $v_n \in S(u_n)$ such that $v_n \rightarrow v$ as $n \rightarrow \infty$. Invoking conditions $(A_3) - (A_5)$, we have

$$\begin{aligned} & \langle G(x_0, u_0), v - u_0 \rangle_{L^2(I;V)} + J^0(u_0; v - u_0) + \phi(v) + \phi(u_0) \\ & \geq \limsup_{n \rightarrow \infty} [\langle G(x_n, u_n), v_n - u_n \rangle_{L^2(I;V)} + J^0(u_n; v_n - u_n) + \phi(v_n) + \phi(u_n)] \\ & \geq \limsup_{n \rightarrow \infty} [-\varepsilon_n \|v_n - u_n\|_{L^2(I;V)}] \geq 0. \end{aligned} \tag{3.20}$$

The arbitrariness of $v \in S(u_0)$ and the facts (3.18)–(3.20) guarantee $(x_0, u_0) \in \Sigma$, thus $\Sigma = \Omega$.

At this point, we know that $\lim_{\varepsilon \rightarrow 0} \mathcal{H}(\Omega(\varepsilon), \Sigma) = 0$ and $\lim_{\varepsilon \rightarrow 0} e(\Omega(\varepsilon), \Sigma) = 0$. Taking account of the compactness of Σ and Theorem 3.9, it concludes that DQHVI is strongly well-posed in the generalized sense. \square

Moreover, using the same arguments as Theorem 3.7 and Remark 3.4, we have the following result.

Theorem 3.11. *Assume that conditions (A_1) , (A_3) , (A_5) , (A_6) , and (B_2) hold or conditions (A_1) , (A_3) , (A_6) , (B_2) , and (B_3) hold. Then, DQHVI is strongly well-posed in the generalized sense, if and only if*

$$\Omega(\varepsilon) \neq \emptyset, \quad \text{for all } \varepsilon > 0 \text{ and } \lim_{\varepsilon \rightarrow 0} \mu(\Omega(\varepsilon)) = 0.$$

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References

[1] J. S. Pang and D. E. Stewart, *Differential variational inequalities*, Math. Program. **113** (2008), 345–424, DOI: 10.1007/s10107-006-0052-x.

- [2] X. Chen and Z. Wang, *Differential variational inequality approach to dynamic games with shared constraints*, Math. Program. **146** (2014), 379–408, DOI: 10.1007/s10107-013-0689-1.
- [3] X. Wang, G. J. Tang, X. S. Li, and N. J. Huang, *Differential quasi-variational inequalities in finite dimensional spaces*, Optimization **64** (2015), no. 4, 895–907, DOI: 10.1080/02331934.2013.836646.
- [4] S. Migórski, Z. H. Liu, and S. D. Zeng, *A class of history-dependent differential variational inequalities with application to contact problems*, Optimization **69** (2019), no. 4, 743–775, DOI: 10.1080/02331934.2019.1647539.
- [5] X. Chen and Z. Wang, *Convergence of regularized time-stepping methods for differential variational inequalities*, SIAM J. Optim. **23** (2013), no. 3, 1647–1671, DOI: 10.1137/120875223.
- [6] X. S. Li, N. J. Huang, and D. O'Regan, *Differential mixed variational inequalities in finite dimensional spaces*, Nonlinear Anal. TMA **72** (2010), no. 9–10, 3875–3886, DOI: 10.1016/j.na.2010.01.025.
- [7] X. S. Li, N. J. Huang, and D. O'Regan, *A class of impulsive differential variational inequalities in finite dimensional spaces*, J. Franklin. I **353** (2016), no. 13, 3151–3175, DOI: 10.1016/j.jfranklin.2016.06.011.
- [8] Z. H. Liu, D. Motreanu, and S. D. Zeng, *Nonlinear evolutionary systems driven by mixed variational inequalities and its applications*, Nonlinear Anal. RWA **42** (2018), 409–421, DOI: 10.1016/j.nonrwa.2018.01.008.
- [9] Z. H. Liu, S. Migórski, and S. D. Zeng, *Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces*, J. Differ. Equations **263** (2017), no. 7, 3989–4006, DOI: 10.1016/j.jde.2017.05.010.
- [10] Z. H. Liu, D. Motreanu, and S. D. Zeng, *On the well-posedness of differential mixed quasi-variational inequalities*, Topol. Method Nonl. Anal. **51** (2018), no. 1, 135–150, DOI: 10.12775/TMNA.2017.041.
- [11] Z. H. Liu, S. D. Zeng, and D. Motreanu, *Evolutionary problems driven by variational inequalities*, J. Differ. Equations **260** (2016), no. 9, 6787–6799, DOI: 10.1016/j.jde.2016.01.012.
- [12] X. J. Long and N. J. Huang, *Metric characterization of α -well-posedness for symmetric quasi-equilibrium problems*, J. Glob. Optim. **45** (2009), 459–471, DOI: 10.1007/s10898-008-9385-8.
- [13] N. V. Loi, *On two-parameter global bifurcation of periodic solutions to a class of differential variational inequalities*, Nonlinear Anal. TMA **122** (2015), 83–99, DOI: 10.1016/j.na.2015.03.019.
- [14] T. V. A. Nguyen and D. K. Tran, *On the differential variational inequalities of parabolic-elliptic type*, Math. Method. Appl. Sci. **40** (2017), no. 13, 4683–4695, DOI: 10.1002/mma.4334.
- [15] N. T. Van and T. D. Ke, *Asymptotic behavior of solutions to a class of differential variational inequalities*, Ann. Pol. Math. **114** (2015), 147–164, DOI: 10.4064/ap114-2-5.
- [16] X. Wang and N. J. Huang, *A class of differential vector variational inequalities in finite dimensional spaces*, J. Optim. Theory Appl. **162** (2014), 633–648, DOI: 10.1007/s10957-013-0311-y.
- [17] X. Wang, Y. W. Qi, C. Q. Tao, and Q. Wu, *Existence result for differential variational inequality with relaxing the convexity condition*, Appl. Math. Comput. **331** (2018), 297–306, DOI: 10.1016/j.amc.2018.03.004.
- [18] X. Wang, Y. W. Qi, C. Q. Tao, and N. J. Huang, *A class of differential fuzzy variational inequalities in finite-dimensional spaces*, Optim. Lett. **11** (2017), 1593–1607, DOI: 10.1007/s11590-016-1066-9.
- [19] Z. H. Liu, S. D. Zeng, and D. Motreanu, *Partial differential hemivariational inequalities*, Adv. Nonlinear Anal. **7** (2018), no. 4, 571–586, DOI: 10.1515/anona-2016-0102.
- [20] S. Migórski and S. D. Zeng, *A class of differential hemivariational inequalities in Banach spaces*, J. Glob. Optim. **72** (2018), 761–779, DOI: 10.1007/s10898-018-0667-5.
- [21] X. W. Li and Z. H. Liu, *Sensitivity analysis of optimal control problems described by differential hemivariational inequalities*, SIAM J. Control Optim. **56** (2018), no. 5, 3569–3597, DOI: 10.1137/17M1162275.
- [22] Y. R. Jiang, N. J. Huang, and Z. C. Wei, *Existence of a global attractor for fractional differential hemivariational inequalities*, Discrete Cont. Dyn. Sys. B **25** (2020), no. 4, 1193–1212, DOI: 10.3934/dcdsb.2019216.
- [23] S. Migórski, A. A. Khan, and S. D. Zeng, *Inverse problems for nonlinear quasi-hemivariational inequalities with application to mixed boundary value problems*, Inverse Probl. **36** (2019), no. 2, 024006, DOI: 10.1088/1361-6420/ab44d7.
- [24] S. D. Zeng, Z. H. Liu, and S. Migórski, *A class of fractional differential hemivariational inequalities with application to contact problem*, Z. Angew. Math. Phys. **69** (2018), 36, DOI: 10.1007/s00033-018-0929-6.
- [25] S. Migórski and S. D. Zeng, *Hyperbolic hemivariational inequalities controlled by evolution equations with application to adhesive contact model*, Nonlinear Anal. RWA **43** (2018), 121–143, DOI: 10.1016/j.nonrwa.2018.02.008.
- [26] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.
- [27] Z. Denkowski, S. Migórski, and N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
- [28] Z. Denkowski, S. Migórski, and N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
- [29] S. Migórski, A. Ochal, and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics 26, Springer, New York, 2013.
- [30] M. Kamemskii, V. Obukhovskii, and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Space*, Water de Gruyter, Berlin, 2001.
- [31] C. Kuratowski, *Topology vol. I and II*, Academic Press, New York, 1966.