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ON THE WELL-POSEDNESS OF THE KIRCHHOFF STRING

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ABSTRACT. Let us consider the Cauchy problem for the quasilinear hyperbolic integro-differential equation

$$\begin{aligned} &u_{tt} - m\left(\int_\Omega |\bigtriangledown_x u|^2\,dx\right)\triangle_x u = f(x,t) \qquad (x \in \Omega,\ t > 0),\\ &u(\cdot,t)_{|\partial\Omega} = 0 \qquad\qquad (t \geq 0), \end{aligned}$$

where Ω is an open subset of $\mathbb{R}^n$ and m is a positive function of one real variable which is continuously differentiable. We prove the well-posedness in the Hadamard sense (existence, uniqueness and continuous dependence of the local solution upon the initial data) in Sobolev spaces of low order.

1. INTRODUCTION

Let $m(\cdot)\colon [0,+\infty[\longrightarrow]0,+\infty[$ be any continuos function and let Ω be any open subset of $\mathbb{R}^n$. We are concerned with the following integro-differential equation, which as in [AG] we call *Kirchhoff equation*:

$$\begin{aligned} &u_{tt} - m\left(\int_\Omega |\bigtriangledown_x u|^2\,dx\right)\triangle_x u = f(x,t) \qquad (x \in \Omega,\ t > 0),\\ &u(\cdot,t)_{|\partial\Omega} = 0 \qquad\qquad (t \geq 0). \end{aligned} \tag{1}$$

Eq. (1) for $n = 1$ and

$$m(r) = c_0^2 + r$$

was introduced in 1876 by KIRCHHOFF [Ki] as a nonlinear model of the free transversal vibrations of a clamped string. G. F. CARRIER [Ca] and R. NARASHIMA [Na] recovered Eq.(1), without quoting Kirchhoff. See also [Ka] [NM] [Am]. D. W. OPLINGER [Op] compared numerical and experimental results. We refer the reader to the survey [Ar] for references about Eq.(1) from a physical point of view.

From the pure mathematical point of view, Eq.(1) may be considered as the simplest example of quasilinear evolution equation of hyperbolic type.

There are therefore two good reasons to study Eq.(1): for its intrinsic physical meaning, and as a prototype for more complicated equations. A variety of papers have been produced on the existence and uniqueness of the solution of the Cauchy

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problem for Eq.(1): we refer the reader to the survey [Ar] for a (possibly incomplete) commented list. For reasons of space, we do not consider here any generalization of Eq.(1) and we limit ourselves to list known papers about Eq.(1) in three classes:
1) local existence in Sobolev spaces: [Be] [Di1] [Me] [Ri] [MM] [AG] [Ma];
1′) same as above, but degenerate $(m \geq 0)$: [Ni] [Eb] [EMM] [Ya] [Yam] [Ga] [AG];
2) special cases of global existence in Sobolev spaces. Special $m(\cdot)$: [Po2]. Travelling wave solutions in $\Omega := \mathbb{R}^n$: [Di2]. $\Omega := \mathbb{R}^n$ and small data: [GH] [DS2]. See also [BL].
3) global existence for analytic initial data: [Be] [Po1] [Ni] [Li] [AS] [Nis] [DS1] [Ar].

The outstanding question for Eq.(1) is the global solvability for C^∞ initial data. In this paper we treat another question which seems to be not yet explicitly considered: the well-posedness in Hadamard's classical sense of the Cauchy problem for Eq.(1). Actually, a straightforward application of T. KATO's general theory [K2] for abstract evolution equations would permit us to decide the question affirmatively. However we felt the exigency of examining the question in its details, and of providing accurate estimates of the solution, for the purpose of investigating the continuity modulus of the

$$\text{resolvent map: (initial data)} \longmapsto \text{(solution)}. \tag{2}$$

We first prove accurate *a priori* estimates for the energy of a solution, which yield local well-posedness in Hadamard's sense for the Cauchy problem for Eq.(1) (Theorems 4.1-4.2). As a byproduct, Ritz-Galerkin approximants converge uniformly in the phase space on a suitable time interval to a function, which is the unique solution (Corollaries 4.1-4.2). Such a proof of existence and uniqueness differs substantially from KATO's point of view, which adopts Banach's contraction principle. This result improves all the existence/uniqueness results of the group 1) above. For instance: in [AG] the open set Ω was assumed to satisfy Poincaré's inequality and the technique of proof was based on a fixed point argument, so no mention was made of Ritz-Galerkin approximations; Matos in [Ma] removed the coerciveness assumption on the operator $-\triangle$, allowing Ω to be any open subset of $\mathbb{R}^n$, but uniqueness was established only for higher Sobolev exponent and the convergence of the Ritz-Galerkin sequence was proven to hold true only in a weak sense. Also, our method provides an accurate estimate of the energy, which should permit us, for instance, to treat in a future paper also the mildly degenerate case $m \geq 0,\ m\left(\int_\Omega |\bigtriangledown_x u_0|^2\, dx\right) > 0$, in the spirit of [Ga].

We discover that the modulus of continuity of the map (2) depends only upon the fixed initial data (and may actually behave very badly), and we describe subsets of initial data with uniform modulus. In particular we investigate conditions which guarantee the Hölder continuity of the map (2).

We note that the two crucial points of our proof of the well-posedness are: (i) a splitting of the difference of two different solutions (this device appears already in [K3]); (ii) the use of the spectral decomposition of the operator $-\triangle$.

Because of the presence of the integro-differential term, we have found that the most profitable way to study Eq.(1) is to reformulate it as an evolution equation in a Hilbert space.

The plan of the paper is as follows:
Section 2. Statement of the main result (the well-posedness of the Cauchy-Dirichlet problem for Eq.(1)).
Section 3. Preliminaries: abstract framework.
Section 4. *A priori* estimates.
Section 5 and Section 6. Proof of the main result.
Section 7. Study of the continuity modulus of the resolvent map (2), which in general is not Hölder continuous.
Appendix.

2. Statement of the main result: well-posedness of the Cauchy-Dirichlet problem for Eq.(1)

Since we want to study Eq.(1) in Sobolev spaces of fractional order, we recall the concrete [1] characterization of the domains

$$V_\alpha(\Omega) := D(A^{\alpha/2})$$

of the fractional powers of the operator

$$D(A) := H^2(\Omega) \bigcap H_0^1(\Omega),$$
$$A := -\triangle_x := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

where Ω is either (i) any open bounded subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of C^∞ class, or (ii) the half-space $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$, or (iii) $\Omega \equiv \mathbb{R}^n$.

In the cases (i), (ii) the characterization is due to D. FUJIWARA [Fu] (cf. [LM]). Let us recall it.

For $\alpha \geq 0$, $\alpha = 2k + \alpha'$, $k \in \mathbb{N}$, $0 \leq \alpha' < 2$, the space V_α as follows is defined:

$$V_\alpha(\Omega) = \{u \in H^\alpha(\Omega) : \ (-\triangle_x)^j u \in H_0^1(\Omega) \text{ for } 0 \leq j \leq k-1, \quad (-\triangle_x)^k u \in B^{\alpha'}\},$$

where

$$B^{\alpha'}(\Omega) := H^{\alpha'}(\Omega) \ \text{ for } \ 0 \leq \alpha' < 1/2,$$
$$B^{\alpha'}(\Omega) := \{u \in H^{\alpha'}(\Omega) : u_{|\partial\Omega} = 0\} \ \text{ for } 1/2 < \alpha' < 2, \ \alpha' \neq 3/2.$$

In the exceptional cases $\alpha' = 1/2$, $\alpha' = 3/2$ we have:

$$B^{1/2}(\Omega) := \{u \in H^{1/2}(\Omega) : \ \int_\Omega d^{-1}(x)|u(x)|^2 \, dx \ < \ \infty\},$$

$$B^{3/2}(\Omega) := \{u \in H^{3/2}(\Omega) \cap H_0^1(\Omega) : \ \int_\Omega d^{-1}(x)|\tilde{D}_j u(x)|^2 \, dx \ < \ \infty, \, 0 \leq j \leq n-1\},$$

[1] For any bounded Ω, the space $V_\alpha(\Omega)$ may be characterized in the following abstract way. Let $\{v_k\}$ be an orthonormal basis such that for each $k = 1, 2, \dots$

$$Av_k = \lambda_k^2 v_k;$$

we have that $u \in V_\alpha(\Omega)$ if and only if

$$\sum_{k=1}^{+\infty} k^{2\alpha/n} \left| \int_\Omega u(x)\overline{v_k(x)} \, dx \right|^2 < +\infty.$$

where $d(x) := \operatorname{dist}(x, \partial\Omega)$, and $(\tilde{D}_j)_{0\le j\le n-1}$ is a system of first order differential operators, the restriction of which to $\partial\Omega$ forms a basis of the tangent space to $\partial\Omega$.

Finally, in case (iii) we simply have

$$V_\alpha(\mathbb{R}^n) := H^\alpha(\mathbb{R}^n). \tag{3}$$

In the following we denote by $|\cdot|$ the norm in $L^2(\Omega)$ and by $|\cdot|_{V_\alpha}$ the seminorm $|A^{\alpha/2}(\cdot)|$. As for the norm in the space $V_\alpha(\Omega)$, we set

$$\|\cdot\|_{V_\alpha} := (c|\cdot|^2 + |\cdot|^2_{V_\alpha})^{1/2}$$

where c is any positive number in the cases (ii), (iii) and $c = 0$ in the case (i).

We need the following definition

Definition 2.1. We say that the function m is *coercive* if and only if there exists a constant ν such that

$$m(s) \ge \nu > 0 \qquad (s \ge 0).$$

We say that the function m is *coercive at* ∞ if and only if

$$m(s) > 0 \qquad (s \ge 0)$$

and

$$\int_0^{+\infty} m(s)\, ds = +\infty. \tag{4}$$

Let us set

$$M(r) := \int_0^r m(s)\, ds \qquad (r \ge 0).$$

Given a pair (u_0, u_1) in $H^1(\Omega) \times L^2(\Omega)$ let us define the Hamiltonian [Be]

$$\mathcal{H}(u_0, u_1) := M\Big(\int_\Omega |\bigtriangledown_x u_0|^2\, dx\Big) + \int_\Omega |u_1(x)|^2\, dx \tag{5}$$

and the quantities $\nu(u_0, u_1)$ and $L(u_0, u_1)$ as the best constants such that

$$M(r) \le \mathcal{H}(u_0, u_1) \implies m(r) \ge \nu(u_0, u_1), \tag{6}$$

$$\Big[M(r) \le \mathcal{H}(u_0, u_1),\ M(s) \le \mathcal{H}(u_0, u_1)\Big] \implies |m(r) - m(s)| \le L(u_0, u_1)\,|r - s|. \tag{7}$$

Note that if m is a locally Lipschitz function coercive at ∞, we have (see Proposition 6.2 below)

$$\nu(u_0, u_1) > 0, \qquad L(u_0, u_1) < +\infty.$$

We remark that part (a) of the following theorem, *i. e.* existence and uniqueness in Sobolev spaces of order $\ge 3/2$, was already proven, by using a fixed point technique, in [AG] under the assumption: Ω contained in a bounded strip of $\mathbb{R}^n$, namely $-\triangle$ strictly positive. Moreover in [Ma] the existence of a solution in Sobolev spaces of order $\ge 3/2$ was proven without restrictions on the set Ω, but the uniqueness was proven only for solutions in Sobolev spaces of order ≥ 2. The technique of proof in [Ma] was based on the spectral decomposition of the operator $-\triangle$, in order to construct a sequence of Ritz-Galerkin approximations converging in a weak star topology to a solution.

We also note that the life span exhibited in [AG] is improved by a factor 4.

Theorem 2.1 (main result)**.** *Let Ω be either any bounded open subset of $\mathbb{R}^n$ with C^∞ boundary or the half-space $\mathbb{R}^n_+$ or $\Omega \equiv \mathbb{R}^n$. Let m be any locally Lipschitz function coercive at ∞, and let $\alpha \geq 3/2$.*

Then (a) *for any pair of initial data $(u_0, u_1) \in V_\alpha(\Omega) \times V_{\alpha-1}(\Omega)$ and any function $f \in L^1_{loc}([0, +\infty[; V_{\alpha-1}(\Omega))$, there exists a unique solution*

$$u \in C^0([0, T^{**}[; V_\alpha(\Omega)) \cap C^1([0, T^{**}[; V_{\alpha-1}(\Omega)),$$

of the problem

$$\begin{cases} u_{tt} - m\left(\displaystyle\int_\Omega |\bigtriangledown_x u|^2\, dx\right) \triangle_x u = f(x,t) & (x \in \Omega,\ t > 0), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & (x \in \Omega), \\ u(\cdot,t) \in V_\alpha(\Omega), \quad u_t(\cdot,t) \in V_{\alpha-1}(\Omega) & (t \geq 0). \end{cases}$$

Let us introduce the time T^*

$$T^* := \frac{\nu(u_0,u_1)^{3/2}}{L(u_0,u_1)\left(m(\displaystyle\int_\Omega |\bigtriangledown_x u_0|^2\, dx)|u_0|^2_{V_{3/2}(\Omega)} + |u_1|^2_{V_{1/2}(\Omega)}\right)}.$$

The critical time T^{**} is implicitly defined by the formula

$$T^* = T^{**}\Bigg\{1 + \left(m(\int_\Omega |\bigtriangledown_x u_0|^2\, dx)|u_0|^2_{V_{3/2}(\Omega)} + |u_1|^2_{V_{1/2}(\Omega)}\right)^{-1/2} \times \int_0^{T^{**}} |f(\cdot,s)|_{V_{1/2}(\Omega)}\, ds\Bigg\}.$$

Moreover: (b) *the solution is the uniform limit of the whole sequence of its Ritz-Galerkin approximations*; (c) *the solution depends continuously upon the data*; *namely, for every $T < T^*$, the map*

$$\textit{data} \longmapsto \textit{solution}$$

$$V_\alpha \times V_{\alpha-1} \times L^1([0,T]; V_{\alpha-1}(\Omega)) \longrightarrow C^0([0,T]; V_\alpha) \cap C^1([0,T]; V_{\alpha-1})$$

is continuous at the point (u_0, u_1, f).

The above statement follows by a straightforward application of the abstract results given in Section 4 (note: in the cases (ii), (iii) one works with an arbitrary $c > 0$, and then lets $c \to 0^+$).

3. Preliminaries: abstract framework

To reformulate Eq. (1) as an evolution equation in a Hilbert space, we need some preliminaries and notations.

Let V and H be real or complex Hilbert spaces, normed respectively by $\|\cdot\|$ and $|\cdot|$, $V \subseteq H$. If V' denotes the (anti)dual of V, we have then $V \subseteq H \subseteq V'$, in the sense that the duality bracket $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{V \times V'}$ coincides with the inner product $(\cdot, \cdot) := (\cdot, \cdot)_H$ on $H \times V$.

Let $A : V \longrightarrow V'$ be a linear bounded operator, symmetric in the sense that

$$\langle Av, w \rangle = \overline{\langle Aw, v \rangle} \qquad (w \in V,\ v \in V).$$

Moreover assume that for some $c(A) \geq 0$ and $\eta(A) > 0$,

$$\begin{aligned} &\langle Av, v\rangle \geq 0 && (v \in V),\\ &\langle Av, v\rangle + c(A)|v|^2 \geq \eta(A)\|v\|^2 && (v \in V). \end{aligned} \tag{8}$$

The operator $(A + c(A))$ turns out to be an isomorphism of V onto V'. We denote $D := D(A) := \{v \in V : Av \in H\}$. It is $D(A+cI_H) = D$ for every $c \in \mathbb{R}$. The operator

$$A_c := (A + cI_H) : D \longrightarrow H \qquad (c \geq c(A)),$$

turns out to be a self-adjoint positive definite operator in H, hence $A : D \longrightarrow H$ is a self-adjoint nonnegative operator in H.

In this paper we establish results for the Cauchy problem for the abstract evolution equation, $(\cdot)' := \frac{d}{dt}$,

$$u'' + m\left(\langle Au, u\rangle\right) Au = f \qquad (t > 0), \tag{9}$$

but for the sake of clearness in calculations, we will give complete proofs only in the case when $f \equiv 0$, *i.e.* for the equation

$$u'' + m\left(\langle Au, u\rangle\right) Au = 0 \qquad (t > 0). \tag{10}$$

with initial data

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{11}$$

If we consider any open set $\Omega \subseteq \mathbb{R}^n$ and we set

$$V := H_0^1(\Omega), \qquad H := L^2(\Omega) \qquad \text{and} \qquad A := -\triangle_x := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

then Eq. (9) reads as Eq. (1). Moreover we may choose

$$c(A) = \begin{cases} 0 & \text{in the case (i)},\\ \text{any } c > 0 & \text{in the cases (ii), (iii)}. \end{cases}$$

Hereafter we set for simplicity's sake

$$c := c(A).$$

We will base our spectral analysis upon the operator $A^{1/2}$, which in the concrete case of the Kirchhoff string (1) is a first order (pseudodifferential) operator.

For every $\alpha \in \mathbb{R}$, we will consider the power $A_c^{\alpha/2}$ (namely the $(\alpha/2)^{th}$ power of A_c), see *e.g.* [Ru]. For $\alpha \geq 0$, the space $V_\alpha := D(A_c^{\alpha/2}) = D(A^{\alpha/2})$ is made a Hilbert space under the norm

$$\|u\|_\alpha := |A_c^{\alpha/2}u|.$$

We note that $V_\alpha \subseteq V_\beta$ for $\beta \leq \alpha$, with continuous and dense injection. For $\alpha < 0$, V_α is defined as the (anti)dual space of $V_{-\alpha}$, endowed with the (anti)dual norm, which will be still denoted by $\|\cdot\|_\alpha$. In any case V_α and H turn out to be isomorphic via $(A_c^{\alpha/2})^t$, where $(\cdot)^t$ denotes transposition (for $\alpha \geq 0$ this means that $(A_c^{\alpha/2})^t$ extends $A_c^{\alpha/2}$, *i.e.* $(A_c^{\alpha/2})^t_{|V_\alpha} = A_c^{\alpha/2}$). More generally we have

$$\begin{aligned} (A_c^{\beta/2})^t_{|V_\alpha} &= (A_c^{(\beta-\alpha)/2})^t A_c^{\alpha/2} && \text{for } 0 \leq \alpha \leq \beta,\\ (A_c^{\beta/2})^t_{|V_\alpha} &= A_c^{(\beta-\alpha)/2}(A_c^{\alpha/2})^t && \text{for } \beta \leq \alpha \leq 0, \end{aligned}$$

so that V_α and $V_{\alpha-\beta}$ are isomorphic via $(A_c^{\beta/2})^t$ for each $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$.

The phase space for Eq.(10) will be

$$\mathcal{E}_\alpha := V_\alpha \times V_{\alpha-1} \qquad (\alpha \in \mathbb{R}),$$

and we set, for any continuous nonnegative function $a(\cdot)$ and $u \in C^0([0,T];\mathcal{E}_\alpha)$, [2]

$$\begin{aligned} E_{\alpha,a}(t,u) &:= a(t)\|A^{1/2}u(t)\|^2_{\alpha-1} + \|u'(t)\|^2_{\alpha-1} \\ &= a(t)|A_c^{(\alpha-1)/2}A^{1/2}u(t)|^2 + |A_c^{(\alpha-1)/2}u'(t)|^2 \qquad (t \ge 0). \end{aligned}$$

4. A priori estimates

From now on, we set for brevity

$$m_{(u)}(t) := m(\langle Au(t), u(t)\rangle).$$

Let f be any function in $L^1_{loc}([0,+\infty[;V_{1/2})$ and let $(u_0,u_1) \in \mathcal{E}_{3/2}$. In the following statement we will need the notation $(t \ge 0)$

$$\eta(t;f,u_0,u_1) := \left\{1 + \left(m(\langle Au_0,u_0\rangle)\|A^{1/2}u_0\|^2_{1/2} + \|u_1\|^2_{1/2}\right)^{-1/2}\int_0^t \|f(s)\|_{1/2}\,ds\right\}^2.$$

Theorem 4.1 (a priori estimates)**.** *Let us assume that the operator A satisfies condition* (8)*, and that there exist two numbers ν and L such that*

$$\text{(12)} \qquad \begin{cases} m(r) \ge \nu > 0 & (r \ge 0), \\ |m(r) - m(s)| \le L\,|r-s| & (r \ge 0,\ s \ge 0). \end{cases}$$

Let $u \in C^0([0,T];\mathcal{E}_{\alpha_0})$, $\alpha_0 \ge 3/2$, be a solution of (9)–(11)*, i.e.*

$$\begin{cases} u'' + m_{(u)}(t)Au = f(t) & (t>0), \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

and let $w \in C^0([0,T];\mathcal{E}_\beta)$, $\beta \in \mathbb{R}$, be a solution of

$$\text{(13)} \qquad w'' + m_{(u)}(t)Aw = g(t) \qquad (t>0), \qquad \text{with } g \in L^1(0,T;V_{\beta-1}).$$

Let us introduce the quantity $T^* := T^*(u_0,u_1) \in]0,+\infty]$ as [3]

$$\text{(14)} \qquad T^* := T^*(u_0,u_1) := \frac{\nu^{3/2}}{L\,E_{3/2,m_{(u)}}(0,u)},$$

and let $T^{**} := T^{**}(u_0,u_1,f)$ be the quantity implicitly defined as

$$T^* = T^{**}\,\eta(T^{**};f,u_0,u_1).$$

Then, for every $\alpha \le \alpha_0$, the following a priori estimates hold true for $0 \le t < T^{**}$:

$$\text{(15)} \qquad E_{\alpha,m_{(u)}}(t,u) \le \frac{\left((E_{\alpha,m_{(u)}}(0,u))^{1/2} + \int_0^t \|f(s)\|_{\alpha-1}\,ds\right)^2}{1 - \eta(t;f,u_0,u_1)t/T^*},$$

[2] This is a short notation to mean that $(u,u') \in C^0([0,T];\mathcal{E}_\alpha)$.

[3] When $T^* = +\infty$ any fraction with T^* in the denominator is meant to vanish.

$$\big(E_{\beta,m_{(u)}}(t,w)\big)^{1/2} \le \big(E_{\beta,m_{(u)}}(0,w)\big)^{1/2} + \int_0^t \left(\frac{\big(E_{\beta,m_{(u)}}(s,w)\big)^{1/2}}{2(T^*\,\eta(s;f,u_0,u_1)^{-1} - s)} + \|g(s)\|_{\beta-1} \right) ds, \tag{16}$$

and

$$E_{\beta,m_{(u)}}(t,w) \;\le\; \frac{\left((E_{\beta,m_{(u)}}(0,w))^{1/2} + \int_0^t \|g(s)\|_{\beta-1}\, ds \right)^2}{1 - \eta(t;f,u_0,u_1)\, t/T^*}. \tag{17}$$

Proof. In order that the schema of proof appear clear, we give the proof only for the case $f \equiv 0$. Let us set

$$y_\beta(t) := \left(E_{\beta,m_{(u)}}(t,w) \right)^{1/2},$$

$$\varphi(t) := \frac{L}{2\nu^{3/2}}\, E_{3/2,m_{(u)}}(t,u).$$

Let us assume for the moment that w belongs to $C^0([0,T];\mathcal{E}_{\beta+1})$: then by Eq.(13) we get that for a. a. $t \in [0,T]$

$$\frac{1}{2}\,\frac{d}{dt}(y_\beta^2) = m'(\langle Au,u\rangle)\,\mathrm{Re}(Au,u')|A_c^{(\beta-1)/2}A^{1/2}u|^2 + \mathrm{Re}(A_c^{(\beta-1)/2}g, A_c^{(\beta-1)/2}u')$$

$$\le \varphi y_\beta^2 \;+\; \|g(s)\|_{\beta-1}\, y_\beta.$$

By dividing by y_β we get

$$\frac{d}{dt}(y_\beta) \le \varphi y_\beta + \|g(s)\|_{\beta-1}.$$

By integrating we get for $0 \le t \le T$

$$y_\beta(t) \;\le\; y_\beta(0) + \int_0^t \big(\varphi(s) y_\beta + \|g(s)\|_{\beta-1}\big)\, ds. \tag{18}$$

Now, due to the fact that Eq.(13) is linear in w, estimate (18) holds true also for any function w which is simply in $C^0([0,T];\mathcal{E}_\beta)$. We can therefore apply (18) for the choice $w = u$, $g \equiv 0$ and any $\alpha \le \alpha_0$ in place of β, to get

$$\big(E_{\alpha,m_{(u)}}(t,u)\big)^{1/2} \le \big(E_{\alpha,m_{(u)}}(0,u)\big)^{1/2} + \int_0^t \varphi(s)\big(E_{\alpha,m_{(u)}}(s,u)\big)^{1/2}\, ds. \tag{19}$$

In particular for $\alpha = 3/2$ we get, by a standard comparison principle for ordinary differential equations and by (14), that

$$\varphi(t) \le \frac{1}{\frac{1}{\varphi(0)} - 2t} = \frac{1}{2(T^* - t)} \qquad (0 \le t < T^* \wedge T), \tag{20}$$

hence by (19), via the Gronwall lemma, that (15) holds true for any $\alpha \le \alpha_0$.

Condition (16) follows immediately by (18) and (20); (17) follows in turn from the Gronwall lemma. Q.E.D.

Remark 4.1. Estimate (15) is the best estimate that we are able to provide for Eq.(10), however we are not able to prove that it is sharp. As a matter of fact *no* counterexample to the global existence for the Cauchy problem for Eq.(10) has been exhibited up to now .

5. Proof of the main result

Now we state, in abstract form, our basic result. In order to make the exposition easier, in this section we will make some simplifying assumptions, namely:
1) m is globally (not merely locally) Lipschitz continuous, and it is bounded;
2) the source term f vanishes identically.

We will discuss the general case in the next section.

For any subset $W \subseteq \mathcal{E}_{3/2}$ and any function m satisfying (12), let us introduce the following quantities:

$$E := E(W) := \sup_{(u_0,u_1)\in W} m(\langle Au_0, u_0\rangle)\|A^{1/2}u_0\|^2_{1/2} + \|u_1\|^2_{1/2}, \tag{21}$$

$$T^* := T^*(W) := \frac{\nu^{3/2}}{LE} \in [0, +\infty]. \tag{22}$$

If W is bounded in $\mathcal{E}_{3/2}$ it is not difficult to prove that (see Proposition 6.2 in the next section)

$$E < \infty \quad \text{and} \quad 0 < T^* \leq \infty.$$

Theorem 5.1 (continuous dependence upon the data)**.** *Let us assume that the operator A satisfies condition* (8)*, and that*

(i) *m satisfies* (12) *and*

$$m(r) \leq \Lambda \qquad (r \geq 0); \tag{23}$$

(ii) *W is a bounded set of $\mathcal{E}_{3/2}$.*

Let the quantities E and T^* be defined as in (21), (22)*. Let α_0 and T be real numbers such that*

$$\alpha_0 \geq 3/2, \tag{24}$$

$$0 < T < T^*. \tag{25}$$

Consider any given solution $u \in C^0([0,T];\mathcal{E}_{\alpha_0})$ to the problem (10)–(11) *with initial data in W. Let finally α be given such that*

$$1/2 \leq \alpha \leq \alpha_0. \tag{26}$$

Then, for every $\varepsilon > 0$, there exists a number δ which depends only upon the quantities ε, ν, Λ, u_0, u_1, α, T/T^*, such that

$$E_{\alpha,\nu}(0, v-u) < \delta \implies E_{\alpha,\nu}(t, v-u) < \varepsilon \qquad (0 \leq t \leq T),$$

for every v in $C^0([0,T];\mathcal{E}_{\alpha_0})$ which solves problem (10)–(11) *with initial data in W.*

Special case: for $\alpha \leq 1/2$ we have a Lipschitz dependence upon the initial data, i.e. the number δ depends in a linear way upon the quantity ε (with coefficient depending only upon the quantities Λ/ν, T/T^*).

Proof. We prove the statement under the following simplifying assumption:

$$\alpha_0 = 3/2.$$

Let u and v be as in the statement, and let us denote by (u_0, u_1) and respectively (v_0, v_1) their initial data. Following [K3], in order to estimate the α^{th}-energy of the difference $v - u$, we split it in two parts. For this we need an auxiliary function z in $C^0([0,T]; \mathcal{E}_\alpha)$, defined as the solution to the linear "hybrid" problem:

$$\begin{cases} z'' + m_{(v)} A z = 0 & (0 < t < T), \\ z(0) = u_0, \\ z'(0) = u_1. \end{cases}$$

Then we set

$$w_1 := v - z \qquad \text{and} \qquad w_2 := z - u,$$

so that

$$w_1 + w_2 = v - u. \tag{27}$$

The first term w_1 is an "innocuous" one. Indeed it satisfies the following linear problem with *vanishing source term*

$$\begin{cases} w_1'' + m_{(v)} A w_1 = 0 & (0 < t < T), \\ w_1(0) = v_0 - u_0, \\ w_1'(0) = v_1 - u_1, \end{cases}$$

so that we may apply Theorem 4.1 (namely estimate (17)) to get for any $\beta \le 3/2$

$$E_{\beta, m_{(v)}}(t, w_1) \le \frac{E_{\beta, m_{(v)}}(0, v-u)}{1 - t/T^*} \qquad (0 \le t \le T). \tag{28}$$

On the other hand, w_2 solves the problem

$$\begin{cases} w_2'' + m_{(v)} A w_2 = (m_{(u)} - m_{(v)}) A u & (0 < t < T), \\ w_2(0) = 0, \\ w_2'(0) = 0. \end{cases}$$

To simplify the notation, we set, for any $\beta \le 3/2$,

$$y_\beta(t) := \left(E_{\beta, m_{(v)}}(t, w_2) \right)^{1/2}. \tag{29}$$

We can apply Theorem 4.1 (estimate (16)) for any

$$\beta \le 1/2 \tag{30}$$

to get

$$y_\beta(t) \le \int_0^t \left(\frac{y_\beta(s)}{2(T^* - s)} + |(m_{(u)} - m_{(v)})(s)| \cdot \|Au(s)\|_{\beta - 1} \right) ds. \tag{31}$$

By arguing as in [AG] and by using condition (12), the split (27), estimates (15) and (28) for $\beta = 1/2$, we have for $0 \le t \le T$

(32)
$$\begin{aligned}|(m_{(u)} - m_{(v)})(t)| &\le L||A^{1/2}v|^2 - |A^{1/2}u|^2| \\ &= L|(A_c^{1/4}A^{1/2}(u+v), A_c^{-1/4}A^{1/2}(v-u))| \\ &\le \frac{L}{\nu^{1/2}}\left((E_{3/2,m_{(v)}}(t,v))^{1/2} + (E_{3/2,m_{(u)}}(t,u))^{1/2}\right) \\ &\qquad \times \left(\|A^{1/2}w_1\|_{-1/2} + \|A^{1/2}w_2\|_{-1/2}\right) \\ &\le \frac{2LE^{1/2}}{\nu(1-t/T^*)^{1/2}}\left(\frac{(E_{1/2,m_{(v)}}(0,v-u))^{1/2}}{(1-t/T^*)^{1/2}} + y_{1/2}(t)\right).\end{aligned}$$

Moreover, since

$$\|A\cdot\|_{\beta-1} \le \|A^{1/2}\cdot\|_{\beta},$$

thanks to (15) we have

$$\|Au(t)\|_{\beta-1} \le \frac{(E_{\beta+1,m_{(u)}}(0,u))^{1/2}}{\nu^{1/2}(1-t/T^*)^{1/2}} \qquad (0 \le t \le T). \tag{33}$$

By putting (32) and (33) into (31) we obtain

$$\begin{aligned}y_\beta(t) &\le \frac{1}{2}\int_0^t \frac{y_\beta(s)\,ds}{(T^*-s)} \\ &\quad + \frac{2LE^{1/2}}{\nu^{3/2}}(E_{\beta+1,m_{(u)}}(0,u))^{1/2}\int_0^t\left(\frac{(E_{1/2,m_{(v)}}(0,v-u))^{1/2}}{(1-s/T^*)^{3/2}} + \frac{y_{1/2}(s)}{1-s/T^*}\right)ds.\end{aligned} \tag{34}$$

Thanks to (24) and (30) we can choose $\beta := 1/2$ in (34). From the definitions of E and T^* we then get

$$y_{1/2}(t) \le \frac{5}{2}\int_0^t \frac{y_{1/2}(s)\,ds}{(T^*-s)} + 2(T^*\,E_{1/2,m_{(v)}}(0,v-u))^{1/2}\int_0^t \frac{ds}{(T^*-s)^{3/2}}.$$

Now by a standard comparison technique we obtain for $0 \le t \le T$,

$$y_{1/2}(t) \le (E_{1/2,m_{(v)}}(0,v-u))^{1/2}\frac{(2-t/T^*)}{(1-t/T^*)^{5/2}}\ t/T^*. \tag{35}$$

If we re-insert the estimate (35) into (34) and we set now $\beta = \alpha$, thanks to the trivial equality

$$1 + \frac{(2-x)x}{(1-x)^2} = \frac{1}{(1-x)^2}, \tag{36}$$

we get the estimate for $0 \le t \le T$,

$$\begin{aligned}y_\alpha(t) &\le \frac{1}{2}\int_0^t \frac{y_\alpha(s)\,ds}{(T^*-s)} \\ &\quad + \frac{2LE^{1/2}}{\nu^{3/2}}\left(E_{\alpha+1,m_{(u)}}(0,u)E_{1/2,m_{(v)}}(0,v-u)\right)^{1/2}\int_0^t \frac{ds}{(1-s/T^*)^{7/2}}.\end{aligned}$$

If $T^* = \infty$, *i.e.* by (22) $LE = 0$, the above inequality gives immediately $y_\alpha \equiv 0$. In the case $T^* < \infty$, by a standard comparison result and (22) we get the following generalization of (35):

(37)
$$y_\alpha(t) \le \left(\frac{E_{\alpha+1,m_{(u)}}(0,u) E_{1/2,m_{(v)}}(0,v-u)}{E} \right)^{1/2} \cdot \frac{(2 - t/T^*)}{(1 - t/T^*)^{5/2}}\ t/T^*.$$

By using (27), (28) for $\beta = \alpha$, and (37) we get that (in the case when v and u have the same initial data the right-hand side of the following inequality is meant to vanish)

$$\begin{aligned} (E_{\alpha,\nu}(t,v-u))^{1/2} &\le (E_{\alpha,m_{(v)}}(t,v-u))^{1/2} \\ &\le \frac{(E_{\alpha,m_{(v)}}(0,v-u))^{1/2}}{(1-t/T^*)^{5/2}} \\ &\quad \times \max\left\{ 1, \left(\frac{E_{\alpha+1,m_{(u)}}(0,u)}{E} \right)^{1/2} \cdot \left(\frac{E_{1/2,m_{(v)}}(0,v-u)}{E_{\alpha,m_{(v)}}(0,v-u)} \right)^{1/2} \right\}. \end{aligned} \tag{38}$$

If $\alpha = 1/2$, then by (23) we get

$$E_{1/2,\nu}(t,v-u) \le \frac{\Lambda}{\nu}\ \frac{E_{1/2,\nu}(0,v-u)}{(1-t/T^*)^5}.$$

This provides our assertion for the case $\alpha = 1/2$. Unfortunately, (38) does not (apparently) permit us to prove it for any other value of α.

The difficulty is related to the fact that w_2 satisfies no equation with vanishing source term. However, we have found that by combining the estimates (35) and (28) with the spectral decomposition of w_2 we can overcome this difficulty. The idea is that the approximation of w_2 (up to a fixed frequency n) may be controlled in a Lipschitz fashion, with Lipschitz constant growing as the accuracy of the approximation increases. On the other hand, the rest of w_2 becomes smaller and smaller as the frequency n increases. Then, one concludes by a standard $\varepsilon - \delta$ argument.

Let $E(\lambda)$ denote the spectral measure associated to the self-adjoint nonnegative operator $A^{1/2}$ (see *e.g.* [Ru]) and χ_G the characteristic function of a subset G of the real line. For each $n \in \mathbb{N}$, let us define the following orthogonal projections in H:

$$P_n := \int \chi_{[0,n]}(\lambda)\, dE(\lambda), \quad Q_n := \int \chi_{]n,+\infty[}(\lambda)\, dE(\lambda). \tag{39}$$

Since $P_n + Q_n = I_H$, we may split

$$w_2 = P_n w_2 + Q_n w_2.$$

We may split the energy

$$E_{\alpha,\nu}(t,w_2) = E_{\alpha,\nu}(t,P_n w_2) + E_{\alpha,\nu}(t,Q_n w_2). \tag{40}$$

Moreover, for $\alpha \geq 1/2$, thanks to (12), (29), (35), (23) and (36) we have

$$\begin{aligned} E_{\alpha,\nu}(t, P_n w_2) &\leq (n^2+c)^{\alpha-1/2}\, E_{1/2,\nu}(t, P_n w_2) \\ &\leq (n^2+c)^{\alpha-1/2}\, E_{1/2,\nu}(t, w_2) \\ &\leq (n^2+c)^{\alpha-1/2}\, y_{1/2}^2(t) \\ &\leq (n^2+c)^{\alpha-1/2}\, \frac{\Lambda}{\nu}\, \frac{E_{1/2,\nu}(0, v-u)}{(1-t/T^*)^5} \end{aligned} \tag{41}$$

On the other hand, we note that $Q_n u$ and $Q_n z$ solve the following problems with vanishing source term:

$$\begin{cases} (Q_n u)'' + m_{(u)} A Q_n u = 0, \\ Q_n u(0) = Q_n u_0, \\ Q_n u'(0) = Q_n u_1, \end{cases} \quad \text{and} \quad \begin{cases} (Q_n z)'' + m_{(v)} A Q_n z = 0, \\ Q_n z(0) = Q_n u_0, \\ Q_n z'(0) = Q_n u_1. \end{cases}$$

Let us define for each $n \in \mathbb{N}$

$$h_{n,\alpha}(u_0, u_1) := \nu\, |Q_n A_c^{(\alpha-1)/2} A^{1/2} u_0|^2 + |Q_n A_c^{(\alpha-1)/2} u_1|^2. \tag{42}$$

Thanks to (12) and estimate (17) of Theorem 4.1 and (23), we get

$$E_{\alpha,\nu}(t, Q_n u) \leq E_{\alpha, m_{(u)}}(t, Q_n u) \leq \frac{E_{\alpha, m_{(u)}}(0, Q_n u)}{1 - t/T^*} \leq \frac{\Lambda}{\nu}\, \frac{h_{n,\alpha}(u_0, u_1)}{1 - t/T^*}.$$

The analogous estimate holds true for $Q_n z$. Summing up, $Q_n w_2 = Q_n z - Q_n u$ satisfies

$$E_{\alpha,\nu}(t, Q_n w_2) \,\leq\, \frac{4\Lambda}{\nu}\, \frac{h_{n,\alpha}(u_0, u_1)}{1 - t/T^*}. \tag{43}$$

We note that *the relevant feature of* (43) *is that the right hand side does not depend upon* (v_0, v_1), *which can vary in all of* W, *but only upon* (u_0, u_1) , *which is fixed.*

Therefore, by (40), (41) and (43) we get for $0 \leq t \leq T$

$$E_{\alpha,\nu}(t, w_2) \leq (n^2+c)^{\alpha-1/2}\, \frac{\Lambda}{\nu} \frac{E_{1/2,\nu}(0, v-u)}{(1-t/T^*)^5} + \frac{4\Lambda}{\nu}\, \frac{h_{n,\alpha}(u_0, u_1)}{1 - t/T^*},$$

and finally from (27), (28) for $\beta = \alpha$ and (23) $(0 \leq t \leq T)$

$$\begin{aligned} E_{\alpha,\nu}(t, v-u) \leq \frac{2\Lambda}{\nu(1 - T/T^*)} \Bigg\{ & (n^2+c)^{\alpha-1/2}\, \frac{E_{1/2,\nu}(0, v-u)}{(1-T/T^*)^4} \\ & + E_{\alpha,\nu}(0, v-u) \,+ 4 h_{n,\alpha}(u_0, u_1) \Bigg\}. \end{aligned} \tag{44}$$

Now fix a number $\varepsilon > 0$. Since (u_0, u_1) belongs to $\mathcal{E}_{3/2}$ and (26) holds true, there exists a natural number $\bar{n}$ such that

$$\frac{8\Lambda}{\nu(1 - T/T^*)} h_{\bar{n},\alpha}(u_0, u_1) \,<\, \frac{\varepsilon}{3}.$$

Moreover we have still by (26)

$$E_{1/2,\nu}(0, v-u) \leq K^{1/2-\alpha} E_{\alpha,\nu}(0, v-u), \qquad \text{where } K := \frac{\|i\|_{V,H}^2}{\eta(A)}. \tag{45}$$

Then any number $\delta > 0$ such that

$$\delta \frac{2\Lambda}{\nu(1-T/T^*)}\left(1+\left(\frac{\bar{n}^2+c}{K}\right)^{\alpha-1/2}\cdot\frac{1}{(1-T/T^*)^4}\right) < \frac{2}{3}\varepsilon$$

satisfies the thesis. Q.E.D.

In Theorem 5.1 we have employed the energy

$$E_{\alpha,\nu}(t,u) := \nu\|A^{1/2}u(t)\|^2_{\alpha-1} + \|u'(t)\|^2_{\alpha-1}$$

since this quantity provides neat estimates. Unfortunately, it has a drawback: when $c > 0$, that is, in the concrete case when Ω does not satisfy the Poincaré inequality, $E_{\alpha,\nu}$ may be not positive definite. However, we can slightly modify $E_{\alpha,\nu}$ to have a positive definite quantity which fits Theorem 5.1 as well as $E_{\alpha,\nu}$. Let us introduce the following modified energy for a function $u \in C^0([0,T];\mathcal{E}_\alpha)$:

$$\tilde{E}_{\alpha,\nu}(t,u) := \nu\|u(t)\|^2_\alpha + \|u'(t)\|^2_{\alpha-1}$$

Theorem 5.2. *The statement of Theorem* 5.1 *holds true too with* $\tilde{E}_{\alpha,\nu}(t,u)$ *in place of* $E_{\alpha,\nu}(t,u)$.

Proof. For every $w \in C^0([0,T];\mathcal{E}_\alpha)$, we have

$$\tilde{E}_{\alpha,\nu}(t,w) = E_{\alpha,\nu}(t,w) + c\nu\|w(t)\|^2_{\alpha-1}, \tag{46}$$

and

$$\|w(t)\|_{\alpha-1} \le \|w(0)\|_{\alpha-1} + \int_0^t \|w'(t)\|_{\alpha-1}\,ds.$$

Then

$$(\tilde{E}_{\alpha,\nu}(t,w))^{1/2} \le (E_{\alpha,\nu}(t,w))^{1/2} + (c\nu)^{1/2}\left(\|w(0)\|_{\alpha-1} + \int_0^t (E_{\alpha,\nu}(s,w))^{1/2}\,ds\right). \tag{47}$$

Let u and v be as in the statement of Theorem 5.1 and let us put $w := v - u$ in (47). Let us fix a number $\varepsilon > 0$, and let us set

$$\varepsilon' := \frac{\varepsilon}{(2+(c\nu)^{1/2}\,T)^2}.$$

Now by Theorem 5.1 there exists a positive number δ $(\le \varepsilon')$ such that

$$E_{\alpha,\nu}(0,v-u) < \delta \implies E_{\alpha,\nu}(t,v-u) < \varepsilon' \qquad (0 \le t \le T).$$

It is easy to check, by the estimate (47) and by the equality (46), that the number δ is as required. Q.E.D.

As a first consequence of the above theorem we obtain the uniqueness and the existence of the solution to problem (10)–(11).

Corollary 5.1 (uniqueness)**.** *Let us assume that the operator* A *satisfies condition* (8)*, and that* m *satisfies* (12) *and* (23).

Let $T > 0$ *be given, and let* u *and* v *in* $C^0([0,T];\mathcal{E}_{3/2})$ *solve Eq.* (10)*, with the same initial data* (u_0,u_1). *Then* u *coincides with* v.

Proof. Set

$$\overline{T} := \max\{t \in [0,T] : \ u \equiv v \text{ on } [0,t]\}.$$

We claim that $\overline{T} = T$. If, for a contradiction, $\overline{T} < T$, we may restart the Cauchy problem at time $\overline{T}$ with initial data $(u(\overline{T}), u'(\overline{T})) = (v(\overline{T}), v'(\overline{T}))$. Then we apply Theorem 5.1 for $\alpha_0 = 3/2$, $\alpha = 1$ and $W = \{(u(\overline{T}), u'(\overline{T}))\}$. We get, for t in some right neighborhood of $\overline{T}$,

$$|(u-v)'(t)|^2 \ \le \ E_{1,\nu}(t, u-v) = 0,$$

hence $u(t) = v(t)$, which contradicts the assumption $\overline{T} < T$. Q.E.D.

We remark that Theorem 5.1 provides a mere continuity result of the resolvent map

$$(u(0), u'(0)) \in \mathcal{E}_\alpha \ \longmapsto \ u \in C^0([0,T]; \mathcal{E}_\alpha). \tag{48}$$

In the following theorem we obtain, under a suitable condition (which will be discussed in the Appendix) on the set W, the uniform continuity of the map in (48).

Theorem 5.3 (uniformly continuous dependence)**.** *Let us assume that the operator A satisfies condition* (8)*, and that*

(i) *m satisfies* (12) *and* (23);

(ii) *W is a bounded set of $\mathcal{E}_{3/2}$.*

Let the quantity T^* be defined as in (22) *and let α_0 and T be real numbers such that* (24) *and* (25) *holds true.*

Moreover assume that for each $n \in \mathbb{N}$

$$h_{n,\alpha} := \nu |Q_n A_c^{(\alpha-1)/2} A^{1/2} u_0|^2 + |Q_n A_c^{(\alpha-1)/2} u_1|^2 \le \sigma(n)$$

with

$$\lim_{n\to\infty} \sigma(n) \ = \ 0.$$

Then, the number δ in the thesis of Theorem 5.1 (*resp.*: *in the thesis of Theorem* 5.2) *may be choosen in such a way to depend only upon ε, ν, Λ, $\sigma(\cdot)$, α, T/T^*.*

Proof. It suffices to substitute in the estimate (44) of the proof of Theorem 5.1 the quantity $\sigma(n)$ for h_n. Q.E.D.

An immediate consequence of Theorem 5.3 is the local existence of the solution of the Cauchy problem (10)–(11). Given any $(u_0, u_1) \in \mathcal{E}_{3/2}$, let us set

$$\mathcal{H}(u_0, u_1) := M(\langle Au_0, u_0\rangle) + |u_1|^2.$$

Corollary 5.2 (existence)**.** *Let us assume that the operator A satisfies condition* (8)*, and that m satisfies* (12) *and* (23)*.*

Let (u_0, u_1) be in $\mathcal{E}_{\alpha_0}$ with $\alpha_0 \ge 3/2$.

Set

$$T^*(u_0, u_1) := \frac{\nu^{3/2}}{L\left(m(\langle Au_0, u_0\rangle)\|A^{1/2}u_0\|_{1/2}^2 + \|u_1\|_{1/2}^2\right)}.$$

Then there exists a unique solution u in $C^0([0, T^*[; \mathcal{E}_{\alpha_0})$ of the Cauchy problem (10)– (11)*.*

Moreover the whole sequence of Ritz-Galerkin approximations converges to the solution u in the norm of the space $C^0([0,T]; \mathcal{E}_{\alpha_0})$ for every $T < T^*(u_0, u_1)$.

Proof. Let P_n be the orthogonal projection given by (39), and let us set

$$W := \{(P_n u_0, P_n u_1) \colon n \in \mathbb{N}\},$$

For each $n \in \mathbb{N}$ we consider the Ritz-Galerkin approximating solution u_n corresponding to the initial data $(P_n u_0, P_n u_1)$, *i.e.* the solution of the problem

$$\begin{cases} u_n'' + m(\langle Au_n, u_n\rangle) Au_n = 0 \qquad (t > 0), \\ u_n(0) = P_n u_0, \\ u_n'(0) = P_n u_1. \end{cases}$$

Note that u_n exists (and is unique) for all $t \geq 0$, by the classical Cauchy-Lipschitz theorem and by the conservation of the Hamiltonian $\mathcal{H}(t, u_n)$, (see Proposition 6.1 below), which implies that $|u_n'(t)| \leq \text{Const}$.

Let T be any number such that

$$0 < T < T^*(u_0, u_1).$$

Let us observe that $T^*(W) = T^*(u_0, u_1)$, so we have $T < T^*(W)$.

We claim that (u_n) is a Cauchy sequence in $C^0([0,T]; \mathcal{E}_{\alpha_0})$. Since the set W satisfies the hypoteses of Theorem 5.3, for every $\varepsilon > 0$ there exists a $\delta > 0$ which satisfies the thesis of Theorem 5.3. Then we choose $\bar{n}$ large enough to have

$$\tilde{E}_{\alpha_0,\nu}(0, u_n - u_m) < \delta \qquad (n \geq \bar{n},\ m \geq \bar{n}).$$

By Theorem 5.3 we have therefore

$$\tilde{E}_{\alpha_0,\nu}(t, u_n - u_m) < \varepsilon \qquad \text{for } 0 \leq t \leq T.$$

This is exactly our claim. Therefore (u_n) converges to a function $u \in C^0([0,T]; \mathcal{E}_\alpha)$ for each $\alpha \leq \alpha_0$, in particular for $\alpha = 1$. By the continuity of the function m, we have

$$m(\langle Au_n(\cdot), u_n(\cdot)\rangle) \longrightarrow m(\langle Au(\cdot), u(\cdot)\rangle) \qquad \text{uniformly on } [0,T].$$

To achieve the existence it suffices to pass to the limit in Eq.(10). The uniqueness was already established in Corollary 5.1. Q.E.D.

6. Extensions of the results of section 5

In this section we will prove that the simplifying assumptions made in section 4 may be dispensed with.

First we prove that, due to the conservative character of the Kirchhoff equation, it is enough to require (12) and (23) in a local sense.

To this aim we recall the Hamiltonian (5), which in abstract form reads as

$$\mathcal{H}(t,u) := M(\langle Au(t), u(t)\rangle) + |u'(t)|^2 \qquad (t \geq 0).$$

We have

Proposition 6.1 ([Be])**.** *Let $u \in C^0([0,T]; \mathcal{E}_{3/2})$ be a solution to the Cauchy problem* (10)– (11)*. Then the Hamiltonian $\mathcal{H}(t,u)$ is constant with respect to t.*

For any subset $W \subseteq \mathcal{E}_{3/2}$, let us introduce the following (possibly infinite) quantities. First we set

$$\mathcal{H} := \mathcal{H}(W) := \sup_{(u_0,u_1)\in W} M(\langle Au_0, u_0\rangle) + |u_1|^2. \tag{49}$$

Then we introduce $\nu := \nu(W)$, $\Lambda := \Lambda(W)$ and $L := L(W)$ as the best constants such that

$$M(r) \le \mathcal{H} \implies \nu \le m(r) \le \Lambda, \tag{50}$$

$$\Big[M(r) \le \mathcal{H},\ M(s) \le \mathcal{H}\Big] \implies |m(r) - m(s)| \le L\,|r - s|. \tag{51}$$

Finally let $E(W)$ and $T^*(W)$ be defined as in (21), (22).

Proposition 6.2. *Let us assume that the operator A satisfies condition* (8)*, and that*

(i) $m\colon [0,+\infty[\longrightarrow]0,+\infty[$ *is a locally Lipschitz function coercive at* ∞ (*Def.* 1.1);

(ii) W *is a bounded subset of* $\mathcal{E}_{3/2}$.

Let the quantities $\mathcal{H}$, E, ν, Λ, L and T^* be defined as above. Then

$$\mathcal{H} + E + \Lambda + L < \infty,$$

$$0 < \nu,$$

$$0 < T^* \le \infty.$$

Moreover, if we restrict ourselves to solutions in $C^0([0,T];\mathcal{E}_{3/2})$, for initial data in W the Cauchy problem (10)–(11) *is equivalent to an analogous problem for a modified function $\tilde{m}$ satisfying* (12) *and*

$$\tilde{m}(s) \le \Lambda \qquad (s \ge 0).$$

Proof. Assumption (ii) means that the quantity

$$\|u_0\|^2_{3/2} + \|u_1\|^2_{1/2}, \qquad (u_0,u_1) \in W,$$

is bounded. The same holds true for the quantity $\langle Au_0, u_0\rangle$ ($(u_0,u_1) \in W$), since, thanks to (8) and the spectral decomposition of the operators $A^{1/2}$ and $A_c^{1/2}$,

$$\langle Au_0, u_0\rangle \le K\|u_0\|^2_{3/2}, \qquad \text{where } K := \frac{\|i\|_{V,H}}{\eta(A)^{1/2}}.$$

Therefore the quantities $\mathcal{H}$ and E are finite. Then by assumption (i) it follows that the quantities Λ, L and T^* are finite, and that $\nu > 0$.

Let $u \in C^0([0,T];\mathcal{E}_{3/2})$ be any solution of the problem (10)–(11) with initial data in W. By Proposition 6.1,

$$\mathcal{H}(t,u) = \mathcal{H}(0,u) =: \mathcal{H},$$

so we have

$$M(\langle Au(t), u(t)\rangle) \le \mathcal{H} \qquad (0 \le t \le T). \tag{52}$$

Again by (i) (cf. (4)) there exists a number $\tilde{r}$ such that

$$M(r) > \mathcal{H} \qquad \text{for } r > \tilde{r}. \tag{53}$$

From (52) and (53) it follows that

$$\langle Au(t), u(t)\rangle \le \tilde{r} \qquad (0 \le t \le T).$$

If we truncate the function m in the following manner:

$$\tilde{m} := \begin{cases} m(r) & \text{if } 0 \le r < \tilde{r}, \\ m(\tilde{r}) & \text{if } r \ge \tilde{r}, \end{cases}$$

then u satisfies also the equation

$$u'' + \tilde{m}(\langle Au(t), u(t)\rangle)Au = 0 \qquad (0 < t < T).$$

If moreover $\tilde{r}$ is choosen as the smallest number such that (53) holds true (*i.e.* $\tilde{r} = M^{-1}(\mathcal{H})$), we have

$$\inf_{[0,+\infty[} \tilde{m} = \inf_{[0,\tilde{r}]} m =: \nu,$$

$$\sup_{[0,+\infty[} \tilde{m} = \sup_{[0,\tilde{r}]} m =: \Lambda,$$

$$\sup_{\substack{r,s\in[0,\tilde{r}]\\ r\neq s}} \frac{|\tilde{m}(r) - \tilde{m}(s)|}{|r-s|} = \sup_{\substack{r,s\in[0,\tilde{r}]\\ r\neq s}} \frac{|m(r) - m(s)|}{|r-s|} =: L.$$

The converse statement may be proven in an analogous way. Q.E.D.

Thanks to Proposition 6.2, it follows that *in the statements of Theorems* 5.1, 5.2, 5.3 *and Corollaries* 5.1, 5.2, *one might assume* m *locally Lipschitz and coercive at* ∞.

Then we pass to consider the case when a source term f is present. When we are dealing with Eq.(9), there are a few complications in the statement of the theorems. More explicitly, let W be a bounded set of $\mathcal{E}_{3/2}$ and let F be a subset of $L^1([0,T^*]; V_{\alpha_0 - 1})$, F bounded in $L^1([0,T^*]; V_{1/2})$, where $T^* = T^*(W)$ is given by (22). We assume that f varies in F. In this case the Hamiltonian $\mathcal{H}(t)$ (see (5) is no longer constant anyway, we have the estimate

$$\mathcal{H}'(t) = 2\,\mathrm{Re}(u', f) \le 2\,|u'|\,|f| \le 2\,\mathcal{H}(t)^{1/2}|f|.$$

Therefore

$$\mathcal{H}(t) \le \left(\mathcal{H}(0)^{1/2} + \int_0^t |f(s)|\,ds\right)^2.$$

The presence of the source term forces us to modify the constants ν, Λ, L and the critical time T^*. First of all, we set

$$\tilde{\mathcal{H}} := \tilde{\mathcal{H}}(W,F) := \sup_{g\in F}\left(\mathcal{H}(W)^{1/2} + \int_0^{T^*} |g(s)|\,ds\right)^2,$$

then we define $\tilde{\nu}$, $\tilde{\Lambda}$, $\tilde{L}$ as in (50) and (51) with $\tilde{\mathcal{H}}$ in place of $\mathcal{H}(W)$. Finally, the critical time $T^{**} := T^{**}(W,F)$ is implicitly defined by

$$\frac{\tilde{\nu}^{3/2}}{\tilde{L}} = T^{**}\left(E^{1/2} + \sup_{g\in F}\int_0^{T^{**}} \|g\|_{1/2}\,ds\right)^2.$$

Now we are in a position to state the inhomogeneous version of Theorem 5.1, which reads as follows. For every $\varepsilon > 0$, there exists a number δ which depends only upon the quantities ε, $\tilde{\nu}$, $\tilde{\Lambda}$, u_0, u_1, α, T/T^{**}, such that

$$E_{\alpha,\nu}(0, v-u) + \int_0^T \|(g-f)(s)\|_{\alpha-1}\,ds < \delta \implies E_{\alpha,\nu}(t, v-u) < \varepsilon \quad (0 \le t \le T),$$

for every v in $C^0([0,T]; \mathcal{E}_{\alpha_0})$ which solves the Cauchy problem (9)–(11) with initial data in W and source term g in F.

In a similar way one might modify the statements of Theorem 5.2, Corollary 5.1, Theorem 5.3, Corollary 5.2.

7. Study of the continuity modulus of the resolvent map (3), which in general is not Hölder continuous

Theorem 5.1 guarantees the continuity of the resolvent map (48), but by no means gives informations about its continuity modulus. Actually, the continuity of the resolvent map (48) for the Kirchhoff equation cannot in general be improved to Hölder continuity. To show this we exhibit a counterexample, which is modelled on a previous one of Kato [K1] for the so-called Burgers' equation $u_t + uu_x = 0$.

Let us set $\Omega \equiv \mathbb{R}$ (we might as well consider a bounded interval of $\mathbb{R}$) and $m(r) := 1 + r$. If we are looking for solutions of Eq.(1) (with $f \equiv 0$) of the form $u_0(x - ct)$, for some positive constant c, we find the conditions

$$c = c(u_0) := \left(1 + \int_{-\infty}^{+\infty} \left|\frac{du_0}{dx}\right|^2 dx\right)^{1/2}, \tag{54}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = -c\,\frac{du_0}{dx}(x) \qquad (x \in \mathbb{R}). \tag{55}$$

Given a function u_0, we call $-c(u_0)\,du_0/dx$ the *compatible velocity* for u_0.

Now let us define

$$f(x) := \begin{cases} \dfrac{x^{3/2}}{\log x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$$

let us consider a function $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi \equiv 1$ for $|x| \le 1/2$, and finally let us consider for $\delta \in [-1,1]$, the family of functions $u_0^\delta \in H^2(\mathbb{R})$ defined as

$$u_0^\delta(x) := (\delta + f(x))\phi(x) \qquad (x \in \mathbb{R}).$$

Note that

$$\|u_0^\delta - u_0^0\|_{H^2} = |\delta|\,\|\phi\|_{H^2}. \tag{56}$$

If we choose as initial data for the Kirchhoff equation the family u_0^δ with the relative compatible velocities, we get the following family of solutions:

$$u^\delta(x,t) = (\delta + f(x - c_\delta t))\phi(x - c_\delta t),$$

where $c_\delta := c(u_0^\delta)$ (see (54)).

A simple computation gives for two suitable constants $k_1,\, k_2 > 0$

$$k_1\,|\delta| \;\ge\; |c_\delta - c_0| \;\ge\; k_2\,|\delta|^2 \qquad \text{as } \delta \to 0. \tag{57}$$

Let us evaluate the H^2-norm of the difference of the solutions corresponding to u_0^δ and u_0^0. Let us fix $T > 0$. If δ is so small that $|c_\delta - c_0|T < 1$, we have

$$\begin{aligned} \sup_{t\in[0,T]} \|u^\delta(t) - u^0(t)\|_{H^2}^2 &\ge \int_0^{|c_\delta - c_0|T} \left|\frac{d^2 f}{dx^2}(x)\right|^2 dx \\ &= \int_0^{|c_\delta - c_0|T} x^{-1}\left(\frac{3}{4\log x} + \frac{2}{\log^2 x} - \frac{2}{\log^3 x}\right)^2 dx. \end{aligned}$$

Then for δ small enough, by (57) the right-hand side behaves like

$$\left(\frac{3}{4}\right)^2 \int_0^{|c_\delta - c_0|T} \frac{dx}{x \log^2 x} = \left(\frac{3}{4}\right)^2 \frac{1}{-\log(|c_\delta - c_0|T)}$$
$$\geq \left(\frac{3}{4}\right)^2 \frac{1}{-\log(k_2|\delta|^2 T)}.$$

By (56) one cannot therefore expect Hölder continuity of the map (48) for any positive exponent.

To conclude, let us investigate which conditions guarantee the Hölder continuity of the map (48). For example, via an interpolation estimate, it is not difficult to prove that if $W \subseteq \mathcal{E}_{\alpha+\beta}$ for some $\beta \in]0,1]$, then the map (48) is Hölder continuous with exponent β. In the next theorem we prove the stronger result that *it suffices to take the fixed initial datum* (u_0, u_1) *in* $\mathcal{E}_{\alpha+\beta}$ *to have Hölder continuity of the map* (48) *at* (u_0, u_1).

Theorem 7.1. *Let us assume that the operator* A *satisfies condition* (8)*, and that*

(i) $m : [0, +\infty[\longrightarrow]0, +\infty[$ *is a locally Lipschitz function coercive at* ∞ ;

(ii) W *is a bounded set of* $\mathcal{E}_{3/2}$.

Let the quantities E, T^* $\mathcal{H}$, ν, Λ *and* L *be defined as in* (21), (22), (49)–51). *Let* α *and* T *be real numbers such that*

$$\alpha \geq 3/2,$$

$$0 < T < T^*.$$

Assume that the initial datum (u_0, u_1) *belongs to* $\mathcal{E}_{\alpha+\beta}$ *with* $0 < \beta \leq 1$.

Then

(58)
$$E_{\alpha,\nu}(t, u - v) \leq C\left\{(E_{\alpha,\nu}(0, u - v))^\beta + E_{\alpha,\nu}(0, u - v)\right\} \qquad (0 \leq t \leq T),$$

where the constant C *depends upon the quantities* ν, Λ, T/T^*, $E_{\alpha+\beta,\nu}(0, u)$, E *and* α.

The exponent β *in* (58) *cannot be improved.*

Remark 7.1. In the case when $\beta = 1$, we have Lipschitz continuity of the map (48) (note that this agrees with the last statement of Theorem 5.1 for the case $\alpha = 1/2$, since the initial data belong at least to $\mathcal{E}_{3/2}$).

Proof. Let $n \in [0, +\infty[$ and let $h_{n,\alpha}(u_0, u_1)$ be defined as in (42). We have for $n^2 + c > 0$

$$h_{n,\alpha}(u_0, u_1) \leq (n^2 + c)^{-\beta} E_{\alpha+\beta,\nu}(0, u). \tag{59}$$

Let us put $\alpha + \beta - 1$ in place of α in the estimate (37) of the proof of Theorem 5.1. We get, by (29) and (36), for $0 \leq t \leq T$

$$E_{\alpha+\beta-1,m(v)}(t, w_2) =: y^2_{\alpha+\beta-1}(t) \leq \frac{E_{\alpha+\beta,m_{(u)}}(0, u) E_{1/2,m_{(v)}}(0, v - u)}{E\,(1 - T/T^*)^5}.$$

By following the lines of the estimate (41) with $\alpha+\beta-1$ in place of $1/2$, we have then

$$\begin{aligned} E_{\alpha,\nu}(t,P_nw_2) &\leq (n^2+c)^{1-\beta}\,E_{\alpha+\beta-1,m_{(v)}}(t,w_2) \\ &\leq \frac{\Lambda}{\nu}\,(n^2+c)^{1-\beta}\,\frac{E_{\alpha+\beta,m_{(u)}}(0,u)E_{1/2,\nu}(0,v-u)}{E(1-t/T^*)^5}. \end{aligned} \tag{60}$$

Moreover, thanks to (43) and (59) we have for $n^2+c>0$

$$E_{\alpha,\nu}(t,Q_nw_2) \leq \frac{4\Lambda}{\nu}\,(n^2+c)^{-\beta}\,\frac{E_{\alpha+\beta,\nu}(0,u)}{1-t/T^*}. \tag{61}$$

Therefore, by (27), (28), (40), (60) , (61) and (45) for $\beta=\alpha$, we get for $0\leq t\leq T$

$$\begin{aligned} E_{\alpha,\nu}(t,v-u) &\leq \frac{2\Lambda}{\nu(1-T/T^*)}\Bigg\{(n^2+c)^{1-\beta}\,\frac{E_{\alpha+\beta,m_{(u)}}(0,u)\,E_{1/2,\nu}(0,v-u)}{E(1-T/T^*)^4} \\ &\qquad + 4(n^2+c)^{-\beta}\,E_{\alpha+\beta,\nu}(0,u) + E_{\alpha,\nu}(0,v-u)\Bigg\} \\ &\leq C\Bigg\{(n^2+c)^{1-\beta}\,E_{\alpha,\nu}(0,v-u) + (n^2+c)^{-\beta} + E_{\alpha,\nu}(0,v-u)\Bigg\}, \end{aligned}$$

where C is a constant depending only upon Λ/ν, T/T^*, $E_{\alpha+\beta,\nu}(0,u)$, E and α.

If we minimize with respect to the (real) variable n, and then maximize with respect to $\beta\in]0,1]$, we get

$$\begin{aligned} E_{\alpha,\nu}(t,u-v) &\leq C\Bigg\{\frac{(E_{\alpha,\nu}(0,u-v))^\beta}{\beta^\beta(1-\beta)^{1-\beta}} + E_{\alpha,\nu}(0,u-v)\Bigg\} \\ &\leq 2\,C\Bigg\{(E_{\alpha,\nu}(0,u-v))^\beta + E_{\alpha,\nu}(0,u-v)\Bigg\} \end{aligned}$$

This proves estimate (58).

To show that the exponent β is optimal we will exhibit, for each β, an initial datum belonging to $\mathcal{E}_{\alpha+\beta}$, for which condition (58) fails to hold true for any $\beta'>\beta$.

We will do it only for $\alpha=2$, by adapting the previous counterexample. Let us define for $\beta>0$

$$f_\beta(x) := \begin{cases} \dfrac{x^{3/2+\beta}}{\log x} & \text{if } x>0, \\ 0 & \text{if } x\leq 0, \end{cases}$$

and let us consider the following family of functions $u_0^{\delta,\beta}$, $\delta\in[-1,1]$, given by

$$u_0^{\delta,\beta}(x) := (\delta+f_\beta(x))\phi(x).$$

In order to improve (57) to

$$k_1|\delta| \,\geq\, |c_\delta-c_0| \,\geq\, k_3\,|\delta| \qquad \text{as } \delta\to 0 \ \ (k_3>0),$$

we assume now that

$$\int_0^{+\infty}\frac{d\phi}{dx}\,\frac{d(\phi f)}{dx}\,dx\neq 0.$$

Since $u_0^{\delta,\beta} \in H^{2+\beta}(\mathbb{R})$, the pair $(u_0^{\delta,\beta}, c(u_0^{\delta,\beta})du_0^{\delta,\beta}/dx)$ belongs to $\mathcal{E}_{2+\beta}$ (see (3)).

On the other hand, by arguing as in the previous counterexample we get, for $\delta \to 0^+$

$$\frac{(k_2\,|\delta|\,T)^\beta}{-\log^2(k_2|\delta|\,T)} = \mathrm{O}\left(\sup_{t\in[0,T]} \|u^{\delta,\beta}(t) - u^{0,\beta}(t)\|_{H^2}\right).$$

The above inequality proves that (58) may not hold for any $\beta' > \beta$. Q.E.D.

Appendix A.

Let H be any real or complex Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$. Let $B : D(B) \subseteq H \to H$ be any self-adjoint operator. We denote by $E(\lambda)$ the spectral decomposition relative to B; we refer to [Ru] for the definition and the properties of $E(\lambda)$. For any $x \in H$, let $E_{x,x}$ be the positive measure given by

$$E_{x,x}(\omega) = (E(\omega)x, x) \qquad \text{for any Borel subset } \omega \text{ of } \mathbb{R}.$$

For every $s \geq 0$, let us define the following orthogonal projection in H

$$Q_s := \int \chi_{]s,+\infty[}\, dE(\lambda),$$

where χ_G denotes the characteristic function of a set $G \subseteq \mathbb{R}$.

Theorem A.1. *For any bounded subset W of H, the following conditions are equivalent:*

(A1) $$\lim_{s\to+\infty} |Q_s x| = 0 \quad \textit{uniformly in } x \in W.$$

(A2) *There exists a measurable real function f such that*

$$\lim_{r\to+\infty} f(r) = +\infty, \qquad W \textit{ is a bounded subset of } D(f(B)).$$

Proof. (A2) $\Longrightarrow$ (A1). We have for every $x \in W$, for s large enough

$$\begin{aligned} |Q_s x|^2 &= \int \chi_{]s,+\infty[}\, dE_{x,x} \\ &\leq \frac{1}{\inf_{]s,+\infty[} f^2} \int \chi_{]s,+\infty[} f^2\, dE_{x,x} \\ &\leq \frac{1}{\inf_{]s,+\infty[} f^2}\, |f(B)x|^2. \end{aligned}$$

Now (A2) implies that

$$|f(B)x|^2 \leq Const. \qquad (x \in W),$$

and

$$\lim_{s\to+\infty} \frac{1}{\inf_{]s,+\infty[} f^2} = 0.$$

(A1) $\Longrightarrow$ (A2). Let k_0 be the first natural number such that

$$\sup_{x\in W} |Q_{k_0} x| \leq 1,$$

and by induction for $i = 1, 2, \dots$ let us define k_i as the first natural number such that

$$k_i > k_{i-1} \quad \text{and} \quad \sup_{x \in W} |Q_{k_i} x| \le 2^{-i}.$$

We construct the function f in the following way:

$$f(r) := \begin{cases} i & \text{if } r \in]k_i, k_{i+1}], \\ 0 & \text{if } r \le k_0. \end{cases}$$

For every $x \in W$, we have

$$|f(B)x|^2 = \int f^2 \, dE_{x,x} = \sum_{i=0}^{+\infty} \int \chi_{]k_i, k_{i+1}]} f^2 \, dE_{x,x} \le \sum_{i=0}^{+\infty} i^2 \, 2^{-i} < +\infty.$$

Since $\lim_{r \to +\infty} f(r) = +\infty$, the proof is complete . Q.E.D.

The next theorem states two conditions under which (A1) (equivalently: (A2)) holds true.

Theorem A.2. *For any bounded subset W of H, let us consider the following conditions:*

(A3) *W is a relatively compact subset of H.*

(A4) *There exists a nonnegative Borel measure μ on $\mathbb{R}$, such that*

$$\mu(\mathbb{R}) < +\infty \qquad \text{and}$$

on some positive half-line: $E_{x,x} \le \mu$ for each $x \in W$.

Then each of the above conditions are sufficient to imply (A1)*, but none is necessary (but if B^{-1} exists and is a compact operator, then* (A3) *is necessary).*

Proof. (A3) $\Longrightarrow$ (A1). Since for every $x \in H$

$$|Q_s x|^2 = \int \chi_{]s, +\infty[} \, dE_{x,x} = E_{x,x}(]s, +\infty[) \to 0 \quad \text{as } s \to +\infty,$$

it follows that $|Q_s x| \to 0$ uniformly on any relatively compact subset of H.

(A4) $\Longrightarrow$ (A1). We have for s large enough

$$|Q_s x|^2 = E_{x,x}(]s, +\infty[) \le \mu(]s, +\infty[) \qquad (x \in W).$$

Since μ is finite we have $\lim_{s \to +\infty} \mu(]s, +\infty[) = 0$, so that (A1) holds true.

We remark that between the conditions (A3) and (A4) there is in general no implication, as the following examples show. *A fortiori* none of these conditions is necessary to have (A1).

(A3) $\not\Longrightarrow$ (A4). Let B be any unbounded positive operator. Then there exists a sequence (ω_k) of subsets of $\mathbb{R}$ such that:

(i) the ω_k are mutually disjoint;

(ii) $E(\omega_k) \neq 0$ for each $k \in \mathbb{N}$;

(iii) for each $s \in \mathbb{R}$ there exists $\overline{k}(s)$ such that $\omega_k \subseteq [s, +\infty[$ for each $k \ge \overline{k}(s)$ (*e.g.* $\omega_k = [\lambda_k, \lambda_{k+1}[$, where (λ_k) is a suitable sequence of real numbers $\nearrow +\infty$).

By (ii) there exists $x_k \in H$, $E(\omega_k)x_k \neq 0$. We set

$$v_k := k^{-1/2}\,\frac{E(\omega_k)x_k}{|E(\omega_k)x_k|} \qquad (k \in \mathbb{N}).$$

We have $|v_k| = k^{-1/2}$, so that $W := \{v_k : k \in \mathbb{N}\}$ is relatively compact in H. On the other hand, if μ is any nonnegative Borel measure such that $E_{v_k,v_k} \leq \mu$ $(k \in \mathbb{N})$ on some half-line $[s, +\infty[$, then by (iii)

$$k^{-1} = |v_k|^2 = E_{v_k,v_k}(\mathbb{R}) = E_{v_k,v_k}(\omega_k) \leq \mu(\omega_k) \qquad (k \geq \overline{k}(s)),$$

hence by (i)

$$\mu(\mathbb{R}) \;\geq\; \sum_{k\geq 0} \mu(\omega_k) \;\geq\; \sum_{k\geq \overline{k}} k^{-1} = +\infty.$$

(A4) $\not\Longrightarrow$ (A3). Let $\dim H = \infty$, B any bounded operator, $W := \{x \in H : |x| = 1\}$ and $\mu \equiv 0$.

To finish, we prove that (A3) is a necessary condition to have (A1), if B^{-1} is a compact operator. Indeed if (A1) holds true then

$$W = P_s W + Q_s W.$$

Now BP_s is a bounded operator, hence $P_s = B^{-1}BP_s$ is a compact operator. Therefore $P_s W$ is precompact. Since $Q_s W$ is arbitrarily small as $s \to +\infty$, it follows by a standard argument that W is precompact. Q.E.D.

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