53 (2008) APPLICATIONS OF MATHEMATICS No. 6, 583–598

ON THE WORST SCENARIO METHOD: A MODIFIED CONVERGENCE THEOREM AND ITS APPLICATION TO AN UNCERTAIN DIFFERENTIAL EQUATION*

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(Received July 16, 2007)

Abstract. We propose a theoretical framework for solving a class of worst scenario problems. The existence of the worst scenario is proved through the convergence of a sequence of approximate worst scenarios. The main convergence theorem modifies and corrects the relevant results already published in literature. The theoretical framework is applied to a particular problem with an uncertain boundary value problem for a nonlinear ordinary differential equation with an uncertain coefficient.

Keywords: worst scenario problem, nonlinear differential equation, uncertain input parameters, Galerkin approximation

MSC 2010: 34B15, 47H05, 47J05, 65L60

1. Introduction

This paper

(a) deals with the worst scenario method for a class of problems with uncertain input data,

(b) presents and correctly proves a modified fundamental convergence result, and

(c) applies this result to a particular worst scenario problem.

In brief, the worst scenario problem is characterized by a state operator equation $A_a u = f$ dependent on an input parameter a belonging to an admissible set $\mathcal{U}_{\rm ad}$ that is related to the amount of uncertainty in a. The a-dependent state solution $u(a)$ is then evaluated by a criterion functional. The goal is to maximize the criterion functional over $\mathcal{U}_{\rm ad}$.

* This research was supported in part by the project MSM4781305904 from the Ministry of Education, Youth and Sports of the Czech Republic.

The general abstract framework of the worst scenario method can be found in [4, Chapter II]. One of the goals of the worst scenario analysis is to show the convergence of the state solutions $u(a_n)$ if $a_n \to a$ and, analogously, to show the convergence of approximate state solutions $u_h(a^M)$ if $h \to 0$ and $M \to \infty$, where a^M is the approximate input parameter and h as well as M are the parameters that control the finite-dimensional space of u_h and the approximate admissible set $\mathcal{U}_{\rm ad}^M$, respectively. To show that $u_h(a^M) \to u$, a relationship between h and M mediated through a function μ is introduced in [4, Chapter II]. However, it has turned out that the convergence is not that straightforward unless additional, stronger assumptions are made. In this paper, the μ-based concept has been abandoned and a reshaped convergence theorem as well as its correct proof are presented.

Quasilinear elliptic boundary value problems with uncertain coefficients were studied in [2], [3], [6], [7], see also [4, Chapter III]. In these works the coefficient of the state equation is a u-dependent function. The state problem that has motivated this paper is different: the coefficient is a function of the squared derivative of the state solution u. Equations of this kind describe some electromagnetic phenomena, fluid flow phenomena, and the elastoplastic deformation of a body, see [8, p. 212].

Although the existence of the state solution to the above problems can be proved rather easily, see this paper, the existence of the worst scenario solution is a more challenging problem. Indeed, one of the corner-stones of the convergence analysis (see [1, p. 290], [4, Section 4], [5, p. 178]) is the following convergence result: if $a_n \to a$ uniformly, then $u(a_n) \to u(a)$ strongly or at least weakly in a relevant Sobolev space, where $u(a)$ is the state solution related to the limit parameter $a \in \mathcal{U}_{\rm ad}$. If a is u-dependent, then the Rellich theorem can be used to prove the above convergence, see the above-mentioned references. For the problem analyzed in this paper, however, the Rellich theorem is useless and, consequently, the standard technique for proving the existence of the worst scenario fails.

This is why the $u(a_n) \to u(a)$ convergence is avoided in this paper and the existence of the worst scenario is proved via the convergence of the approximate worst scenarios. In this respect, this approach also differs from that used in [4].

The paper is organized as follows: Section 2 introduces the abstract framework of the worst scenario method, the main convergence result and its proof are presented in Section 3, and Section 4 deals with a relevant application.

2. Worst scenario problem

Let V be a real, separable, and reflexive Banach space and let V^* denote its dual space. We deal with the nonlinear operator state equation

$$A(a)u = b, \quad u \in V, \tag{2.1}$$

where $A(a)\colon V \to V^*$, $b \in V^*$. We assume that the operator $A(a)$ depends on a parameter a that belongs to a set of admissible input parameters $\mathcal{U}_{\rm ad} \subset U$, where U is a Banach space. We assume that

(i) the set $\mathcal{U}_{\rm ad}$ is compact in U;

(ii) a unique state solution $u(a)$ of equation (2.1) exists for any parameter $a \in \mathcal{U}_{\rm ad}$;

(iii) a criterion-functional $\Phi\colon \mathcal{U}_{\rm ad} \times V \to \mathbb{R}$ is given such that: if $a_n \in \mathcal{U}_{\rm ad}$, $a_n \to a$ in U and $v_n \to v$ in V as $n \to \infty$, then

$$\Phi(a_n, v_n) \to \Phi(a, v).$$

The goal is to solve the following worst scenario maximization problem: Find $a^0 \in \mathcal{U}_{\rm ad}$ such that

$$a^0 = \arg \max_{a \in \mathcal{U}_{\rm ad}} \Phi(a, u(a)). \tag{2.2}$$

Due to the difficulties mentioned in the introduction, we will prove the existence of a solution to problem (2.2) by means of a sequence of solutions to approximate worst scenario problems, see (2.3) below.

We resort to a discretization of both the set $\mathcal{U}_{\rm ad}$ and the space V. Let $\mathcal{U}^M_{\rm ad} \subset \mathcal{U}_{\rm ad} \subset U$ be a finite-dimensional approximation of the set $\mathcal{U}_{\rm ad}$ and let V_h be a finite-dimensional subspace of V. Let us consider the Galerkin approximation $u_h(a) \in V_h$ of the state solution $u(a)$. We set the following approximate worst scenario problem: Find $a_h^{M0} \in \mathcal{U}^M_{\rm ad}$ such that

$$a_h^{M0} = \arg \max_{a^M \in \mathcal{U}^M_{\rm ad}} \Phi(a^M, u_h(a^M)). \tag{2.3}$$

Next, we assume that

(iv) the set $\mathcal{U}^M_{\rm ad}$ is compact in U;

(v) for any $a \in \mathcal{U}_{\rm ad}$, there exists a unique Galerkin approximation $u_h(a)$ of the state solution $u(a)$;

(vi) if $a_n \in \mathcal{U}_{\rm ad}$ and $a_n \to a$ in U as $n \to \infty$, then $u_h(a_n) \to u_h(a)$ in V_h;

(vii) if $a_n \in \mathcal{U}_{\rm ad}$, $a_n \to a$ in U as $n \to \infty$, and if $h_n \to 0$ as $n \to \infty$, then $u_{h_n}(a_n) \to u(a)$ in V, where $\{u_{h_n}(a_n)\}$ is an n-controlled sequence of the Galerkin approximations;

(viii) for any $a \in \mathcal{U}_{\rm ad}$, there exists a sequence $\{a^M\}$, $a^M \in \mathcal{U}^M_{\rm ad}$, $M \to \infty$, such that $a^M \to a$ in U as $M \to \infty$.

Except for (vii), the above assumptions appear in [4, Chapter II], too. To show that the approximate worst scenario problem (2.3) has at least one solution, we can proceed analogously to the proof of [4, Theorem 3.3].

3. Main result

The goal of this section is to prove the existence and convergence theorem for the worst scenario. Let us formulate an analogue to [4, Theorem 3.4].

Theorem 3.1. *Let* $\{V_h\}$, $h \to 0$, *be a sequence of finite-dimensional subspaces of the space* V. *For any fixed* $h > 0$, *let* $\{a_h^{M0}\}$, *where* $a_h^{M0} \in \mathcal{U}_{\rm ad}^M$ *and* $M \to \infty$, *be a sequence of solutions to the approximate worst scenario problem* (2.3). *Let the assumptions* (i)–(viii) *be fulfilled. Then there exists a sequence* $\{a_{h_n}^{M_n 0}\}$, $a_{h_n}^{M_n 0} \in \mathcal{U}_{\rm ad}^{M_n}$ *such that* $h_n \to 0$ *and* $M_n \to \infty$ *as* $n \to \infty$, *and*

$$a_{h_n}^{M_n 0} \to a^0 \quad \text{in } U, \tag{3.1}$$

$$u_{h_n}(a_{h_n}^{M_n 0}) \to u(a^0) \quad \text{in } V, \tag{3.2}$$

$$\Phi\big(a_{h_n}^{M_n 0}, u_{h_n}(a_{h_n}^{M_n 0})\big) \to \Phi(a^0, u(a^0)) \tag{3.3}$$

as $n \to \infty$, *where* $a^0 \in \mathcal{U}_{\rm ad}$ *solves problem* (2.2) *and* $u(a^0)$ *is the corresponding state solution mentioned in* (ii).

Proof. We fix a subspace V_h for a while and consider a sequence $\{a_h^{M0}\}$, $a_h^{M0} \in \mathcal{U}_{\rm ad}^M$, $M \to \infty$, i.e., a sequence of solutions of the approximate worst scenario problem (2.3). Since $\{a_h^{M0}\} \subset \mathcal{U}_{\rm ad}$ and $\mathcal{U}_{\rm ad} \subset U$ is compact, there exists a convergent subsequence $\{a_h^{M_k^h 0}\} \subset \{a_h^{M0}\}$ such that

$$a_h^{M_k^h 0} \to a_h^0 \quad \text{in } U \quad \text{as } k \to \infty, \tag{3.4}$$

where $a_h^0 \in \mathcal{U}_{\rm ad}$. The subsequence $\{M_k^h\}$ may depend on h, which is now fixed. By virtue of assumption (vi) of the previous section, we obtain

$$u_h\big(a_h^{M_k^h 0}\big) \to u_h(a_h^0) \quad \text{in } V_h \quad \text{as } k \to \infty. \tag{3.5}$$

Let $a \in \mathcal{U}_{\rm ad}$ be arbitrary and chosen independently of h. It follows from assumption (viii) that there exists a sequence $\{a^M\}$, $a^M \in \mathcal{U}_{\rm ad}^M$, such that

$$a^M \to a \quad \text{in } U \quad \text{as } M \to \infty. \tag{3.6}$$

By virtue of assumption (vi), we infer

$$u_h(a^M) \to u_h(a) \quad \text{in } V_h \quad \text{as } M \to \infty. \tag{3.7}$$

For any k, we have

$$\Phi\big(a_h^{M_k^h 0}, u_h\big(a_h^{M_k^h 0}\big)\big) \geqslant \Phi(a^{M_k^h}, u_h(a^{M_k^h})). \tag{3.8}$$

By virtue of (3.4)–(3.7) and assumption (iii), we obtain

$$\Phi(a_h^0, u_h(a_h^0)) \geqslant \Phi(a, u_h(a)). \tag{3.9}$$

Inequality (3.9) is valid for any $h > 0$.

Let us release h and consider the sequences $\{a_h^0\}$, $\{u_h(a_h^0)\}$, and $\{u_h(a)\}$, where $h \to 0$. Since $\{a_h^0\} \subset \mathcal{U}_{\text{ad}}$ and $\mathcal{U}_{\text{ad}} \subset U$ is compact, there exists a convergent subsequence $\{a_{h_l}^0\} \subset \{a_h^0\}$, $h_l \to 0$ as $l \to \infty$, such that

$$a_{h_l}^0 \to a^0 \quad \text{in } U \quad \text{as } l \to \infty, \tag{3.10}$$

where $a^0 \in \mathcal{U}_{\text{ad}}$. By virtue of assumption (vii), we get for the corresponding sequence of the Galerkin approximations

$$u_{h_l}(a_{h_l}^0) \to u(a^0) \quad \text{in } V \quad \text{as } l \to \infty. \tag{3.11}$$

If we set $a_n := a \in \mathcal{U}_{\text{ad}}$ for $n = 1, 2, \ldots$, then it follows from assumption (vii) that

$$u_{h_l}(a) \to u(a) \quad \text{in } V \quad \text{as } l \to \infty. \tag{3.12}$$

By virtue of (3.9)–(3.12) and assumption (iii), we obtain

$$\Phi(a^0, u(a^0)) \geqslant \Phi(a, u(a)). \tag{3.13}$$

Inequalities (3.8), (3.9), and (3.13) hold for any $a \in \mathcal{U}_{\text{ad}}$, so that a^0 is a solution of problem (2.2).

The existence of the sequence $\{a_{h_n}^{M_n 0}\}$ appearing in (3.1) is a direct consequence of the existence of the solution a^0. Indeed, let us introduce a sequence $\{\varepsilon_n\}$, $n \to \infty$, where

$$\varepsilon_n = \frac{1}{n}.$$

By (3.10), for each n we can find an element $a_{h_n}^0 \in \mathcal{U}_{\text{ad}}$ such that

$$\|a_{h_n}^0 - a^0\|_U < \frac{\varepsilon_n}{2}.$$

If we fix n and the related h_n, then it follows from (3.4) that there exists a sequence $\{a_{h_n}^{M_k^{h_n} 0}\}$, $k \to \infty$, such that

$$a_{h_n}^{M_k^{h_n} 0} \to a_{h_n}^0 \quad \text{in } U \quad \text{as } k \to \infty.$$

Therefore, there exists an element $a_{h_n}^{M_n 0} \in \mathcal{U}_{\text{ad}}^{M_n}$ such that

$$\|a_{h_n}^{M_n 0} - a_{h_n}^0\|_U < \frac{\varepsilon_n}{2},$$

so that

$$\|a_{h_n}^{M_n0} - a^0\|_U < \varepsilon_n.$$

The sequence $\{a_{h_n}^{M_n0}\}$, $n \to \infty$, is convergent to a^0 in U. By virtue of assumption (vii), we infer (3.2), and by assumption (iii), we obtain (3.3). $\square$

Remark 3.1. We can replace the strong convergence $v_n \to v$ in (iii) and $u_{h_n}(a_n) \to u(a)$ in (vii) by the weak convergence. Then, the assertion of Theorem 3.1 is valid if we replace the strong convergence in (3.2) by the weak convergence.

4. Application

In this section, we apply the proposed theoretical framework to a concrete state problem motivated by the following boundary value problem: Find a function $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$-(a(u'^2)u')' = f \quad \text{in } \Omega, \tag{4.1}$$

$$u = 0 \quad \text{on } \Gamma, \tag{4.2}$$

where $\Omega = (0,1)$, $\Gamma = \{0,1\}$, a is a Lipschitz continuous function on $\mathbb{R}_0^+$ (nonnegative real numbers), and $f \in C(\Omega)$. The prime stands for $\mathrm{d}u/\mathrm{d}x$.

Instead of (4.1)–(4.2), we will deal with the following weakly formulated problem: Find $u \in H_0^1(\Omega)$ such that

$$\int_0^1 a(u'^2)u'v' \,\mathrm{d}x = \int_0^1 fv \,\mathrm{d}x \quad \forall\, v \in H_0^1(\Omega), \tag{4.3}$$

where $H_0^1(\Omega)$ is the Sobolev space of absolutely continuous functions on $\overline{\Omega}$ with zero boundary conditions and with a square-integrable generalized derivative on Ω, and $f \in L^2(\Omega)$. We assume that the function a belongs to the admissible set

$$\mathcal{U}_{\mathrm{ad}} := \{a \in \mathcal{U}_{\mathrm{ad}}^0\colon\ 0 < a_{\min} \leqslant a(x) \leqslant a_{\max}\ \forall\, x \in \mathbb{R}_0^+\}$$

which models the uncertainty in a and where

$$\mathcal{U}_{\mathrm{ad}}^0 := \Big\{a \in C^{(0),1}(\mathbb{R}_0^+)\colon\ 0 \leqslant \frac{\mathrm{d}a}{\mathrm{d}x} \leqslant C_{\mathrm{L}} \text{ a.e.},\ a(x) = a(x_{\mathrm{C}}) \text{ for } x \geqslant x_{\mathrm{C}}\Big\},$$

C_{L}, $a_{\min}$, $a_{\max}$, x_{C} are positive constants, and $C^{(0),1}(\mathbb{R}_0^+)$ stands for the Lipschitz continuous functions defined on $\mathbb{R}_0^+$.

We observe that $\mathcal{U}_{\mathrm{ad}} \subset U$, where U is the Banach space of functions continuous on $\mathbb{R}_0^+$ and constant for $x \geqslant x_{\mathrm{C}}$, with the norm $\|w\|_U := \max_{x\in[0,x_{\mathrm{C}}]} |w(x)|$ for $w \in U$.

The operator equation (2.1) stems from (4.3) if we set $V := H^1_0(\Omega)$ and define $A(a)\colon V \to V^*$ and $b \in V^*$ by

$$\langle A(a)u, v\rangle := \int_0^1 a(u'^2)u'v' \,\mathrm{d}x, \tag{4.4}$$

$$\langle b, v\rangle := \int_0^1 fv \,\mathrm{d}x \tag{4.5}$$

where $u, v \in V$. For simplicity we will denote $A(a)$ by A.

The functionals Au and b are obviously linear. Since

$$|\langle b, v\rangle| = \left|\int_0^1 fv \,\mathrm{d}x\right| \leqslant \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)},$$

the functional b is bounded. The functional Au is also bounded:

$$\begin{aligned}|\langle Au, v\rangle| &= \left|\int_0^1 a(u'^2)u'v' \,\mathrm{d}x\right| \leqslant \int_0^1 |a(u'^2)u'v'| \,\mathrm{d}x = \int_0^1 a(u'^2)|u'||v'| \,\mathrm{d}x \\ &\leqslant a_{\max}\int_0^1 |u'||v'| \,\mathrm{d}x \leqslant a_{\max}\|u'\|_{L^2(\Omega)}\|v'\|_{L^2(\Omega)} \leqslant K\|v\|_{H^1(\Omega)},\end{aligned}$$

where $K > 0$.

Lemma 4.1. *The operator A defined by* (4.4) *is continuous on V.*

P r o o f. The function $q\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined as

$$q(x, \xi) = a(\xi^2)\xi$$

satisfies the Carathéodory conditions [1, p. 288]. Moreover, q satisfies the growth condition

$$|q(x, \xi)| \leqslant g(x) + c|\xi|^{p/r},$$

where $g \in L^r(\Omega)$, $c > 0$, and $p, r \in [1, \infty)$ if we set $g(x) = 0$, $c = a_{\max}$, $p = 2$ and $r = 2$. Then the operator

$$\begin{aligned}H\colon\ L^2(\Omega) &\to L^2(\Omega), \\ v &\mapsto a(v^2)v,\end{aligned}$$

the Nemyckii operator associated with q, is continuous, see [1, p. 288].

Let $\{u_n\}$ be a sequence in V such that $u_n \to u$, where $u \in V$. Then $u'_n \to u'$ in $L^2(\Omega)$. Since the operator H is continuous, we have

$$a(u'^2_n)u'_n \to a(u'^2)u' \quad \text{in } L^2(\Omega). \tag{4.6}$$

Now, we show that $\|Au - Au_n\|_{V'} \to 0$. Indeed,

$$\begin{aligned}\|Au - Au_n\|_{V'} &= \sup_{\|v\|_V=1} |\langle Au - Au_n, v\rangle| \\ &= \sup_{\|v\|_V=1} \left|\int_0^1 \big[a(u'^2)u' - a(u_n'^2)u_n'\big] v' \, \mathrm{d}x\right|.\end{aligned}$$

By virtue of $\|v'\|_{L^2(\Omega)} \leqslant \|v\|_{H^1(\Omega)} = 1$ and the Schwarz inequality

$$\begin{aligned}\left|\int_0^1 \big[a(u'^2)u' - a(u_n'^2)u_n'\big] v' \, \mathrm{d}x\right| &\leqslant \big\|a(u'^2)u' - a(u_n'^2)u_n'\big\|_{L^2(\Omega)} \|v'\|_{L^2(\Omega)} \\ &\leqslant \big\|a(u'^2)u' - a(u_n'^2)u_n'\big\|_{L^2(\Omega)}.\end{aligned}$$

Then it follows from this and from (4.6) that

$$\begin{aligned}\|Au - Au_n\|_{V'} &= \sup_{\|v\|_V=1} \left|\int_0^1 \big[a(u'^2)u' - a(u_n'^2)u_n'\big] v' \, \mathrm{d}x\right| \\ &\leqslant \big\|a(u'^2)u' - a(u_n'^2)u_n'\big\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

□

Lemma 4.2. *The operator A defined by (4.4) is strongly monotone, that is,*

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \geqslant C\|u_1 - u_2\|_V^2 \quad \textit{for all } u_1, u_2 \in V, \tag{4.7}$$

where $C > 0$.

Proof. Let us write the left-hand side of (4.7) as

$$\int_0^1 \big[a(u_1'^2)u_1' - a(u_2'^2)u_2'\big](u_1' - u_2') \, \mathrm{d}x \tag{4.8}$$

and define $g(y) := a(y^2)y$. Then (4.8) takes the form

$$\int_0^1 [g(u_1') - g(u_2')](u_1' - u_2') \, \mathrm{d}x.$$

Since a' is a non-negative function (see $\mathcal{U}_{\mathrm{ad}}^0$), we obtain

$$g'(y) = 2a'(y^2)y^2 + a(y^2) \geqslant a_{\min} > 0,$$

so that g is an increasing function. Hence

$$\int_0^1 [g(u_1') - g(u_2')](u_1' - u_2') \, \mathrm{d}x \geqslant a_{\min} \int_0^1 (u_1' - u_2')^2 \, \mathrm{d}x \geqslant C\|u_1 - u_2\|_{H^1(\Omega)}^2,$$

where $C > 0$. □

Theorem 4.1. *Let $a \in \mathcal{U}_{\rm ad}$ be arbitrary. Then problem* (4.3) *has a unique solution.*

P r o o f. The existence of a solution is guaranteed by [10, Theorem 2.K]. Therefore, it is sufficient to verify its assumptions:

(α) The operator $A\colon V \to V^*$ is monotone on the real, separable, reflexive Banach space V, that is,

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \geqslant 0 \quad \text{for all } u_1, u_2 \in V.$$

(β) The operator A is continuous on each finite-dimensional subspace of the Banach space V.

(γ) The operator A is coercive on V, that is,

$$\lim_{\|u\|_V \to \infty} \frac{\langle Au, u\rangle}{\|u\|_V} = +\infty.$$

By Lemma 4.2, the operator A is strongly monotone on V. Consequently, (α) is fulfilled.

By Lemma 4.1, assumption (β) is also fulfilled.

We will show that A is coercive. Since $a(\xi^2)\xi^2 \geqslant a_{\min}\xi^2$, we have

$$\langle Au, u\rangle = \int_0^1 a(u'^2)u'^2\,{\rm d}x \geqslant a_{\min}\int_0^1 u'^2\,{\rm d}x \geqslant C\|u\|^2_{H^1(\Omega)},$$

where C is positive constant. Consequently, (γ) holds.

Since the operator A is strongly monotone, the uniqueness of the state solution follows from [10, p. 93, Corollary 1]. □

Let us pay attention to the *approximation* of equation (4.3) and to the corresponding problem (2.3). To this end, we will define the set $\mathcal{U}^M_{\rm ad} \subset \mathcal{U}_{\rm ad}$ and a finite-dimensional space V_h. Let T_i, $i = 1, \ldots, M$, be equally spaced points in $[0, x_C]$, $T_1 = 0$ and $T_M = x_C$. We define

$$\mathcal{U}^M_{\rm ad} := \{a \in \mathcal{U}_{\rm ad}\colon\ a|_{[T_i, T_{i+1}]} \in P_1([T_i, T_{i+1}]),\ i = 1, \ldots, M-1\},$$

where $P_1([T_i, T_{i+1}])$ denotes the linear polynomials on the interval $[T_i, T_{i+1}]$.

To approximate the space V, we introduce points $x_0, x_1, \ldots, x_{N+1}$ in the interval $[0, 1]$, $x_0 = 0$, $x_{N+1} = 1$. We define the discretization parameter h as

$$h := \max_{i=1,\ldots,N+1}(x_i - x_{i-1}).$$

The space V_h is defined as

$$V_h := \{v_h \in V\colon\ v_h|_{[x_i,x_{i+1}]} \in P_1([x_i,x_{i+1}]),\ i = 0,\dots,N\}. \tag{4.9}$$

Now, we define the Galerkin approximation $u_h(a) \in V_h$ of the solution to problem (4.3) by the identity

$$\int_0^1 a(u_h'^2)u_h'v'\,\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x \quad \forall\, v \in V_h.$$

Theorem 4.2. *Let $a \in \mathcal{U}_{\mathrm{ad}}$ be arbitrary. Then there exists a unique Galerkin approximation $u_h(a)$ of the solution to problem* (4.3).

Proof. The space V_h is a real, separable, and reflexive Banach space. The existence of a unique Galerkin approximation is guaranteed by [10, Theorem 2.K] and [10, p. 93, Corollary 1] applied to (4.3), where we replace V by V_h. □

To be able to apply Theorem 3.1, we have to verify its assumptions. By the Arzelà-Ascoli theorem [9, p. 35] the assumptions (i) and (iv) of Section 2 are fulfilled. By the following theorem, assumption (vi) is fulfilled.

Theorem 4.3. *If $a_n \in \mathcal{U}_{\mathrm{ad}}$ and $a_n \to a$ in U as $n \to \infty$, then $u_h(a_n) \to u_h(a)$ in V_h.*

Proof. The space V_h is fixed. Let us denote the Galerkin approximation $u_h(a_n) \in V_h$ by u_n. By observing that

$$|u_n|^2_{H^1(\Omega)} \leqslant \frac{1}{a_{\min}} \int_0^1 a_n(u_n'^2)u_n'^2\,\mathrm{d}x = \frac{1}{a_{\min}} \int_0^1 fu_n\,\mathrm{d}x \leqslant \frac{\|f\|_{L^2(\Omega)}}{a_{\min}} \|u_n\|_{H^1(\Omega)}$$

and by applying the equivalence of the norm $\|\cdot\|_{H^1(\Omega)}$ and the seminorm $|\cdot|_{H^1(\Omega)}$ in $H^1_0(\Omega)$, we infer that the sequence $\{\|u_n\|_{H^1(\Omega)}\}$ is bounded independently of n.

As a consequence, since V_h is finite-dimensional, the sequence $\{u_n\}$ has a strongly convergent subsequence $\{u_{n_k}\}$; we denote its terms by u_k. Hence $w_h \in V_h$ exists such that

$$u_k \to w_h \quad \text{in } H^1(\Omega) \quad \text{as } k \to \infty. \tag{4.10}$$

Let us note that (4.10) and the dimensionality of V_h imply the convergence of $\{u_k'\}$ in, for instance, the $L^\infty(\Omega)$ space. We will show that $w_h = u_h(a)$.

Let $v \in V_h$ be arbitrary and let us write the approximate state equation as follows:

$$\begin{aligned}\int_0^1 fv\,\mathrm{d}x &= \int_0^1 a_k(u_k'^2)u_k'v'\,\mathrm{d}x \\ &= \int_0^1 a_k(u_k'^2)(u_k'-w_h')v'\,\mathrm{d}x + \int_0^1 \big[a_k(u_k'^2)-a(u_k'^2)\big]w_h'v'\,\mathrm{d}x \\ &\quad+ \int_0^1 \big[a(u_k'^2)-a(w_h'^2)\big]w_h'v'\,\mathrm{d}x + \int_0^1 a(w_h'^2)w_h'v'\,\mathrm{d}x \\ &= I_1+I_2+I_3+I_4.\end{aligned}$$

If $k\to\infty$, the integrals I_1, I_2, and I_3 tend to zero by virtue of (4.10) and the uniform convergence of $\{a_k\}$. Consequently, the left-hand side equals I_4 for any $v\in V_h$, which means that $w_h = u_h(a)$. It follows from the uniqueness of the Galerkin approximation that the entire sequence $\{u_n\}$ converges to $u_h(a)$. □

Lemma 4.3. *Let $\{V_h\}$, $h\to 0$ be a sequence of the finite-dimensional subspaces of V defined by* (4.9) *and such that $h_1 < h_2$ implies $V_{h_2}\subset V_{h_1}$. Then $\bigcup_h V_h$ is dense in V.*

Proof. Let $u\in V$ be arbitrary and let $\varepsilon>0$. There exists a function $v\in C_0^\infty(\Omega)$ such that $\|u-v\|_{H^1(\Omega)}<\varepsilon/2$. The theory of interpolation yields that for a sufficiently small parameter h we can approximate the function v by its interpolant $v_h\in V_h$ such that

$$\|v-v_h\|_{H^1(\Omega)}<\varepsilon/2.$$

Therefore,

$$\|u-v_h\|_{H^1(\Omega)}\leqslant \|u-v\|_{H^1(\Omega)}+\|v-v_h\|_{H^1(\Omega)}<\varepsilon.$$

□

The following lemma is a generalization of [10, p. 94, Lemma 3].

Lemma 4.4. *Let V be a real Banach space, let $A\colon V\to V^*$ be an operator continuous on V, and let $b\in V^*$. Further, let the following assumptions be satisfied for $u\in V$, a sequence $\{u_n\}\subset V$, and a sequence of operators $\{A_n\}$, $A_n\colon V\to V^*$, where A_n are monotone on V:*
(α) $\langle A_nu_n, v\rangle \to \langle b, v\rangle$ *as* $n\to\infty$ $\forall v\in V$,
(β) $\langle A_nu_n, u_n\rangle \to \langle b, u\rangle$ *as* $n\to\infty$,
(γ) $\langle A_nv, u_n\rangle \to \langle Av, u\rangle$ *as* $n\to\infty$ $\forall v\in V$,
(δ) $\langle A_nv, v\rangle \to \langle Av, v\rangle$ *as* $n\to\infty$ $\forall v\in V$.
Then u is a solution of the equation $Au=b$.

P r o o f. We can follow the lines of the proof of [10, p. 94, Lemma 3]. Since each of the operators A_n is monotone, we have

$$\langle A_n u_n, u_n\rangle - \langle A_n v, u_n\rangle - \langle A_n u_n, v\rangle + \langle A_n v, v\rangle = \langle A_n u_n - A_n v, u_n - v\rangle \geqslant 0$$

for all $v \in V$ and all n. Letting $n \to \infty$, we get from (α)–(δ)

$$\langle b, u\rangle - \langle Av, u\rangle - \langle b, v\rangle + \langle Av, v\rangle \geqslant 0 \quad \forall\, v \in V,$$

and hence

$$\langle b - Av, u - v\rangle \geqslant 0 \quad \forall\, v \in V. \tag{4.11}$$

Next, let $v = u - tw$, where $t > 0$ and $w \in V$. It follows from (4.11) that

$$\langle b - A(u - tw), w\rangle \geqslant 0$$

for all $t > 0$ and all $w \in V$. Since A is continuous, we get for $t \to 0$

$$\langle b - Au, w\rangle \geqslant 0 \quad \forall w \in V. \tag{4.12}$$

Since (4.12) is valid for any $w \in V$,

$$\langle b - Au, w\rangle = 0 \quad \forall w \in V.$$

□

Let us pay attention to assumption (vii) of Section 2.

Theorem 4.4. *Let $\{a_n\}$, where $a_n \in \mathcal{U}_{\rm ad}$ and $a_n \to a$ in U as $n \to \infty$ in U, be a sequence of parameters. Let $\{V_{h_n}\}$, $h_n \to 0$ as $n \to \infty$, be a sequence of the subspaces from Lemma (4.3), and let $\{u_{h_n}(a_n)\}$, $u_{h_n}(a_n) \in V_{h_n}$, be the corresponding sequence of Galerkin approximations. Then*

$$u_{h_n}(a_n) \to u(a) \quad \text{in } V,$$

where $u(a)$ is the solution of problem (4.3) for the parameter a.

P r o o f. We can prove, analogously to the proof of Theorem 4.3, that the sequence $\{u_{h_n}(a_n)\}$ is bounded in V.

Then, since V is a reflexive Banach space, the sequence $\{u_{h_n}(a_n)\}$ has a weakly convergent subsequence, let us denote it simply by $\{u_k\}$, such that

$$u_k \rightharpoonup w \quad \text{as } k \to \infty, \tag{4.13}$$

where $w \in V$. We will show that $w = u(a)$.

For any $u \in V$ let us define operators $A, A_k\colon V \to V^*$ by

$$\langle Au, v\rangle := \int_0^1 a(u'^2)u'v' \,\mathrm{d}x \quad \forall\, v \in V,$$
$$\langle A_k u, v\rangle := \int_0^1 a_k(u'^2)u'v' \,\mathrm{d}x \quad \forall\, v \in V.$$

By virtue of Lemma 4.4, we will get $Aw = b$.

It is sufficient to verify assumptions (α)–(δ) of Lemma 4.4.

Assumption (δ) is fulfilled. Indeed, let $v \in V$ be arbitrary. By the uniform convergence of $\{a_k\}$, we get

$$\begin{aligned}\langle A_k v, v\rangle &= \int_0^1 a_k(v'^2)v'^2 \,\mathrm{d}x \\ &= \int_0^1 \big[a_k(v'^2) - a(v'^2)\big] v'^2 \,\mathrm{d}x + \int_0^1 a(v'^2)v'^2 \,\mathrm{d}x \\ &\to \int_0^1 a(v'^2){v'}^2 \,\mathrm{d}x = \langle Av, v\rangle.\end{aligned}$$

Let us focus on assumption (γ). We have

$$\begin{aligned}\langle A_k v, u_k\rangle &= \int_0^1 a_k(v'^2)v'u_k' \,\mathrm{d}x \\ &= \int_0^1 \big[a_k(v'^2) - a(v'^2)\big] v'u_k' \,\mathrm{d}x + \int_0^1 a(v'^2)v'u_k' \,\mathrm{d}x.\end{aligned}$$

Since for given $\varepsilon > 0$ there exists $k(\varepsilon)$ such that

$$\left|\int_0^1 \big[a_k(v'^2) - a(v'^2)\big] v'u_k' \,\mathrm{d}x\right| \leqslant \varepsilon \|v'\|_{L^2(\Omega)} \|u_k'\|_{L^2(\Omega)} \leqslant C_1\varepsilon$$

for $k > k(\varepsilon)$, where $C_1 > 0$, we infer

$$\int_0^1 \big[a_k(v'^2) - a(v'^2)\big] v'u_k' \,\mathrm{d}x \to 0 \quad \text{as } k \to \infty.$$

Since the sequence $\{u_k\}$ weakly converges to w, we obtain

$$\int_0^1 a(v'^2)v'u_k' \,\mathrm{d}x \to \int_0^1 a(v'^2)v'w' \,\mathrm{d}x \quad \text{as } k \to \infty.$$

Therefore,

$$\langle A_k v, u_k\rangle \to \langle Av, w\rangle \quad \text{as } k \to \infty.$$

Assumption (β) is fulfilled because, by (4.13),

$$\begin{aligned}\langle A_k u_k, u_k\rangle &= \int_0^1 a_k(u_k'^2)u_k'^2\,\mathrm{d}x\\ &= \int_0^1 f u_k\,\mathrm{d}x \to \int_0^1 f w\,\mathrm{d}x = \langle b, w\rangle.\end{aligned}$$

We will complete the proof by checking the validity of assumption (α). By using Lemma 4.3, we infer that for any $z \in V$ there exists a sequence $\{v_k\}$ such that $v_k \in V_k \equiv V_{h_{n_k}}$ and

$$v_k \to z \quad \text{in } V \quad \text{as } k \to \infty. \tag{4.14}$$

By definition, we have

$$\langle A_k u_k, v_k\rangle = \langle b, v_k\rangle,$$

so that

$$\langle A_k u_k, v_k\rangle \to \langle b, z\rangle \quad \text{as } k \to \infty. \tag{4.15}$$

Let $\varepsilon > 0$ be arbitrary. We may write

$$|\langle A_k u_k, z\rangle - \langle b, z\rangle| \leqslant |\langle A_k u_k, z - v_k\rangle| + |\langle A_k u_k, v_k\rangle - \langle b, z\rangle| = P_{1k} + P_{2k}.$$

From (4.14) and the boundedness of the sequence $\{u_k\}$, we obtain for any $k > k_1(\varepsilon)$ that

$$P_{1k} = \int_0^1 a_k(u_k'^2)u_k'(z' - v_k')\,\mathrm{d}x \leqslant a_{\max}\|u_k\|_{H^1(\Omega)}\|z - v_k\|_{H^1(\Omega)} \leqslant C_2\varepsilon,$$

where $C_2 > 0$. It follows from (4.15) that

$$P_{2k} \leqslant \varepsilon \quad \forall\, k > k_2(\varepsilon).$$

Summarizing, we arrive at

$$P_{1k} + P_{2k} \leqslant (C_2 + 1)\varepsilon$$

for $k > \max(k_1(\varepsilon), k_2(\varepsilon))$. As a consequence,

$$\lim_{k\to\infty} \langle A_k u_k, z\rangle = \langle b, z\rangle \quad \forall\, z \in V.$$

It follows from the uniqueness of the state solution $u(a)$ (Theorem 4.1) that the entire sequence $\{u_{h_n}(a_n)\}$ converges weakly to $u(a)$.

Now, we will prove that

$$u_{h_n}(a_n) \to u(a) \quad \text{in } V.$$

With respect to Lemma 4.2, it suffices to show that

$$\langle Au_{h_n}(a_n) - Au(a), u_{h_n}(a_n) - u(a)\rangle \to 0. \tag{4.16}$$

Let us denote

$$v_n := u_{h_n}(a_n) - u(a),$$
$$u_n := u_{h_n}(a_n).$$

Since $v_n \rightharpoonup 0$, we have $\langle Au(a), v_n\rangle \to 0$. Further,

$$\langle Au_n, v_n\rangle = \langle A_n u_n, v_n\rangle + \langle Au_n - A_n u_n, v_n\rangle = Q_{1n} + Q_{2n}.$$

By (α) and (β)

$$Q_{1n} = \langle A_n u_n, u_n - u\rangle \to 0.$$

For the second term we get

$$\begin{aligned}|Q_{2n}| &\leqslant \int_0^1 \bigl|a(u_n'^2) - a_n(u_n'^2)\bigr||u_n'||v_n'|\,\mathrm{d}x\\ &\leqslant \|a - a_n\|_U\|u_n\|_{H^1(\Omega)}\|v_n\|_{H^1(\Omega)} \to 0.\end{aligned}$$

Summarizing, we infer (4.16). □

By the following lemma, assumption (viii) of Section 2 is fulfilled.

Lemma 4.5. *Let $a \in \mathcal{U}_{\text{ad}}$ be arbitrary. Then there exists a sequence $\{a^M\}$, $a^M \in \mathcal{U}_{\text{ad}}^M$, such that*

$$a^M \to a \quad \text{in } U \quad \text{as } M \to \infty.$$

Proof. Let M be arbitrary. Let us consider $a^M \in \mathcal{U}_{\text{ad}}^M$ such that

$$a^M(T_i) = a(T_i), \quad i = 1, \ldots, M.$$

The interval $[0, x_C]$ is uniformly subdivided into $M - 1$ subintervals of length ν_M. Since a is Lipschitz continuous, we obtain

$$\|a^M - a\|_U \leqslant C_L \nu_M.$$

If $M \to \infty$, then $\nu_M \to 0$ and $a^M \to a$ in U. □

We have shown that the assumptions of Section 2 are fulfilled. Consequently, it follows from Theorem 3.1 that the worst scenario problem (2.2) with the state equation (4.3) has a solution $a^0 \in \mathcal{U}_{\rm ad}$. Furthermore, there exists a sequence of approximate worst scenarios that converges to a^0.

Acknowledgments. The author would like to thank Dr. J. Chleboun for his great help during the work on this paper. The author is also indebted to an anonymous referee and Dr. P. Krejčí for their suggestions leading to an improved paper.

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